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A FORMULA FOR THE DENSITY OF THE NORM OF STABLE RANDOM VECTORS IN HILBERT SPACES

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Abstract. Let μ be a symmetric p-stable measure on a Hilbert space H. The distribution function of the norm $F(t) = \mu\{x: \|x\| < t\}$ is absolutely continuous on $(0, \infty)$. We prove an explicit formula for the density $F'(t)$ and some of its consequences.

1. Introduction. Let μ be a symmetric p-stable measure on a Banach space $(E, \|\cdot\|)$. Consider the distribution function of the norm, i.e. $F(t) = \mu\{x: \|x\| < t\}$. It is well-known ([3], [8], [9]) that F is absolutely continuous with respect to the Lebesgue measure on $(0, \infty)$ (apart from one possible jump if $1 \leqslant p \leqslant 2$). The properties of the density $F'(t)$ were investigated (even in more general setting) for $0 < p < 1$ in [5]. It was shown that

$$F'(t) = \frac{p}{t} \int_E [\mu(U_t) - \mu(U_t + x)]\, d\nu(x) \quad \text{for } t > 0, \tag{1}$$

where $U_t = \{x: \|x\| < t\}$ and ν is the Lévy measure of μ. The crucial point in the proof of this formula was the fact that the absolute continuity of F implied that, in a neighbourhood of the origin,

$$\mu(U_t) - \mu(U_t + x) \leqslant c_t \|x\|, \tag{2}$$

where c_t are bounded on half-lines (a, ∞). Since, for $0 < p < 1$, the integral

$$\int_{\{\|x\| < 1\}} \|x\|\, d\nu(x)$$

is finite, we could prove formula (1). From (1) we deduced the asymptotic behaviour of $F'(t)$, when t tends to infinity (cf. [5]). If $1 \leqslant p < 2$, the problem is more difficult. The estimate (2) is not strong enough, but it is easy to see that the estimate

$$\mu(U_t) - \mu(U_t + x) \leqslant c_t \|x\|^2 \tag{3}$$

is sufficient, where c_t are bounded on every half-line (a, ∞).

In this paper we show that (3) holds for all $p \in (0, 2]$ if E is a separable Hilbert space. As a corollary we get formula (1) and some of its consequences like boundedness and behaviour at infinity. The problem of boundedness of $F'(t)$ is important for the Berry-Esseen type estimates in the Central Limit Theorem in Banach spaces with the stable limiting law.

In the Hilbert space these densities were examined by Pap [6]. He showed that, for $1 < p < 2$, the density $F'(t)$ is bounded, but he used the Hölder's inequality, hence he could not examine the case $p = 1$. We use his idea to prove our Theorem 2. Later, in [2], Bentkus and Pap investigated the smoothness of $F(t)$ in Banach spaces, when the norm is of a particular form, for example, if it is induced by a bilinear functional. Using characteristic functions, they managed to show that, under additional assumptions, F has a number of derivatives, and if E is a Hilbert space, then $F'(t)$ is bounded for $1 < p < 2$. They also gave an asymptotic estimate of $F'(t)$.

In our paper we obtain formula (1) for $F'(t)$ which enables us to show that $F'(t)$ is bounded for every $p \in (0, 2)$ and to give an asymptotic estimate for $F'(t)$ at infinity. Our methods are quite elementary (especially for Gaussian measures) and do not depend on characteristic functions and sophisticated symmetrisation inequalities used in [2]. We use only the fact that any symmetric p-stable measure can be obtained as a mixture of Gaussian measures (see Proposition 1).

2. Notation and basic facts. Throughout the paper H denotes the separable Hilbert space with its norm $\|\cdot\|$. We write $U_t = \{x : \|x\| < t\}$. We consider symmetric p-stable, $0 < p \leqslant 2$, measures μ on H. If $p = 2$, then this measure is Gaussian and we usually denote it by γ. To avoid triviality, we always assume that dim supp $\mu \geqslant 2$. If μ is a symmetric p-stable measure on H, then there exists a σ-finite measure ν on H, $\nu(V^c) < \infty$, for every open neighbourhood V of the origin and such that $\mu = \lim \exp(\nu|V_n^c)$ for $V_n \searrow \{0\}$. The measure ν is called the *Lévy measure* of μ, and $\nu(rA) = r^{-p}\nu(A)$ for every Borel set A and $r > 0$. There exists a finite measure σ on the unit sphere S_1 in H such that, if $r(x) = \|x\|$ and $s(x) = x/\|x\|$,

$$\nu|_{U_\varepsilon^c}\{x : \|x+y\| \in A\} = \int_{S_1}\int_{\varepsilon}^{\infty} 1_A(\|rs+y\|)\frac{dr}{r^{1+p}}d\sigma(s) \tag{4}$$

for every $\varepsilon > 0$ and a Borel set A. We call σ the spectral measure for μ.

In the sequel all absolute constants will be denoted by $c_1, c_2, \ldots$ By F we denote the distribution function of the norm:

$$F(t) = \mu\{x : \|x\| < t\}.$$

We prove the estimate (3) for Gaussian measures and next apply it to stable measures using the following

PROPOSITION 1 ([4], [8]). *Let X be a symmetric p-stable vector in H with the* *distribution* μ *and with the spectral measure* σ. *Put* $M^p = \sigma(S_1)$ *and*

$$c_p^{-1} = \int_0^\infty x^{-p} \sin x\, dx.$$

Let $X_1(\omega_1), X_2(\omega_1), \ldots$ *be a sequence of i.i.d. random variables with the exponential distribution* $\mathrm{P}\{X_1 \geqslant x\} = \exp(-x)$ *for* $x > 0$; $\Gamma_n = X_1 + \ldots + X_n$. *Let* $(g_i(\omega_2))_{i=1}^\infty$ *denote a sequence of i.i.d. Gaussian random variables with* $Eg_1 = 0$ *and* $E|g_1|^p = 1$, *and let* $Z_1(\omega_1), Z_2(\omega_1), \ldots$ *be a sequence of i.i.d. random vectors with values in* H *and with the distribution* $L(Z_1) = \sigma/\sigma(S_1)$.

Assume also that the three sequences defined above are independent.

Then for every Borel set A *we have*

$$\mathrm{P}\{X \in A\} = E_{\omega_1} \mathrm{P}_{\omega_2}\{c_p M \sum_{i=1}^\infty \Gamma_i(\omega_1)^{-1/p} g_i(\omega_2) Z_i(\omega_1) \in A\}. \tag{5}$$

3. Estimates for the Gaussian measures in R^n and H. Let γ be a symmetric Gaussian measure on H. Assume that supp $\gamma = H$ and that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n \geqslant \ldots$ are the eigenvalues of the covariance operator of γ. It is well known that we can choose an orthonormal basis $\{e_i\}$ in H in such a way that γ is the distribution of a series

$$\sum_{i=1}^\infty \sqrt{\lambda_i}\, \theta_i(\omega) e_i,$$

where $(\theta_i)_{i=1}^\infty$ are i.i.d. with the distribution $N(0, 1)$. We are interested in behaviour of the distribution of the norm of

$$S_n(\omega) = \sum_{i=1}^n (\sqrt{\lambda_i}\, \theta_i(\omega) + r_i) e_i, \quad \text{where } r = (r_1, \ldots, r_n) \in R^n.$$

In the sequel by $\tilde{a}$ we always denote $\min(a, 1)$ for every $a \in R$.

LEMMA 1. *The distribution function of* $\|S_n(\omega)\|$ *is absolutely continuous on* $(0, \infty)$. *If we denote its density by* $f_r(t)$, *then there exist constants* $c_1, c_2 > 0$ *such that, for every* $r \in R^n$, $\|r\| < 1$ *and* $t > 0$,

$$f_r(t) \leqslant \frac{c_1 t}{(\tilde{\lambda}_1 \tilde{\lambda}_2)^{1/2}}, \quad \textit{where } \tilde{\lambda}_i = \min(\lambda_i, 1), \tag{a}$$

and

$$\left|\frac{d}{dt} f_r(t)\right| \leqslant \frac{c_2 t}{(\tilde{\lambda}_1 \tilde{\lambda}_2)^{5/2}}. \tag{b}$$

Proof. For every $s, h > 0$ we have

$$\begin{aligned} \mathrm{P}\{s < \|S_n(\omega)\| < s+h\} &= \mathrm{P}\{s^2 \leqslant \|S_n(\omega)\|^2 \leqslant (s+h)^2\} \\ &= \mathrm{P}\{s^2 \leqslant \sum_{i=1}^n (\sqrt{\lambda_i}\, \theta_i(\omega) + r_i)^2 \leqslant (s+h)^2\}. \end{aligned} \tag{6}$$

The distribution function of $(\sqrt{\lambda_i}\,\theta_i(\omega)+r_i)^2$ is absolutely continuous on $(0, \infty)$, hence $\|S_n(\omega)\|$ has absolutely continuous distribution.

We estimate the density of the distribution of $\|S_2(\omega)\|^2$.

Let us put, for $t > 0$,

$$h_i(t) = \frac{1}{2\sqrt{2\pi\lambda_i t}}\left[\exp\left(\frac{-(\sqrt{t}+r_i)^2}{2\lambda_i}\right)+\exp\left(\frac{-(\sqrt{t}-r_i)^2}{2\lambda_i}\right)\right].$$

The density $g(t)$ of the variable $\|S_2(\omega)\|^2$ is the convolution of h_1 *and* h_2:

$$(7)\quad g(t) = \int_0^t h_1(t-x)h_2(x)\,dx \leqslant \frac{1}{2\pi(\lambda_1\lambda_2)^{1/2}}\int_0^t [(t-x)x]^{-1/2}\,dx = \frac{c_2}{(\lambda_1\lambda_2)^{1/2}}.$$

It is evident that $g(0) = 0$.

Let us denote by $R(x)$ the density of

$$\|S_n(\omega)-S_2(\omega)\|^2 = \sum_{i=3}^{n} (\sqrt{\lambda_i}\,\theta_i(\omega)+r_i)^2.$$

Since

$$\begin{aligned}
\mathrm{P}\{s^2 < \|S_n(\omega)\|^2 < (s+h)^2\} &= \mathrm{P}\{s^2 < \|S_2(\omega)\|^2+\|S_n(\omega)-S_2(\omega)\|^2 < (s+h)^2\}\\
&= \int_0^\infty \mathrm{P}\{s^2-x < \|S_2(\omega)\|^2 < (s+h)^2-x\}\,R(x)\,dx\\
&\leqslant \sup_{x\geqslant 0} \mathrm{P}\{s^2-x < \|S_2(\omega)\|^2 < (s+h)^2-x\}\\
&\leqslant \frac{c_2}{(\lambda_1\lambda_2)^{1/2}}[(s+h)^2-s^2],
\end{aligned}$$

from (6) we get that $f_r(s) \leqslant c_1 s/(\lambda_1\lambda_2)^{1/2}$. We now estimate $f_r'(t)$. Denote by $k_r(t)$ the density of the distribution of $\|S_n(\omega)\|^2$; then

$$k_r(t) = \int_0^t g(t-x)R(x)\,dx.$$

Observe that $k_r(t^2) = f_r(t)$, hence $f_r'(t) = 2tk_r'(t^2)$ and $k_r'(t)$ exists because k_r is a convolution of smooth functions $h_i(t)$. If we show that

$$(8)\qquad |g'(t)| \leqslant \frac{c_4}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}},$$

then, since $g(0) = 0$, we infer that

$$|k_r'(t)| = \left|\int_0^t g'(t-x)R(x)\,dx\right| \leqslant \frac{c_4}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}}$$

and, finally, $|f_r'(t)| \leqslant c_2 t/(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}$. Now we show that (8) holds. Substituting $u = x/t$ in (7), we get

$$g(t) = \frac{1}{8\pi(\lambda_1\lambda_2)^{1/2}}\int_0^1 [(1-u)u]^{-1/2} X_1^t(1-u) X_2^t(u)\,du,$$

where

$$X_i^t(x) = \exp\left(\frac{-t(\sqrt{x}+r_i/\sqrt{t})^2}{2\lambda_i}\right)+\exp\left(\frac{-t(\sqrt{x}-r_i/\sqrt{t})^2}{2\lambda_i}\right).$$

Let

$$Y_i^t(x) = \exp\left(\frac{-t(\sqrt{x}+r_i/\sqrt{t})^2}{2\lambda_i}\right)-\exp\left(\frac{-t(\sqrt{x}-r_i/\sqrt{t})^2}{2\lambda_i}\right).$$

Easy calculations show that

(9) $$g'(t) = \frac{-1}{16\pi(\lambda_1\lambda_2)^{1/2}}\int_0^1 [(1-u)u]^{-1/2}\left[\frac{1}{\lambda_1}(1-u)X_1^t(1-u)X_2^t(u)+\right.$$

$$\left.+\frac{1}{\lambda_1}\frac{r_1\sqrt{1-u}}{\sqrt{t}}Y_1^t(u)X_2^t(u)+\frac{1}{\lambda_2}uX_2^t(u)X_1^t(1-u)+\frac{1}{\lambda_2}\frac{r_2\sqrt{u}}{\sqrt{t}}Y_2^t(u)X_1^t(1-u)\right]du.$$

Let us divide the right-hand side of (9) into four integrals and observe that the absolute value of the first and the third integral is less than $(\lambda_1^{-1}+\lambda_2^{-1})g(t)$. It is easy to see that estimating two remaining integrals it is sufficient to do it for one of them:

$$\sup_{t>0}\int_0^1 [(1-u)u]^{-1/2}\frac{r_1\sqrt{1-u}}{\sqrt{t}}Y_1^t(u)X_2^t(u)\,du = c_5 < +\infty$$

(recall that $|r_1| \leqslant 1$ by assumption).

By elementary inequality $|e^{-x}-e^{-y}| \leqslant |x-y|$ for $x, y > 0$, we get

$$\left|\frac{1}{\sqrt{t}}Y_1^t(u)\right| = \frac{1}{\sqrt{t}}\left|\exp\left(\frac{-t(\sqrt{1-u}-r_1/\sqrt{t})^2}{2\lambda_1}\right)-\exp\left(\frac{-t(\sqrt{1-u}+r_1/\sqrt{t})^2}{2\lambda_1}\right)\right|$$

$$\leqslant \frac{2|r_1|\sqrt{1-u}}{\lambda_1} \leqslant \frac{2}{\lambda_1}.$$

Finally,

$$|g'(t)| \leqslant \frac{1}{8\pi(\lambda_1\lambda_2)^{1/2}}\left(\frac{c_6}{\lambda_1^2}+\frac{c_7}{\lambda_2^2}\right) \leqslant \frac{c_3}{(\lambda_1\lambda_2)^{5/2}},$$

which completes the proof.

Now we prove a theorem which is the crucial point of the paper.

THEOREM 1. *Let γ be a distribution of the series*

$$\sum_{i=1}^{n}\sqrt{\lambda_i}\,\theta_i(\omega)\,e_i,$$

where $(e_i)_{i=1}^n$ *is the standard basis in* R^n *and* $(\theta_i)_{i=1}^n$ *are i.i.d. with the distribution* $N(0, 1)$, *Assume that* $\lambda_1 \geqslant \ldots \geqslant \lambda_n > 0$ *and let* $r \in R^n$ *with* $\|r\| \leqslant 1$.

Then there exists an absolute constant $c > 0$ *such that*

$$\gamma(U_t) - \gamma(U_t + r) \leqslant \frac{c}{\tilde{t}(\tilde{\lambda}_1 \tilde{\lambda}_2)^{5/2}} \|r\|^2. \tag{10}$$

Remark 1. It is obvious that the left-hand side of (10) is less than 1.

Remark 2. Observe that by virtue of the well-known Anderson's inequality we have $\gamma(U_t) - \gamma(U_t + r) \geqslant 0$. In view of Proposition 1 we infer that the same is true for symmetric stable measures.

Proof. For fixed $r \in R^n$ put $r^k = (r_1, \ldots, r_k, 0, \ldots, 0) \in R^n$. Let us write

$$S_n(\omega) = \sum_{i=1}^n \sqrt{\lambda_i}\, \theta_i(\omega) e_i \quad \text{and} \quad S_n^k = \sum_{i \neq k} \sqrt{\lambda_i}\, \theta_i(\omega) e_i.$$

We show that, for $k = 0, 1, \ldots, n-1$,

$$|\mathrm{P}\{S_n(\omega) \in U_t + r^k\} - \mathrm{P}\{S_n(\omega) \in U_t + r^{k+1}\}| \leqslant \frac{c}{\tilde{t}(\tilde{\lambda}_1 \tilde{\lambda}_2)^{5/2}} r_{k+1}^2. \tag{11}$$

By the triangle inequality and (11) we get

$$\mathrm{P}\{S_n(\omega) \in U_t\} - \mathrm{P}\{S_n(\omega) \in U_t + r\} \leqslant \sum_{k=0}^{n-1} \frac{c}{\tilde{t}(\tilde{\lambda}_1 \tilde{\lambda}_2)^{5/2}} r_{k+1}^2 = \frac{c}{\tilde{t}(\tilde{\lambda}_1 \tilde{\lambda}_2)^{5/2}} \|r\|^2.$$

Now we show (11). For fixed $k \in \{0, 1, \ldots, n-1\}$ let f_r^k be the density of the distribution of $\|S_n^{k+1}(\omega) - r^k\|$; here $r^k = (r_1, \ldots, r_k, 0, \ldots, 0) \in R^{n-1}$. We have

$$\begin{aligned} I_1^k &= \mathrm{P}\{S_n(\omega) - r^k \in U_t\} - \mathrm{P}\{S_n(\omega) - r^{k+1} \in U_t\} \\ &= \mathrm{P}\{\|S_n^{k+1}(\omega) - r^k\|^2 + (\sqrt{\lambda_{k+1}}\, \theta_{k+1}(\omega))^2 < t^2\} - \\ &\qquad - \mathrm{P}\{\|S_n^{k+1}(\omega) - r^k\|^2 + (\sqrt{\lambda_{k+1}}\, \theta_{k+1}(\omega) + r_{k+1})^2 < t^2\} \\ &= \int_0^t [\mathrm{P}\{(\sqrt{\lambda_{k+1}}\, \theta_{k+1}(\omega))^2 < t^2 - x^2\} - \\ &\qquad - \mathrm{P}\{(\sqrt{\lambda_{k+1}}\, \theta_{k+1}(\omega) + r_{k+1})^2 < t^2 - x^2\}]\, f_r^k(x)\, dx \\ &= \int_0^t \Bigg[\Phi\Bigg(\sqrt{\frac{t^2 - x^2}{\lambda_{k+1}}}\Bigg) - \Phi\Bigg(-\sqrt{\frac{t^2 - x^2}{\lambda_{k+1}}}\Bigg) - \Phi\Bigg(\sqrt{\frac{t^2 - x^2}{\lambda_{k+1}}} - \frac{r_{k+1}}{\sqrt{\lambda_{k+1}}}\Bigg) + \\ &\qquad + \Phi\Bigg(-\sqrt{\frac{t^2 - x^2}{\lambda_{k+1}}} - \frac{r_{k+1}}{\sqrt{\lambda_{k+1}}}\Bigg)\Bigg] f_r^k(x)\, dx, \end{aligned}$$

where

$$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx.$$

Denoting for simplicity λ_{k+1} by λ and r_{k+1} by a we get

$$I_1^k = \int_0^t H_\lambda(x) f_r^k(x)\, dx,$$

where

$$H_\lambda(x) = 2\Phi\left(\sqrt{\frac{t^2-x^2}{\lambda}}\right) - \Phi\left(\frac{\sqrt{t^2-x^2}+a}{\sqrt{\lambda}}\right) - \Phi\left(\frac{\sqrt{t^2-x^2}-a}{\sqrt{\lambda}}\right).$$

Now the estimate depends on k. Let $k = 0$ or 1. Taking three terms in the Taylor's formula with the Lagrange form of the reminder, we have

$$\sqrt{2\pi}\, H_\lambda(x) = \frac{a^2}{2\lambda}\left[\frac{\sqrt{t^2-x^2}-\delta_1 a}{\sqrt{\lambda}} \exp\left(-\frac{(\sqrt{t^2-x^2}-\delta_1 a)^2}{2\lambda}\right) + \right.$$

$$\left. + \frac{\sqrt{t^2-x^2}+\delta_2 a}{\sqrt{\lambda}} \exp\left(-\frac{(\sqrt{t^2-x^2}+\delta_2 a)^2}{2\lambda}\right)\right] \quad \text{for } 0 < \delta_1, \delta_2 < 1.$$

Because $\max_{x \in R} |xe^{-x^2/2}| = e^{-1/2}$, we have

$$|I_1^k| \leqslant \frac{a^2}{\sqrt{e}\lambda} \int_0^t f_r^k(x)\, dx \leqslant \frac{c_8}{\lambda},$$

hence, for $k = 0$ or 1, we have $|I_1^k| \leqslant c_8/(\tilde{\lambda}_1 \tilde{\lambda}_2)^{5/2}$.

Now, take $k \in \{2, \ldots, n-1\}$ and use the Taylor's formula with the integral form of the reminder:

$$H_\lambda(x) = \frac{a^2}{\lambda} \int_0^1 (1-s)\left[\Phi''\left(\frac{\sqrt{t^2-x^2}-sa}{\sqrt{\lambda}}\right) + \Phi''\left(\frac{\sqrt{t^2-x^2}+sa}{\sqrt{\lambda}}\right)\right] ds.$$

Because $\sqrt{2\pi}\Phi''(x) = -x \exp(-x^2/2)$, we have to show that

$$\sup_{0<\lambda<1} |I_1^k| = c_9 < \infty,$$

where

$$I_1^k = \frac{1}{\lambda} \int_0^t \frac{u f_r^k(\sqrt{t^2-u^2})}{\sqrt{t^2-u^2}} \int_0^1 (1-s)\left[\frac{u+sa}{\sqrt{\lambda}} \exp\left(-\frac{(u+sa)^2}{2\lambda}\right) + \right.$$

$$\left. + \frac{u-sa}{\sqrt{\lambda}} \exp\left(\frac{-(u-sa)^2}{2\lambda}\right)\right] ds\, du.$$

Assuming $0 < a$, we estimate I_1^k using Fubini theorem and Lemma 1. The first term is estimated as follows:

$$(12)\qquad \int_0^t \frac{f_r^k(\sqrt{t^2-u^2})}{\sqrt{t^2-u^2}} \int_0^1 (1-s)\frac{u}{\sqrt{\lambda}}\frac{u+sa}{\sqrt{\lambda}}\frac{1}{\sqrt{\lambda}}\exp\left(\frac{-(u+sa)^2}{2\lambda}\right) ds\, du$$

$$\leqslant \int_0^t \frac{f_r^k(\sqrt{t^2-u^2})}{\sqrt{t^2-u^2}} \int_0^1 \left(\frac{u+sa}{\sqrt{\lambda}}\right)^2 \frac{1}{\sqrt{\lambda}}\exp\left(\frac{-(u+sa)^2}{2\lambda}\right) ds\, du$$

$$= \int_0^1 \int_{sa/\sqrt{\lambda}}^{(t+sa)/\sqrt{\lambda}} \frac{f_r^k(\sqrt{t^2-(\sqrt{\lambda}v-sa)^2})}{\sqrt{t^2-(\sqrt{\lambda}v-sa)^2}} v^2 \exp\left(-\frac{v^2}{2}\right) dv\, ds$$

$$\leqslant \int_0^1 \frac{c_2}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{1/2}} \int_{-\infty}^{+\infty} v^2 \exp\left(-\frac{v^2}{2}\right) dv\, ds \leqslant \frac{c_2}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}}.$$

Now the second term:

$$\frac{1}{\lambda}\int_0^t \frac{u f_r^k(\sqrt{t^2-u^2})}{\sqrt{t^2-u^2}} \int_0^1 (1-s)\frac{u-sa}{\sqrt{\lambda}}\exp\left(\frac{-(u-sa)^2}{2\lambda}\right) ds\, du$$

$$= \int_0^1\int_0^t \frac{f_r^k(\sqrt{t^2-u^2})}{\sqrt{t^2-u^2}} \frac{1-s}{\sqrt{\lambda}} \left(\frac{u-sa}{\sqrt{\lambda}}\right)^2 \exp\left(\frac{-(u-sa)^2}{2\lambda}\right) du\, ds +$$

$$+ \int_0^1\int_0^t \frac{f_r^k(\sqrt{t^2-u^2})}{\sqrt{t^2-u^2}} \frac{sa}{\sqrt{\lambda}}\frac{u-sa}{\sqrt{\lambda}}\frac{1-s}{\sqrt{\lambda}}\exp\left(\frac{-(u-sa)^2}{2\lambda}\right) du\, ds = I_2 + I_3.$$

The intergal I_2 we estimate in the same way as (12). Observe that it is sufficient to prove (10) for r such that $\|r\|^2 \leqslant t/2$; hence we may assume that $a = \|r\| \leqslant t/2$ because, for $0 < a \leqslant 1$, we have $a^2 \leqslant a$. To estimate I_3 we divide it into two integrals; the first one is the following:

$$(13)\qquad I_4 = \int_0^1 \int_{t/2}^t \frac{f_r^k(\sqrt{t^2-u^2})}{\sqrt{t^2-u^2}} \frac{sa}{\sqrt{\lambda}}\frac{u-sa}{\sqrt{\lambda}}\frac{1-s}{\sqrt{\lambda}}\exp\left(\frac{-(u-sa)^2}{2\lambda}\right) du\, ds$$

$$\leqslant \int_0^1 \int_{t/2}^t \frac{c_1}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{1/2}} \frac{sa}{\sqrt{\lambda}}\frac{u-sa}{\sqrt{\lambda}}\frac{1}{\sqrt{\lambda}}\exp\left(\frac{-(u-sa)^2}{2\lambda}\right) du\, ds$$

$$= \frac{c_1}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{1/2}} \int_0^1 \int_{(t/2-sa)/\sqrt{\lambda}}^{(t-sa)/\sqrt{\lambda}} \frac{sa}{\sqrt{\lambda}} x \exp\left(\frac{-x^2}{2}\right) dx\, ds$$

$$\leqslant \frac{c_1}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{1/2}} \frac{a}{\sqrt{\lambda}} \int_0^1 \exp\left(\frac{-(t/2-sa)^2}{2\lambda}\right) ds$$

$$= \frac{c_1}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{1/2}} \int_{(t/2-a)/\sqrt{\lambda}}^{t/2\sqrt{\lambda}} \exp\left(\frac{-x^2}{2}\right) dx \leqslant \frac{c_1\sqrt{\pi/2}}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{1/2}}.$$

Estimating the second integral we apply the mean value theorem,

$$(14)\quad \frac{f_r^k(\sqrt{t^2-(\sqrt{\lambda}\,x+sa)^2})}{\sqrt{t^2-(\sqrt{\lambda}\,x+sa)^2}} = \frac{f_r^k(\sqrt{t^2-(sa)^2})}{\sqrt{t^2-(sa)^2}} +$$

$$+x\sqrt{\lambda}(\sqrt{\lambda}\,\vartheta x+sa)\frac{f_r^k(y_x)-f_r^{k'}(y_x)\sqrt{y_x}}{y_x^{3/2}} = D+x\sqrt{\lambda}(\sqrt{\lambda}\,\vartheta x+sa)E(y_x),$$

where $y_x = t^2-(\sqrt{\lambda}\,\vartheta x+sa)^2$ and $0<\vartheta<1$.

By (14) we obtain the estimate

$$(15)\quad I_3-I_4 = \int_0^1 (1-s) \int_{-sa/\sqrt{\lambda}}^{(t/2-sa)/\sqrt{\lambda}} D\frac{sa}{\sqrt{\lambda}} x\exp\left(\frac{-x^2}{2}\right) dx\,ds +$$

$$+\int_0^1 \int_{-sa/\sqrt{\lambda}}^{(t/2-sa)/\sqrt{\lambda}} (1-s)x\sqrt{\lambda}(\sqrt{\lambda}\,\vartheta x+sa)E(y_x)\frac{sa}{\sqrt{\lambda}} x\exp\left(\frac{-x^2}{2}\right) dx\,ds = I_5+I_6.$$

The integral I_5 can be estimated in the same way as I_4 in (13), hence it remains to estimate I_6 only. From Lemma 1, for $x\in(-sa/\sqrt{\lambda},\,(t/2-sa)/\sqrt{\lambda})$, we have

$$|(\sqrt{\lambda}\,\vartheta x+sa)E(y_x)| \leqslant |\sqrt{\lambda}\vartheta x+sa|\left(\frac{|f_r^{k'}(y_x)|}{y_x} + \frac{f_r^k(y_x)}{y_x^{3/2}}\right)$$

$$\leqslant \frac{c_2}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}} \frac{|\sqrt{\lambda}\vartheta x+sa|}{\sqrt{y_x}} + \frac{\frac{c_1}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{1/2}}|\sqrt{\lambda}\,\vartheta x+sa|}{y_x}$$

$$\leqslant \frac{c_{10}}{(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}} \frac{t/2}{\min(\frac{3}{4}t^2,\sqrt{\frac{3}{4}}t)} \leqslant \frac{c_{11}}{\tilde{t}(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}}.$$

Finally,

$$(16)\quad |I_6| \leqslant \frac{c_{11}}{\tilde{t}(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}} \int_0^1 \int_{-sa/\sqrt{\lambda}}^{(t/2-sa)/\sqrt{\lambda}} sa\cdot x^2\cdot\exp\left(\frac{-x^2}{2}\right) dx\,ds \leqslant \frac{c_{12}}{\tilde{t}(\tilde{\lambda}_1\tilde{\lambda}_2)^{5/2}},$$

which, together with (13), gives the desired result. The proof of the theorem is completed.

Let now γ be a Gaussian measure on H. Assume that $\operatorname{supp}\gamma = H$ and that γ is the distribution of the series $\sum_{i=1}^{\infty} \sqrt{\lambda_i}\,\theta_i(\omega)\,e_i$, where $(e_i)_{i=1}^{\infty}$ is a CONS in H and $\sum_{i=1}^{\infty} \lambda_i < \infty$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$

By γ_n we denote the distribution of $\sum_{i=1}^{n} \sqrt{\lambda_i}\,\theta_i(\omega)\,e_i$.

COROLLARY. 1. *Under the above assumptions there exists an absolute constant $c > 0$ such that, for every $t > 0$ and $r \in H$,*

(17) $$\gamma(U_t) - \gamma(U_t + r) \leqslant \frac{c}{\tilde{t}(\tilde{\lambda}_1 \tilde{\lambda}_2)^{5/2}} \|r\|^2.$$

Proof. For every $r \in H$ the series

$$S(\omega) = \sum_{i=1}^{\infty} \left(\sqrt{\lambda_i}\,\theta_i(\omega) + r_i\right) e_i$$

is convergent with probability 1, hence weakly. Lemma 1 implies that the distribution of $\|S(\omega)\|$ is absolutely continuous on $(0, \infty)$, hence, for every $t > 0$, and $r \in H$,

$$\lim_n \gamma_n(U_t + r) = \gamma(U_t + r).$$

For every $\varepsilon > 0$ we can choose an $n_0 \in N$ such that, for $n > n_0$,

(18) $$|\gamma_n(U_t) - \gamma(U_t)| < \varepsilon \quad \text{and} \quad |\gamma_n(U_t + r) - \gamma(U_t + r)| < \varepsilon.$$

By virtue of Theorem 1 and estimates (18) we have

$$\gamma(U_t) - \gamma(U_t + r) = |\gamma(U_t) - \gamma_n(U_t) + \gamma_n(U_t) - \gamma_n(U_t + r) + \gamma_n(U_t + r) - \gamma(U_t + r)|$$
$$\leqslant \varepsilon + \frac{c}{\tilde{t}(\tilde{\lambda}_1 \tilde{\lambda}_2)^{5/2}} \|r\|^2 + \varepsilon.$$

Taking $\varepsilon \to 0$, we get the desired result.

Remark. Using the Cameron-Martin formula we get an estimate much weaker than (17) and only for r belonging to the RKHS of γ.

4. Estimates for stable measures in *H*. We use an idea belonging to Pap [6]. Let μ be a symmetric p-stable measure on H. By Proposition 1 we can write, for every $t > 0$,

(19) $$\mu(U_t) = E_{\omega_1} \mathrm{P}_{\omega_2} \left\{ c_p M \sum_{i=1}^{\infty} \Gamma_i(\omega_1)^{-1/p} g_i(\omega_2) Z_i(\omega_1) \in U_t \right\}.$$

For fixed ω_1 the series on the right-hand side of (19) represents a symmetric Gaussian measure γ_{ω_1}. Let $\lambda_i(\omega_1)$ denote the eigenvalues of the covariance operator of γ_{ω_1} and assume that $\lambda_1(\omega_1) \geqslant \lambda_2(\omega_1) \geqslant \ldots$ Pap [6] showed that

$$E_{\omega_1} \frac{1}{(\lambda_1(\omega_1)\lambda_2(\omega_1))^{5/2}} < \infty .$$

From this result we deduce our next theorem.

THEOREM 2. *If μ is a symmetric p-stable measure on H, then there exists a constant $C > 0$ such that, for all $r \in \operatorname{supp} \mu$ and $t > 0$,*

$$\mu(U_t) - \mu(U_t + r) \leqslant \frac{C}{t} \|r\|^2. \tag{20}$$

Proof. Sztencel [9] proved that for almost all ω_1 the supports of γ_{ω_1} are equal to the support of μ, so we can assume that supp $\mu = H$, and then supp $\gamma_{\omega_1} = H$ for almost all ω_1.

By Pap's result, (19) and Theorem 1 we see that there exists a constant $C > 0$ such that

$$\mu(U_t) - \mu(U_t + r) = E_{\omega_1}[\gamma_{\omega_1}(U_t) - \gamma_{\omega_1}(U_t + r)]$$
$$\leqslant \frac{C}{t} \|r\|^2 E_{\omega_1} \frac{1}{(\tilde{\lambda}_1(\omega_1)\tilde{\lambda}_2(\omega_1))^{5/2}} = \frac{C}{t} \|r\|^2 .$$

5. Formula for the density. We can now prove our main theorem. In [5] we gave an analogous formula for $p \in (0, 1)$ and measurable seminorms in any Banach space. Here we show the formula in Hilbert space only, but for all $p \in (0, 2)$. All the technical details are very similar to those given in [5] with one exception: now we apply Theorem 2 instead of estimate (2).

THEOREM 3. *Let μ be a symmetric p-stable, $0 < p < 2$, measure on a separable Hilbert space H. Then the distribution function $F(t) = \mu\{x : \|x\| < t\}$ is absolutely continuous and, for every $t > 0$,*

$$F'(t) = \frac{p}{t} \int_H [\mu(U_t) - \mu(U_t + x)]\, d\nu(x), \tag{21}$$

where ν is the Lévy measure of μ.

Proof. The details of the proof may be found in [5], here we sketch only the main ideas. It is easy to see that if $F'(t)$ exists, then

$$F'(t) = \lim_{s \to 0} \frac{p}{t} \int_H [\mu(U_t) - \mu(U_t + x)]\, d\frac{1}{s}\mu_s(x),$$

where $(\mu_s)_{s>0}$ is a symmetric semigroup of p-stable measures such that $\mu_1 = \mu$. For every $t > 0$ the function $f_t(x) = \mu(U_t) - \mu(U_t + x)$ is continuous and bounded and, for every $\varepsilon > 0$,

$$(22)\qquad \frac{1}{s}\mu_s\Big|_{\{x:\,\|x\|>\varepsilon\}} \text{ converges weakly to } \nu\Big|_{\{x:\,\|x\|>\varepsilon\}} \text{ as } s\to 0.$$

By virtue of Theorem 2, for every $\varepsilon > 0$, we have

$$(23)\qquad \int\limits_{\{x:\,\|x\|<\varepsilon\}} [\mu(U_t) - \mu(U_t+x)]\, d\frac{1}{s}\mu_s(x)$$

$$\leqslant \int\limits_{\{x:\,\|x\|<\varepsilon\}} \frac{C}{\tilde{t}}\|x\|^2\, d\frac{1}{s}\mu_s(x) = \frac{C}{\tilde{t}} \int\limits_{\{x:\,\|x\|<\varepsilon s^{-1/p}\}} s^{(2/p)-1}\|x\|^2\, d\mu(x)$$

$$\leqslant \frac{C}{\tilde{t}} s^{(2/p)-1} \int\limits_0^{\varepsilon s^{-1/p}} 2\alpha\mu\{x:\|x\|>\alpha\}\, d\alpha \leqslant \frac{c_{13}}{\tilde{t}}\varepsilon^{2-p},$$

because $\mu\{x: \|x\| > \alpha\} \leqslant c_{14}\alpha^{-p}$ (see e.g. [1]).

Combining (22) and (23) we deduce that, for every $\varepsilon > 0$,

$$\lim_{s\to 0}\int\limits_H [\mu(U_t) - \mu(U_t+x)]\, d\frac{1}{s}\mu_s(x) = \int\limits_H [\mu(U_t) - \mu(U_t+x)]\, d\nu(x),$$

which implies

$$F'(t) = \frac{p}{t}\int\limits_H [\mu(U_t) - \mu(U_t+x)]\, d\nu(x).$$

This formula implies the continuity of $F'(t)$. Indeed, let $t_n \to t > 0$. Then, for every $x \in H$, $f_{t_n}(x) \to f_t(x)$, and this, in turn, implies that $F'(t_n) \to F'(t)$ (cf. [5]). This completes the proof of Theorem 3.

Now we give an asymptotic estimate of $F'(t)$ at infinity. Fix $t > 0$. By Theorems 2 and 3 and by (4) we can estimate as follows:

$$\int\limits_H [\mu(U_t) - \mu(U_t+x)]\, d\nu(x) = \int\limits_{S_1}\int\limits_0^\infty [\mu(U_t) - \mu(U_t+rz)]\, \sigma(dz)\frac{dr}{r^{1+p}}$$

$$\leqslant \sigma(S_1)\int\limits_0^\infty \sup_{z\in S_1}[\mu(U_t) - \mu(U_t+rz)]\frac{dr}{r^{1+p}}$$

$$\leqslant \sigma(S_1)\int\limits_0^\infty \min\left(\frac{C}{\tilde{t}}r^2,\, 1\right)\frac{dr}{r^{1+p}}$$

$$= \sigma(S_1)\left[\int\limits_0^1 \frac{C}{\tilde{t}} r^{1-p}\, dr + \int\limits_1^\infty Cr^{-1-p}\, dr\right] \leqslant c_{15} < +\infty.$$

By formula (21) we obtain the following

COROLLARY 2. *Let μ and F be as in Theorem 3. There exists a constant $K > 0$ such that, for all $t > 1$, $F'(t) \leqslant Kt^{-1}$.*

Remark. In [5] we showed the exact asymptotic behaviour of $F'(t)$. Namely, if $0 < p < 1$, then

$$\lim_{t \to \infty} t^{1+p} F'(t) = \sigma(S_1).$$

We conjecture that in this case the same is true.

In view of formula (1), obtained in [5] for $0 < p < 1$, Ryznar [7] showed that $F'(t)$ is bounded. Repeating his arguments and using (21) instead of (1) one can show that $F'(t)$ is bounded for all $p \in (0, 2)$.

COROLLARY 3. *In the setting of Theorem 3 the density $F'(t)$ is bounded on $(0, \infty)$.*

Proof. It is enough to show that $F'(t)$ is bounded on $(0, 1)$. Let us choose $t < 1$ and fix $m \in N$. We will specify m later on. By virtue of (21) and (20) we have

$$\begin{aligned} F'(t) &= \frac{p}{t} \int_E [\mu(U_t) - \mu(U_t + x)]\, d\nu(x) \\ &\leqslant \frac{p}{t} \sigma(S_1) \left[\int_0^{t^m} \frac{C}{t} r^2 \frac{dr}{r^{1+p}} + \int_{t^m}^{\infty} F(t) \frac{dr}{r^{1+p}} \right] \\ &\leqslant c_{16}\, p\, \sigma(S_1) \left[C t^{(2-p)m-2} + F(t)\, t^{-mp-1} \right]. \end{aligned}$$

Taking m such that $(2-p)m - 2 > 0$ and taking into account that if supp μ is infinite-dimensional, then, for every $n \in N$, $F(t) = o(t^n)$, $t \to 0$ (see [1]), we get that

$$\lim_{t \to 0} F'(t) = 0.$$

If supp μ is finite-dimensional, the result is well-known.

We believe that the estimate of type (3) is valid in any Banach space and that the following conjecture is true:

CONJECTURE. *Let μ be a symmetric p-stable measure on a separable Banach space E and put $t_0 = \inf\{t > 0 : F(t) > 0\}$. Then for every $t > t_0$ there exists a constant $c(t)$ which is bounded on every half-line (a, ∞) and such that $\mu(U_t) - \mu(U_t + x) \leqslant c(t)\|x\|^2$.*

If this conjecture is true, then formula (1) holds, and we can apply it in the investigation of properties of the density $F'(t)$.

Added in proof. It turned out that in general the above conjecture is false. Nevertheless, it is true in some classes of Banach spaces, e.g. in L_p spaces for $p \geqslant 2$. The results are contained in the forthcoming paper "The measure of a translated ball in uniformly convex spaces" written by M. Ryznar and the author.

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