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A REPRESENTATION THEOREM FOR CONTINUOUS ADDITIVE FUNCTIONALS OF ZERO QUADRATIC VARIATION

BY

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Abstract. We study continuous additive functionals of zero quadratic variation of strong Markov continuous local martingales by means of stochastic calculus. We show that they admit a representation as a stochastic integral with respect to local time in the sense of Bouleau and Yor.

1. Introduction. In the theory of Markov processes additive functionals are an important tool and have been studied intensively. For a detailed treatise we refer to the books by Blumenthal and Getoor [2] and Sharpe [20].

A particular challenge consists in finding powerful representations of additive functionals, thus facilitating related stochastic calculus. An important result is the one-to-one correspondence of nonnegative finite continuous additive functionals and certain measures. In the case of d-dimensional Brownian motion it can be found in the book by Dynkin (see [6], Theorem 8.5). It was obtained by Volkonski in [22] and [23] in the one-dimensional case and also by McKean and Tanaka [13]. Meyer [14] extended the characterization of nonnegative additive functionals to Hunt processes.

Revuz [15] developed an explicit description of the unique measure m representing a nonnegative finite continuous additive functional of a Markov process X (see also [16], Chapter X). If X admits a local time $L^X(t, y)$ up to time t at every point y, then a nonnegative finite continuous additive functional A of X can be represented as an integral over local time with respect to the Revuz measure m:

$$A_t = \int L^X(t, y)\, m(dy), \qquad t \geqslant 0. \tag{1}$$

In the case of Brownian motion a representation of this type was originally proved by Tanaka [21]; we also refer to Wang [24].

In the framework of stochastic analysis, Engelbert and Schmidt studied nonnegative additive functionals of strong Markov continuous local martingales using the method of time change (see [8] and [9]). They gave a natural proof of the representation theorem ([8], Theorem 3.1, and [9], Theorem 2) which is strongly based on the Itô-Tanaka formula.

Of course, every nonnegative additive functional is increasing. Since a continuous additive functional of finite variation can be written as the difference of two nonnegative additive functionals (see [16], X.2.22), the representation theorem is easily extended to such functionals if we allow m to be a signed measure.

But, for example in the theory of Dirichlet forms (see [12]), a more general class of additive functionals appears to play an important role, namely, the class of continuous additive functionals of zero quadratic variation. In particular, we find such functionals in the Fukushima decomposition (see [12], Theorem 5.5.1).

In this paper we will realize that local time also allows to establish a representation theorem for such functionals. Let X be a one-dimensional strong Markov continuous local martingale and E its set of absorbing points. We will show that a continuous additive functional of zero quadratic variation A of X up to the first entry time $D_E(X) := \inf\{t \geqslant 0\colon X_t \in E\}$ can be represented as a stochastic integral in the sense of Bouleau and Yor:

$$A_t = \int g(x)\, d_x L^X(t, x) \quad \text{on } \{t < D_E(X)\} \text{ a.s.} \tag{2}$$

In the special case where A has finite variation, (1) and (2) are seen to coincide by integration by parts ([25], 5.1).

Let us recall that Bouleau and Yor [3] defined the stochastic integral of the above type as the unique continuous extension of the mapping which, on step functions, is given by

$$f = \sum_{i=1}^{n} f_i \mathbb{1}_{(a_i, a_{i+1}]} \mapsto \sum_{i=1}^{n} f_i\big(L^X(t, a_{i+1}) - L^X(t, a_i)\big) =: \int f(a)\, d_a L^X(t, a).$$

Thus, they succeeded in establishing a change of variables formula. In [25] the author developed a generalization of the Bouleau–Yor formula and showed that transformations according to the generalized Bouleau–Yor formula map continuous semimartingales to continuous local Dirichlet processes, i.e., processes admitting a decomposition into the sum of a continuous local martingale and a continuous adapted process of zero quadratic variation ([25], Corollary 5.8).

The starting point of this paper is the result on continuous additive functionals of linear Brownian motion B obtained by Tanaka in 1963 (see [21]). Such functionals take the form

$$A_t = f(B_t) - f(B_0) + \int_0^t g(B_s)\, dB_s,$$

where f is continuous and g locally square integrable.

We introduce definitions and prerequisites in Section 2. In Section 3 we derive on the basis of Tanaka's representation that every continuous additive functional of zero quadratic variation of Brownian motion B stopped when leaving (a, b) can be represented as a Bouleau–Yor term $\int g(x)\, d_x L^B(t, x)$.

To this end, we essentially use the generalized Bouleau–Yor formula ([25], 2.4) and results by Engelbert and Wolf on analytical properties of functions transforming stopped Brownian motion into local Dirichlet processes (see [10]). The converse statement that every Bouleau–Yor term $\int g(x)\,d_x L^B(t, x)$ is a continuous additive functional is an immediate consequence of Theorem 5.7 in [25]. In Section 4 we extend the representation theorem to continuous additive functionals of zero quadratic variation of strong Markov continuous local martingales by the method of time change.

2. Definitions and prerequisites. Let $\mathscr{B}(\boldsymbol{R})$ denote the σ-algebra of Borel sets, and $\mathscr{B}^u(\boldsymbol{R})$ its universal completion. We consider a family $(\Omega, \mathscr{F}, \boldsymbol{P}_x;\ x \in \boldsymbol{R})$ of probability spaces such that $(\boldsymbol{P}_x)_{x\in\boldsymbol{R}}$ is a probability kernel from $(\boldsymbol{R}, \mathscr{B}^u(\boldsymbol{R}))$ into $(\Omega, \mathscr{F})$. For every probability measure μ on $(\boldsymbol{R}, \mathscr{B}(\boldsymbol{R}))$ we define a probability measure $\boldsymbol{P}_\mu$ on $(\Omega, \mathscr{F})$ by

$$\boldsymbol{P}_\mu(A) := \int_{\boldsymbol{R}} \boldsymbol{P}_x(A)\,\mu(dx), \qquad A \in \mathscr{F}.$$

Replacing $\mathscr{F}$ by its completion with respect to the family $(\boldsymbol{P}_\mu)$ and extending $\boldsymbol{P}_\mu$ in the natural way we may assume $\mathscr{F}$ to be complete with respect to $(\boldsymbol{P}_\mu)$.

Furthermore, let $\boldsymbol{F} = (\mathscr{F}_t)_{t\geq 0}$ denote a right-continuous filtration with $\mathscr{F}_t \subseteq \mathscr{F}$ such that $\mathscr{F}_0$ (and, consequently, every $\mathscr{F}_t$) is complete with respect to $(\boldsymbol{P}_\mu)$ (see Blumenthal and Getoor [2], I.5.3). We write $(\Omega, \mathscr{F}, \boldsymbol{F}, \boldsymbol{P}_x;\ x \in \boldsymbol{R})$ for the family of filtered probability spaces.

We say that an *assertion holds a.s.* if it holds $\boldsymbol{P}_x$-a.s. for every $x \in \boldsymbol{R}$.

If X is a process on $(\Omega, \mathscr{F}, \boldsymbol{P}_x;\ x \in \boldsymbol{R})$, let $\boldsymbol{F}^0 = (\mathscr{F}_t^0)_{t\geq 0}$ denote the filtration generated by X. We set $\mathscr{F}_\infty^0 := \bigvee_{t\geq 0} \mathscr{F}_t^0$. We define $\mathscr{F}_t^X$ to be the completion of $\mathscr{F}_t^0$ in $\mathscr{F}$ with respect to $(\boldsymbol{P}_\mu)$ and write $\boldsymbol{F}^X = (\mathscr{F}_t^X)_{t\geq 0}$ as well as $\mathscr{F}_\infty^X := \bigvee_{t\geq 0} \mathscr{F}_t^X$.

For a real process Z let $Z_\infty := \limsup_{t\to\infty} Z_t$.

For a filtration $\boldsymbol{H}$, we denote the smallest right-continuous filtration containing $\boldsymbol{H}$ by $\boldsymbol{H}_+$.

DEFINITION 2.1. A stochastic process $X = (X_t)_{t\geq 0}$ is said to admit *(perfect) shift operators* if there exists a semigroup $\Theta = (\theta_t)_{0\leq t\leq\infty}$ of operators $\theta_t\colon \Omega \to \Omega$ such that

$$X_{t+s} = X_s \circ \theta_t, \qquad 0 \leq t < \infty,\ s \geq 0, \text{ a.s.}$$

and

$$X_t \circ \theta_\infty = X_\infty \quad \text{ on } \{\langle X\rangle_\infty < \infty\},\ t \geq 0, \text{ a.s.}$$

DEFINITION 2.2. A continuous adapted process $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathscr{F}, \boldsymbol{F}, \boldsymbol{P}_x;\ x \in \boldsymbol{R})$ is said to be a *strong Markov continuous local martingale* if the following conditions are satisfied:

(i) $\boldsymbol{P}_x(X_0 = x) = 1$ for every $x \in \boldsymbol{R}$.

(ii) X is a continuous local $\boldsymbol{F}$-martingale with respect to every $\boldsymbol{P}_x$, $x \in \boldsymbol{R}$.

(iii) X is a strong Markov process, i.e.,

– X admits shift operators $\Theta = (\theta_t)_{0 \leqslant t \leqslant \infty}$;

– for every nonnegative $\mathscr{F}^X_\infty$-measurable random variable Z we have

$$E_x(Z \circ \theta_t \mid \mathscr{F}_t) = E_{X_t}(Z) \;\; P_x\text{-a.s. for every } x \in I;$$

– for every F^X-stopping time T and every nonnegative $\mathscr{F}^X_\infty$-measurable random variable Z we have

$$E_x(Z \circ \theta_T \mathbf{1}_{\{T<\infty\}} \mid \mathscr{F}^X_T) = E_{X_T}(Z) \mathbf{1}_{\{T<\infty\}} \;\; P_x\text{-a.s. for every } x \in I.$$

We remark that, by 1.18 in [7], X also is a strong Markov continuous local martingale with respect to F_+ and we have $F^X_+ = F^X$. Throughout this paper X denotes a strong Markov continuous local martingale. A point X is called *absorbing* if $P_x(X_t = x) = 1$ for every $t \geqslant 0$. The set E of absorbing points is closed and we have (see [7], 1.14)

$$X_{D_E} \in E \quad \text{on } \{D_E < \infty\} \text{ a.s.}$$

as well as

$$X_t = X_{t \wedge D_E} \quad \text{for every } t \geqslant 0, \text{ a.s.},$$

where D_E denotes the first entry time of X in E. Furthermore, we assume that $\theta_s = \theta_{s \wedge D_E}$ for every $0 \leqslant s < \infty$.

Definition 2.3. Let U be an F^X-stopping time. An F^X-adapted process A taking values in $\boldsymbol{R} \cup \{\pm\infty\}$ is called a *continuous additive functional* of X on $[0, U)$ if

(i) $A_0 = 0$ a.s.,

(ii) A is continuous on $[0, U)$ a.s.,

(iii) $A_s + A_t \circ \theta_s = A_{s+t}$ holds on $\{s+t < U\}$ a.s. for all $s, t \geqslant 0$.

If the equality in (iii) holds simultaneously for all s and t, then A is said to be *perfect*. In the case $U = \infty$ we briefly call A a *continuous* (*perfect*) *additive functional.*

We recall the representation theorem for nonnegative continuous additive functionals.

Theorem 2.4 ([8], Theorem 3.1). *Let A be a right-continuous nonnegative process. If A is a continuous perfect additive functional of X on $[0, D_E)$, then there exists a nonnegative measure m such that*

$$A_t = \int_{\boldsymbol{R}} L^X(t, a)\, m(da) \quad \textit{on } \{t < D_E\} \textit{ a.s.}$$

This theorem extends to continuous perfect additive functionals of finite variation. Then m appears to be a signed measure (see [16], X.2.22).

We introduce the notion of "zero quadratic variation" in the framework of stochastic integration by Russo and Vallois [17]–[19]. This framework has proved to be highly appropriate for studying local Dirichlet processes (see [11], [18], [19], [25], [26]).

DEFINITION 2.5. (i) A continuous adapted process $Q = (Q_t)_{t \geq 0}$ has *zero energy* if

$$\lim_{\varepsilon \to 0} E \frac{1}{\varepsilon} \int_0^\infty (Q_{s+\varepsilon} - Q_s)^2 \, ds = 0.$$

(ii) A continuous adapted process Q has *zero quadratic variation up to the stopping time* τ if there exists a nondecreasing sequence of stopping times $(T_n)_{n \in N}$ with $\lim_{n \to \infty} T_n = \tau$ a.s. such that, for each $n \in N$, the stopped process Q^{T_n} has zero energy. If $\tau = +\infty$, we briefly say that Q has *zero quadratic variation.*

(iii) A *continuous Dirichlet process* is defined to be a continuous adapted process admitting a decomposition

$$Y = Y_0 + M + Q,$$

where M is a continuous square integrable martingale with $M_0 = 0$ and Q is a continuous adapted process of zero energy with $Q_0 = 0$.

(iv) A continuous adapted process Y is a *continuous local Dirichlet process up to the stopping time* τ if there exists a nondecreasing sequence of stopping times $(T_n)_{n \in N}$ with $\lim_{n \to \infty} T_n = \tau$ a.s. such that, for each $n \in N$, the stopped process Y^{T_n} is a continuous Dirichlet process. We say that $(T_n)_{n \in N}$ *reduces* Y. If $\tau = +\infty$, we briefly call Y a *continuous local Dirichlet process.*

Remark 2.5 ([24], 4.4, 4.5). (i) A continuous adapted process Y is a continuous local Dirichlet process if and only if it admits a decomposition

$$Y = Y_0 + M + Q,$$

where M is a continuous local martingale with $M_0 = 0$ and Q is a continuous adapted process of zero quadratic variation with $Q_0 = 0$.

(ii) A continuous process of finite variation has zero quadratic variation. Therefore, the class of continuous local Dirichlet processes extends the class of continuous semimartingales.

Remark 2.7. (i) In the framework of stochastic analysis by Russo and Vallois [17]–[19] the generalized bracket of two continuous processes X, Y is defined by

$$[X, Y] := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\cdot} (X_{s+\varepsilon} - X_s)(Y_{s+\varepsilon} - Y_s) \, ds$$

uniformly on compacts in probability. If X and Y are continuous semimartingales, then Proposition 1.1 of [17] states that the generalized bracket coincides with the classical covariation process given by

$$\langle X, Y \rangle_t := \lim_{\|\Pi\| \to 0} \sum_{k=1}^{m} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}})$$

in probability, where $\Pi: 0 = t_0 < t_1 < \ldots < t_k = t$ is a partition of $[0, t]$, $t \geqslant 0$ (see [16], IV.1.20).

We abbreviate $[X] := [X, X]$ and $\langle X \rangle := \langle X, X \rangle$.

(ii) If the continuous process Q has zero quadratic variation, then the generalized bracket $[Q, Q]$ exists and we have $[Q, Q] \equiv 0$ (see [25], 4.2).

An important classical result in stochastic analysis states that a continuous local martingale of finite variation is constant ([4], Theorem V.39). In our proofs the following generalization plays a central role:

THEOREM 2.8. *Every continuous local martingale M of zero quadratic variation is constant.*

Proof. By Remark 2.7 (ii), the generalized bracket $[M]$ vanishes. Therefore, by Remark 2.7 (i), so does the classical variation process $\langle M \rangle$. But $\langle M \rangle$ is known to be the increasing process in the Doob–Meyer decomposition of M^2, i.e., the unique continuous increasing process vanishing at zero such that $M^2 - \langle M \rangle$ is a local martingale (see [16], IV.1.8). Thus, $(M_t - M_0)^2$ is a non-negative continuous local martingale, hence a supermartingale. Therefore we have

$$0 \leqslant E(M_t - M_0)^2 \leqslant E(M_0 - M_0)^2 = 0.$$

Thus, M must be constant. ■

We need the following version of the generalized Bouleau–Yor formula (see [25], 2.2). For an interval $J \subseteq \boldsymbol{R}$ let $L^2_{\text{loc}}(J)$ denote the set of all functions that are square integrable on every compact subset of J.

THEOREM 2.9 ([25], Corollary 2.4). *Let M be a continuous local martingale and $L^M(t, x)$ its local time.*

(i) *For every finite stopping time T the mapping*

$$f = \sum_{i=1}^{n} f_i \mathbb{1}_{(x_i, x_{i+1}]} \mapsto \sum_{i=1}^{n} f_i \big(L^M(T, x_{i+1}) - L^M(T, x_i)\big) =: \int f(x)\, d_x L^M(t, x)$$

can be extended uniquely to a continuous mapping from $L^2_{\text{loc}}(\boldsymbol{R})$ to $L^0(\Omega, \boldsymbol{P})$, endowed with the topology of convergence in probability.

(ii) *If $f \in L^2_{\text{loc}}(\boldsymbol{R})$ and $F(x) := \int_0^x f(y)\, dy$, $x \in \boldsymbol{R}$, then*

$$F(M_T) = F(M_0) + \int_0^T f(M_s)\, dM_s - \frac{1}{2} \int f(x)\, d_x L^M(T, x)$$

holds for every finite stopping time T.

(iii) ([25], 5.7) *The Bouleau–Yor term $\big(\int f(x)\, d_x L^M(t, x)\big)_{t \geqslant 0}$ is a continuous process of zero quadratic variation.*

Unfortunately, it is not clear whether time changes leave the properties "zero quadratic variation" and "local Dirichlet process" invariant in the sense of Definition 2.5. Since time changes constitute the main tool of Section 4, we will also deal with the following slightly strengthened definition which can be traced back to Bertoin [1].

DEFINITION 2.10. (i) A finite sequence of F-stopping times τ: $0 = T_0 \leqslant T_1 \leqslant \ldots \leqslant T_k$ is called a *random partition* with respect to F.

(ii) A sequence τ_n: $0 = T_0^n \leqslant T_1^n \leqslant \ldots \leqslant T_{k_n}^n$ of random partitions is said to *tend to the identity* if

$$\lim_{n\to\infty} \sup_l T_l^n \mathbf{1}_{\{T_l^n < \infty\}} = \infty \text{ a.s.},$$

$$\|\tau_n\| := \sup_l [(T_{l+1}^n - T_l^n) \mathbf{1}_{\{T_{l+1}^n < \infty\}}] \to 0 \text{ a.s.}$$

(iii) A continuous adapted process Q has *zero energy* (*in the strong sense*) if

$$\lim_{n\to\infty} E \sum_{l:0\leqslant l\leqslant k, T_{l+1}^n < \infty} (X_{T_{l+1}^n} - X_{T_l^n})^2 = 0$$

holds for every sequence τ_n: $0 = T_0^n \leqslant T_1^n \leqslant \ldots \leqslant T_{k_n}^n$ of random partitions tending to identity.

(iv) A continuous adapted process Q has *zero quadratic variation up to the stopping time* τ (*in the strong sense*) if there exists a nondecreasing sequence of stopping times $(T_n)_{n\in N}$ with $\lim_{n\to\infty} T_n = \tau$ a.s. such that, for each n, the stopped process Q^{T_n} has zero energy in the strong sense. If $\tau = +\infty$, we briefly say that Q has *zero quadratic variation.*

(v) ([11], 2.3, 2.4) A continuous process Y is said to be a *continuous strong local Dirichlet process up to the stopping time* τ if it admits a decomposition $Y = Y_0 + M + Q$, where M is a continuous local martingale up to τ with $M_0 = 0$ and Q a continuous adapted process with $Q_0 = 0$ having zero quadratic variation up to τ in the strong sense. If $\tau = +\infty$, we briefly call Y a *continuous strong local Dirichlet process.*

Remark 2.11. The properties "zero energy", "zero quadratic variation in the strong sense" and "strong local Dirichlet process" imply their analogues determined in Definition 2.5.

The last preparatory remark aims at facilitating the proofs. It applies to both definitions of "zero quadratic variation" and can be shown exactly in the same way as VI.30e in [5].

Remark 2.12. A continuous process Q has zero quadratic variation up to the stopping time U if and only if there exists a nondecreasing sequence of stopping times $(T_n)_{n\in N}$ with $\lim_{n\to\infty} T_n = U$ a.s. such that each Q^{T_n}, $n \in N$, has zero quadratic variation.

3. Additive functionals of Brownian motion. In 1963 H. Tanaka established a representation theorem for continuous additive functionals of linear Brownian motion. Looking at Tanaka's proof we realize the following slightly strengthened version without any additional efforts.

THEOREM 3.1 ([21], Theorem 2). *Let* $-\infty \leqslant a < b \leqslant \infty$, X *a Brownian motion and* $U := \inf\{t \geqslant 0\colon X_t \notin (a, b)\}$. *Then every continuous perfect additive functional* A *of* X *on* $[0, U)$ *takes the form*

$$A_t = f(X_t) - f(X_0) + \int_0^t g(X_s)\, dX_s \quad \text{on } \{t < U\} \text{ a.s.},$$

where $f \in C((a, b))$ *and* $g \in L^2_{\text{loc}}((a, b))$.

Using Tanaka's theorem, the generalized Bouleau–Yor formula, analytical properties of Dirichlet functions for stopped Brownian motion and the fact that a Bouleau–Yor term has zero quadratic variation we arrive at the following representation theorem for additive functionals of zero quadratic variation of Brownian motion:

THEOREM 3.2. *Let* $-\infty \leqslant a < b \leqslant \infty$, X *be a Brownian motion and* $U := \inf\{t \geqslant 0\colon X_t \notin (a, b)\}$. *An* F^X*-adapted process* A *with* $A_0 = 0$ *a.s. is a perfect continuous additive functional of* X *on* $[0, U)$ *having zero quadratic variation up to* U *(in the sense of Definition 2.5) if and only if*

$$A_t = \int g(x)\, d_x L^X(t, x) \quad \text{on } \{t < U\} \text{ a.s.},$$

where $g \in L^2_{\text{loc}}((a, b))$.

Proof. (1) First we suppose that $A_t = \int g(x)\, d_x L^X(t, x)$ on $\{t < U\}$ a.s. for some $g \in L^2_{\text{loc}}((a, b))$. Adopting the usual conventions for the arithmetics with $\pm\infty$ we define for each $n \in N$

$$T_n := \inf\{t \geqslant 0\colon X_t \notin ((a+1/n) \vee (-n), (b-1/n) \wedge n)\},$$

$$g_n(x) := \begin{cases} g(x) & \text{for } x \in ((a+1/n) \vee (-n), (b-1/n) \wedge n), \\ 0 & \text{otherwise,} \end{cases}$$

$$G_n(x) := \int_{x_0}^{x} g_n(y)\, dy, \quad x \in \boldsymbol{R}, \text{ for some fixed } x_0 \in (a, b).$$

By hypothesis, g_n, $n \in N$, are square integrable and T_n, $n \in N$, form a non-decreasing sequence of stopping times satisfying $\lim_{n\to\infty} T_n = U$ a.s. and $T_n < U$ on $\{U > 0\}$ a.s. By the Bouleau–Yor formula in Theorem 2.9 (ii), we consequently obtain

$$A_t^{T_n} = \int g(x)\, d_x L^{X^{T_n}}(t, x) = \int g_n(x)\, d_x L^{X^{T_n}}(t, x)$$

$$= \left(-2G_n(X_0) + 2G_n(X_t^{T_n})\right) + 2 \int_0^{t \wedge T_n} g_n(X_s)\, dX_s, \quad t \geqslant 0, \text{ a.s.}$$

Since both summands on the right-hand side are perfect continuous additive functionals of X on $[0, T_n)$ (see [8], 2.20 (iii)) and $\int g_n(x)\, d_x L^{X^{T_n}}(\cdot, x)$ has zero quadratic variation by Theorem 2.9 (iii), we conclude by Remark 2.12 that A is a perfect continuous additive functional of zero quadratic variation of X on $[0, U)$.

(2) Conversely, suppose that A is a perfect continuous additive functional of X on $[0, U)$ having zero quadratic variation up to U. By Theorem 3.1, A can be written as

$$A_t = \tilde{f}(X_t) - \tilde{f}(X_0) + \int_0^t g(X_s)\,dX_s \quad \text{on } \{t < U\} \text{ a.s.}$$

with some $\tilde{f} \in C((a, b))$ and $g \in L^2_{\mathrm{loc}}((a, b))$. We define $T_n, g_n, G_n, n \in N$, in the same way as in (1) and set

$$f(x) := \begin{cases} \tilde{f}(x) & \text{for } x \in (a, b), \\ 0 & \text{otherwise,} \end{cases}$$

as well as

$$J_n := ((a+1/n) \vee (-n), (b-1/n) \wedge n).$$

Then

$$A_t^{T_n} = f(X_t^{T_n}) - f(X_0) + \int_0^{t \wedge T_n} g_n(X_s)\,dX_s, \quad t \geqslant 0, \text{ a.s.}$$

is a continuous process. By hypothesis and Remark 2.12, there exists a sequence $(S_m)_{m \in N}$ of F^X-stopping times with $S_m \nearrow U$ such that, for each $m \in N$, the stopped process A^{S_m} has zero quadratic variation with respect to every P_x, $x \in R$. Then $R_m := S_m + 1_{\{S_m \geqslant T_n\}} \cdot \infty$, $m \in N$, form a sequence of F^X-stopping times with $R_m \nearrow \infty$ because of $T_n < U$ on $\{U > 0\}$. By 4.5 (iii) in [25], each $(A^{T_n})^{R_m} = A^{S_m \wedge T_n}$, $m \in N$, has zero quadratic variation and, consequently, by Remark 2.12, so does A^{T_n}. Thus,

$$f(X_t^{T_n}) - f(X_0) = -\int_0^{t \wedge T_n} g_n(X_s)\,dX_s + A_t^{T_n}, \quad t \geqslant 0, \text{ a.s.,} \tag{3}$$

is seen to be a continuous local Dirichlet process with respect to every P_x, $x \in R$, in view of Remark 2.6 (i). Hence f is a Dirichlet function (see [10], 2.3) for the Brownian motion X^{T_n} stopped when leaving J_n. By Theorem 3.4 in [10], for each n, f is absolutely continuous on J_n admitting a density f' which is locally square integrable on J_n. Consequently, we have $f' \in L^2_{\mathrm{loc}}((a, b))$. Therefore, the generalized Bouleau–Yor formula in Theorem 2.9 (ii) applies to the transformation of X^{T_n} with f. This yields

$$f(X_t^{T_n}) - f(X_0) = \int_0^{t \wedge T_n} f'(X_s)\,dX_s - \frac{1}{2} \int f'(x)\,d_x L^{X^{T_n}}(t, x), \quad t \geqslant 0, \text{ a.s.} \tag{4}$$

Since the Bouleau–Yor term $(\int f'(x)\,d_x L^{X^{T_n}}(t, x))_{t \geqslant 0}$ has zero quadratic variation by Theorem 2.9 (iii), we now realize in view of (3) and (4) that

$$A_t^{T_n} + \frac{1}{2} \int f'(x)\,d_x L^{X^{T_n}}(t, x) = \int_0^{t \wedge T_n} (f' + g_n)(X_s)\,dX_s, \quad t \geqslant 0,$$

is a continuous local martingale of zero quadratic variation. Hence, by Theorem 2.8, it must be constant. This entails

$$A_t^{T_n} = -\frac{1}{2}\int f'(x)\, d_x L^{X^{T_n}}(t, x), \qquad t \geqslant 0, \text{ a.s.}$$

for each $n \in N$. So, we arrive at

$$A_t = -\frac{1}{2}\int f'(x)\, d_x L^X(t, x) \quad \text{on } \{t < U\} \text{ a.s.},$$

thus completing the proof. ∎

Since $\int_0^{\cdot} f(X_s)\, dX_s$ is known to be a continuous perfect additive functional of X (see [8], 2.20 (iii)), the proof of Theorem 3.2 immediately yields the following characterization of continuous perfect additive functionals of Brownian motion that are local Dirichlet processes.

THEOREM 3.3. *Let* $-\infty \leqslant a < b \leqslant \infty$, X *be a Brownian motion and* $U := \inf\{t \geqslant 0 : X_t \notin (a, b)\}$. *Furthermore, let* A *be an* F^X*-adapted process with* $A_0 = 0$ *a.s. Then* A *is a perfect continuous additive functional of* X *on* $[0, U)$ *which is a continuous local Dirichlet process up to* U *(in the sense of Definition 2.5) if and only if*

$$A_t = \int_0^t f(X_s)\, dX_s + \int g(x)\, d_x L^X(t, x) \quad \text{on } \{t < U\} \text{ a.s.},$$

where $f, g \in L^2_{\text{loc}}((a, b))$.

4. Additive functionals of strong Markov continuous local martingales. We extend the representation theorem of Section 3 to continuous additive functionals of zero quadratic variation of strong Markov continuous local martingales by the method of time change. In this section we therefore deal with the notion "zero quadratic variation" in the sense of Definition 2.10.

Let X be a strong Markov continuous local martingale according to Definition 2.2, and E the set of its absorbing points. Let T denote the right inverse of $\langle X \rangle$. Then the time changed process $W := X \circ T$ is a strong Markov continuous local martingale with respect to the filtration $F^X \circ T = F^W$ ([8], Theorem 3.34 (i)). Its set of absorbing points is E and we have

$$\langle W \rangle_t = t \wedge \langle X \rangle_\infty = t \wedge \langle X \rangle_{D_E(X)} = t \wedge D_E(W) \text{ a.s.}$$

Thus, W is a Brownian motion stopped at $D_E(W)$. The operators $\tilde{\theta}_s := \theta_{T_s}$, $0 \leqslant s \leqslant \infty$, are perfect shift operators for W (see [8], Chapter 3.3).

THEOREM 4.1. *Let* A *be a continuous perfect additive functional of* X *on* $[0, D_E(X))$ *which has zero quadratic variation up to* $D_E(X)$ *in the sense of Definition 2.10. Then* A *can be represented in the form*

$$A_t = \int g(x)\, d_x L^X(t, x) \quad \text{on } \{t < D_E(X)\} \text{ a.s.}$$

for some $g \in L^2_{\text{loc}}(\boldsymbol{R} \backslash E)$.

Proof. The process $\tilde{A} := A \circ T$ is $F^X \circ T = F^W$-adapted and satisfies $\tilde{A}_0 = 0$ a.s. Since $\langle X \rangle$ strictly increases on $[0, D_E(X))$ (see [8], 1.23) and, consequently, T is continuous and finite on $[0, \langle X \rangle_{D_E(X)}) = [0, D_E(W))$, $\tilde{A}$ is continuous on $[0, D_E(W))$. Furthermore, we deduce from 2.33 and 2.27 in [8] that

$$T_s + T_t \circ \theta_{T_s} = T_{s+t}$$

holds for all $0 \leqslant s, t \leqslant +\infty$ on $\{T_{s+t} < D_E(X)\} = \{s+t < D_E(W)\}$ a.s. We compute

$$\begin{aligned}
\tilde{A}_{s+t}(\omega) &= A_{T_{s+t}(\omega)}(\omega) = A_{T_s(\omega)+(T_{s+t}(\omega)-T_s(\omega))}(\omega) \\
&= A_{T_s(\omega)}(\omega) + A_{T_{s+t}(\omega)-T_s(\omega)}(\theta_{T_s}(\omega)) \\
&= A_{T_s(\omega)}(\omega) + A_{T_t(\theta_{T_s}(\omega))}(\theta_{T_s}(\omega)) = A_{T_s(\omega)}(\omega) + \tilde{A}_t \circ \theta_{T_s}(\omega) \\
&= \tilde{A}_s(\omega) + \tilde{A}_t \circ \tilde{\theta}_s(\omega) \quad \text{on } \{s+t < D_E(W)\} \text{ for all } s, t \geqslant 0 \text{ a.s.}
\end{aligned}$$

Thus, $\tilde{A}$ is a perfect continuous additive functional of W on $[0, D_E(W))$.

By hypothesis, there exists a sequence of F^X-stopping times $(T_n)_{n \in N}$ with $T_n \nearrow D_E(X)$ a.s. such that the stopped processes A^{T_n} have zero quadratic variation with respect to every P_x, $x \in R$. Now, $\tilde{T}_n := \langle X \rangle_{T_n}$, $n \in N$, form a sequence of F^W-stopping times with $\tilde{T}_n \nearrow D_E(W)$ such that $\tilde{A}^{\tilde{T}_n}$ have zero quadratic variation in the sense of Definition 2.10 with respect to every P_x, $x \in R$ (see [11], 3.7). So, $\tilde{A}$ is a perfect continuous additive functional of W having zero quadratic variation up to $D_E(W)$ and, consequently, up to each $D_{(a_i,b_i)^c}(W)$, where (a_i, b_i) denote the components of the open set $R \backslash E$, since $D_{(a_i,b_i)^c}(W) \leqslant D_E(W)$ holds a.s. By Remark 2.11, for each i, Theorem 3.2 now yields a real function g_i with $g_{i|(a_i,b_i)^c} = 0$ and $g_{i|(a_i,b_i)} \in L^2_{\text{loc}}((a_i, b_i))$ such that

$$\tilde{A}_t = \int g_i(x)\, d_x L^W(t, x)$$

holds a.s. on $\{t < D_{(a_i,b_i)^c}(W)\}$. We set $g := \left(\sum_i g_i\right)\big|_{(R \backslash E)}$. Then $g \in L^2_{\text{loc}}(R \backslash E)$ and, for each $x \in (a_i, b_i)$, we have

$$\begin{aligned}
\tilde{A}_t &= \int g_i(x)\, d_x L^W(t, x) \\
&= \int g(x)\, d_x L^W(t, x) \quad \text{on } \{t < D_{(a_i,b_i)^c}(W)\} = \{t < D_E(W)\} \ P_x\text{-a.s.}
\end{aligned}$$

This means

$$\tilde{A}_t = \int g(x)\, d_x L^W(t, x) \quad \text{on } \{t < D_E(W)\} \text{ a.s.}$$

We conclude

$$\begin{aligned}
A_t = \tilde{A}_{\langle X \rangle_t} &= \int g(x)\, d_x L^W(\langle X \rangle_t, x) \\
&= \int g(x)\, d_x L^X(t, x) \quad \text{on } \{t < D_E(X)\} \text{ a.s. } \blacksquare
\end{aligned}$$

By Theorem 2.9 (iii) and Remark 2.12 we immediately realize the following converse of Theorem 4.1.

Remark 4.2. Let $g \in L^2_{\text{loc}}(\boldsymbol{R} \setminus E)$. Then

$$A_t = \int g(x)\, d_x L^X(t, x), \qquad t < D_E(X),$$

is a continuous perfect additive functional of X on $[0, D_E(X))$ having zero quadratic variation up to $D_E(X)$ in the sense of Definition 2.5.

Using Theorem 3.3 and the fact that time changes leave the class of continuous strong local Dirichlet processes invariant (see [11], 2.5) as well as Theorem 2.9 (iii) we obtain the following result on continuous perfect additive functionals which are (strong) local Dirichlet processes.

THEOREM 4.3. (i) *Let A be a continuous perfect additive functional of X on $[0, D_E(X))$ which is a strong local Dirichlet process up to $D_E(X)$ with respect to every $\boldsymbol{P}_x$, $x \in \boldsymbol{R}$, in the sense of Definition* 2.10. *Then A takes the form*

$$(5) \qquad A_t = \int_0^t f(X_s)\, dX_s + \int g(x)\, d_x L^X(t, x) \quad \text{on } \{t < D_E(X)\} \text{ a.s.},$$

where $f, g \in L^2_{\text{loc}}(\boldsymbol{R} \setminus E)$.

(ii) *Conversely, an F^X-adapted process A satisfying* (5) *is a continuous perfect additive functional of X on $[0, D_E(X))$ that is a continuous local Dirichlet process up to $D_E(X)$ in the sense of Definition* 2.5.

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