PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 22, Fasc. 2 (2002), pp. 343–354

# THE RATE OF CONVERGENCE IN THE PRECISE LARGE DEVIATION THEOREM

BY

ALEKSANDRAS BALTRŪNAS (VILNIUS)

*Abstract.* Let $X_1, X_2, \ldots$ be i.i.d. random variables with a common d.f. $F$. Let $S_n = X_1 + \ldots + X_n$, $n \geqslant 1$, and $M_n = \max_{k \leqslant n} X_k$, $n \geqslant 1$. In this paper for a large class of subexponential distributions we estimate the rate of convergence

$$\Delta_n(t) = \mathbf{P}(S_n > t) - \mathbf{P}(M_n > t),$$

where $n \geqslant 1$ and $t \geqslant 0$. We close this paper with some examples.

**2000 AMS Subject Classification:** Primary 60F10; Secondary 60E05.

**Key words and phrases:** Limit theorems, large deviations, subexponential distributions.

## 1. INTRODUCTION

Let $X_1, X_2, \ldots$ be i.i.d. real random variables with a common distribution function (d.f.) $F(t)$, $t \in \boldsymbol{R}$, which has the mean $\mathbf{E}X_1 = 0$.

DEFINITION. We say that the d.f. $F$ belongs to the *class $S$ of subexponential distributions* if its tail $\bar{F} := 1 - F$ satisfies

$$(1.1) \qquad \lim_{t \to \infty} \bar{F}(x+y)/\bar{F}(x) = 1, \qquad y \in \boldsymbol{R},$$

and

$$(1.2) \qquad \lim_{t \to \infty} \overline{F * F}(t)/\bar{F}(t) = 2,$$

where, as usual, $*$ denotes the Stieltjes convolution of $F$ with itself.

The class $S$ of subexponential distributions was introduced by Chistyakov [3] (in the case $F(0) = 0$).

It is well known (see [3], Theorem 2) that if $F(0) = 0$, then (1.2) implies (1.1).

We denote by $\mathfrak{L}$ a class of heavy tailed distributions for which the relation (1.1) is satisfied.

Let $S_n = \sum_{k=1}^{n} X_k$ and $M_n = \max_{k \leqslant n} X_k$, $n \in N$.

By definition it follows that if $F \in S$, then

$$\mathbf{P}(S_n > t) \sim \mathbf{P}(M_n > t) \quad \text{as } t \to \infty.$$

Thus, we infer that if d.f. $F$ is subexponential, then there exists a positive sequence $t_n$, $n \in N$, such that

$$\mathbf{P}(S_n > t) \sim \mathbf{P}(M_n > t) \quad \text{as } n \to \infty \tag{1.3}$$

uniformly in $t \in (t_n, \infty)$.

This means that in the investigation of precise large deviations for subexponential distributions the main problem becomes finding the intervals $(t_n, \infty)$.

Many papers are devoted (see [12] and the references contained therein) to search conditions for which the relation (1.3) holds as $n \to \infty$ uniformly for $t \in (t_n, \infty)$. There are but a few papers that consider the rate of convergence in the relation (1.3). Perhaps the most important paper among them is [2] in which Borovkov has established the rate of convergence in a theorem of large deviations for a class of subexponential distributions, the so-called semiexponential distributions. In the present paper we shall investigate the rate of convergence in (1.3) for one rather wide subclass of subexponential distributions.

## 2. PRELIMINARIES

Let us define the hazard function $R_F$ of $F$ by

$$R_F(t) = -\log \bar{F}(t), \quad t \in \boldsymbol{R}.$$

Assume that there exists a non-negative function $q_F$: $\boldsymbol{R}^+ \to \boldsymbol{R}$ such that

$$R_F(t) = R_F(0) + \int_0^t q_F(u)\,du, \quad t \in \boldsymbol{R}^+.$$

The function $q_F$ is called the *hazard rate* of $F_0 = F \cdot U_0$, where $U_0$ is the d.f. concentrated at 0.

It is well known (see [7]) that if for some $F_0 \in \mathfrak{L}$ the hazard rate $q_F$ or $\lim_{t \to \infty} q_F(t)$ does not exist, one can always construct a d.f. $H_0$ such that $\bar{H}_0(t) \sim \bar{F}_0(t)$ as $t \to \infty$, and $q_H(t) \to 0$ as $t \to \infty$, where $q_H$ is the hazard rate of $H_0$.

Let us define

$$\alpha = \sup\{k\colon\ \mathbf{E}(X_1^k,\, X_1 > 0) < \infty\},$$

$$\beta = \sup\{k\colon\ \mathbf{E}(|X_1|^k,\, X_1 < 0) < \infty\}, \quad \gamma = \min(\alpha, \beta).$$

Moreover, let us define the hazard ratio index

$$r := \limsup_{t\to\infty} tq_F(t)/R_F(t).$$

LEMMA 2.1. *Assume that* $\gamma > 2$ *and* $\mathbf{E}X_1 = 0$. *Then for* $z > 0$ *we have*

$$\Big|\int_{-\infty}^{1/z} e^{zu}\,dF(u) - 1\Big| < C_0 z^2.$$

Proof. We note that

$$\int_{-\infty}^{1/z} e^{uz}\,dF(u) - 1 = \int_{-\infty}^{1/z} (e^{uz}-1-zu)\,dF(u) - \bar{F}(1/z) + z\int_{-\infty}^{1/z} u\,dF(u).$$

Since $\mathbf{E}X_1 = 0$, we have

$$\int_{-\infty}^{1/z} u\,dF(u) = -\int_{1/z}^{\infty} u\,dF(u).$$

Hence

$$\Big|\int_{-\infty}^{1/z} e^{uz}\,dF(u) - 1\Big| \leqslant z^2 \int_{-\infty}^{1/z} u^2\,dF(u) + z\int_{1/z}^{\infty} u\,dF(u) + \bar{F}(1/z) \leqslant 5z^2\,\mathbf{E}X_1^2.$$

The proof is complete.

## 3. MAIN RESULTS

In this section we study the rate of convergence in (1.3). For further use, let us define

$$\Delta_n(t) = \mathbf{P}(S_n > t) - \mathbf{P}(M_n > t),$$

where $n \in N$ and $t \geqslant 0$.

Put $s := s(t) = R_F(t)/t$, $t > 0$.

We have

(3.1) $$\Delta_n(t) = \mathbf{P}(S_n > t,\ M_n > t) - \mathbf{P}(M_n > t) + \mathbf{P}(S_n > t,\ M_n \leqslant t) := L_1 + L_2.$$

Our first preliminary result is used to estimate the term $L_1$ in (3.1).

LEMMA 3.1. *If* $z > 0$ *is small enough, then*

(3.2) $$0 \geqslant L_1 \geqslant -\mathbf{P}(M_n > t)\Big(\int_t^{t+1/z} q_F(u)\,du + \mathbf{P}(|S_n| \geqslant 1/z) + \mathbf{P}(X_1 > t)\Big).$$

Proof. Let us put $A_n^k = \{1, \ldots, n\}\backslash\{k\}$ and $S_n^k = \sum_{k\in A_n^k} X_k$, $n \in N$. From (3.1) it follows that

$$\mathbf{P}(M_n > t) \geqslant \sum_{k=1}^{n} \mathbf{P}(S_n > t,\ M_n > t,\ M_n = X_k)$$

$$\geqslant \sum_{k=1}^{n} \int_{-1/z}^{\infty} \mathbf{P}(X_k > \max(t-u, t))\, d\mathbf{P}(S_n^k < u)$$

$$\geqslant \sum_{k=1}^{n} \mathbf{P}(X_k > t+1/z)\mathbf{P}(S_n^k \geqslant -1/z)$$

$$\geqslant \mathbf{P}(M_n > t+1/z) - \mathbf{P}(M_n > t)\mathbf{P}(|S_n| \geqslant 1/z) - \mathbf{P}(M_n > t)\mathbf{P}(X_1 > t).$$

Since $z > 0$ is small enough, we have

$$\mathbf{P}(M_n > t) - \mathbf{P}(M_n > t+1/z)$$

$$\leqslant \mathbf{P}(M_n > t)\big(1 - \exp\big(-\int_t^{t+1/z} q_F(u)\, du\big)\big) \leqslant \mathbf{P}(M_n > t) \int_t^{t+1/z} q_F(u)\, du.$$

The proof is complete.

Let $X_{1:n} \leqslant X_{2:n} \leqslant \ldots \leqslant X_{n-1:n} \leqslant X_{n:n} = M_n$ denote the order statistics of the sample.

Define

$$b(r) = \begin{cases} 2 & \text{if } r = 0, \\ 4/(1-r) & \text{if } r \neq 0. \end{cases}$$

Our main result is the following

THEOREM 3.2. *Assume that*

(i) $\mathbf{E}X_1 = 0$;
(ii) $\liminf_{t\to\infty} tq_F(t) > 2$;
(iii) $r < 1$;
(iv) $\beta > 2$, $\alpha > b(r)$.

*Then for n and t large enough*

$$(3.3) \quad -\mathbf{P}(M_n > t)(c_0 n^{1-\gamma/2} + c_1\sqrt{n \log n}s) \leqslant \Delta_n(t)$$

$$\leqslant \mathbf{P}(M_n > t)\big(\exp(c^* ns^2)/t^2 + C_1\sqrt{n \log n}s + C_2 ns^2 + C_3 n^{1-\gamma/2}\big),$$

*where* $c_0 > 0$, $c_1 > 0$, $c^* > 0$, $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ *are some constants.*

Remarks. 1. Let $t_n$, $n \in N$, be a sequence such that

$$\lim_{n\to\infty} \sqrt{n \log n}\, s(t_n) = 0.$$

From (3.3) it follows that under the conditions of Theorem 3.2 we have

$$\Delta_n(t) = o(1)\mathbf{P}(M_n > t) \quad \text{as } n \to \infty$$

uniformly with respect to $t \in (t_n, \infty)$.

2. Moreover, we can see that in this large deviation result the assumption of the concavity of a hazard function $R_F$ can be removed.

For the proof of Theorem 3.2 we first need the next lemma.

LEMMA 3.3. *Assume that*

$$r := \limsup_{t\to\infty} tq_F(t)/R_F(t) < 1.$$

*Then*

$$\int_{1/s}^{t} \exp(su)\, dF(u) \leqslant C < \infty. \tag{3.4}$$

Proof. Using the partial integration, we have

$$\int_{1/s}^{t} \exp(su)\, dF(u) \leqslant s \int_{1/s}^{t} \exp(su)\, \bar{F}(u)\, du + e\bar{F}(1/s) := \mathrm{I} + \mathrm{II}.$$

Let us put $r_\varepsilon = r+\varepsilon$, where $\varepsilon$ is small enough and $r_\varepsilon < 1$. From the relation

$$\limsup_{t\to\infty} tq_F(t)/R_F(t) < 1$$

it follows that for $u$ large enough

$$\big(R_F(u)/u\big)' = \big(uq_F(u) - R_F(u)\big)/u^2 < -(1-r_\varepsilon) R_F(u)/u^2 < 0,$$

so that $R_F(t)/t$ is non-increasing. Then for $u$ such that $1/s \leqslant u \leqslant t$ we obtain

$$\begin{aligned} su - R_F(u) &= \frac{R_F(t)\, u}{t} - R_F(u) \\ &\leqslant -(1-r_\varepsilon)\, u \int_{u}^{t} \big(R_F(v)/v^2\big)\, dv \leqslant -(1-r_\varepsilon)\frac{R_F(t)}{t^2}\, u(t-u). \end{aligned} \tag{3.5}$$

Consequently, from (3.5) it follows that

$$\mathrm{I} = s \int_{1/s}^{t} \exp\big(su - R_F(u)\big)\, du \leqslant 4s/(1-r_\varepsilon)\, s.$$

Moreover, we have

$$\mathrm{II} < e.$$

The proof is complete.

Proof of Theorem 3.2. Let us define $y$ as follows:

$$y = \max\left\{u > 0\colon \frac{2\log u}{R_F(u)} \leqslant (1-r_\varepsilon)\frac{t-u}{t}\right\}.$$

It is known that if $r = 0$, then $y > \delta t$ for some $\delta > 0$. In the case $r \neq 0$ we can see that $y > (1/2+\delta_0)\, t$ for some $\delta_0 > 0$.

Let $\xi$ be the number of summands $X_k$, $k = 1, \ldots, n$, in $S_n$ such that $X_k \geqslant y$. Since the random variable $\xi$ has the Bernoulli distribution with parameters $n$ and $\bar{F}(y)$, we may write

$$L_2 = \mathbf{P}(S_n > t,\, M_n \leqslant t) = \mathbf{P}(S_n > t,\, \xi = 0) + \mathbf{P}(S_n > t,\, \xi = 1,\, M_n \leqslant t)$$
$$+\mathbf{P}(S_n > t,\, \xi \geqslant 2,\, M_n \leqslant t) := \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

We have

$$\mathrm{III} \leqslant \mathbf{P}(X_{n-1:n} > y,\ M_n \leqslant t) = O(1)\,\mathbf{P}^2(M_n > y)$$
$$= O(1)\,\mathbf{P}(M_n > t)\, n \exp\big(-2R_F(y) + R_F(t)\big).$$

Under our assumptions we obtain

$$R_F(t) - R_F(y) \leqslant r_\varepsilon s(y)(t-y),$$

where $r_\varepsilon$ is the same as in Lemma 3.3. Hence

$$R_F(t) - 2R_F(y) \leqslant -R_F(y) + r_\varepsilon s(y)(t-y) = -R_F(y)\left(1 - r_\varepsilon \frac{t-y}{y}\right).$$

Since $\varepsilon$ is an arbitrarily small positive quantity, in the case $r = 0$ we obtain

$$R_F(t) - 2R_F(y) \leqslant -R_F(y) + r_\varepsilon s(y)(t-y)$$
$$\leqslant -R_F(\delta t)(1-\varepsilon) \leqslant -2\log t + O(1).$$

In the case $r \neq 0$ we have

$$R_F(t) - 2R_F(y) \leqslant -R_F(y) + r_\varepsilon s(y)(t-y) = -R_F(y)\left(1 - r_\varepsilon \frac{t-y}{y}\right)$$
$$\leqslant -R_F(t/2)(1-r) \leqslant -2\log t + O(1).$$

Consequently, we obtain

$$\mathrm{III} = O(1)\,\mathbf{P}(M_n > t)\, n/t^2 = o(1)\,\mathbf{P}(M_n > t)\, ns^2.$$

Next we consider I. Let us define

$$V_k = \begin{cases} X_k & \text{for } X_k < y, \\ 0 & \text{for } X_k \geqslant y, \end{cases} \qquad U_n = \sum_{k=1}^{n} V_k.$$

Let $\delta_1, \delta_2, \ldots$ be a sequence of i.i.d. random variables with common d.f. $F_s$ which equals

$$F_s(u) = \min\Big\{1, \Big(\int_{-\infty}^{u} \exp(sv)\, dF(v)\Big)\Big(\int_{-\infty}^{y} \exp(sv)\, dF(v)\Big)^{-1}\Big\}.$$

So, to estimate the term I, we use the Cramer equality (see e.g. [9]): for any $u > 0$ we have

$$\mathbf{P}(S_n > u,\ \xi = 0) = \big(\mathbf{E}(\exp(sV_1))\big)^n \int_u^\infty e^{-sv}\, d\mathbf{P}\big(\sum_{i=1}^n \delta_i < v\big).$$

Hence

$$\mathbf{P}(S_n > u,\ \xi = 0) \leqslant \exp(-su)\big(\mathbf{E}(\exp(sV_1))\big)^n\, \mathbf{P}\big(\sum_{j=1}^n \delta_j \geqslant u\big). \tag{3.6}$$

We have

$$\mathbf{E}\exp(sV_1) = \Big(\int_{-\infty}^{1/s} + \int_{1/s}^{y}\Big)\exp(su)\,dF(u)$$
$$\leqslant J_1 + s\int_{1/s}^{y} \exp\big(su - R_F(u)\big)\,du := J_1 + sJ_2.$$

Using the condition $\gamma > 2$, from Lemma 2.1 we get

$$J_1 = 1 + O(1)s^2.$$

Now we consider $J_2$. We have

$$J_2 \leqslant s^2 \int_{1/s}^{y} u^2 \exp\big(su - R_F(u)\big)\,du.$$

Let us define the function $Q_1$ as follows:

$$Q_1(t) = R_F(t) - 2\log t, \qquad t \geqslant t_1 \geqslant 1.$$

Since $\liminf_{t\to\infty} tq_F(t) > 2$, we infer that $Q_1$ is a hazard function. Let us put

$$q_1(t) = \frac{d}{dt}Q_1(t), \qquad t \geqslant t_1 \geqslant 1.$$

We can show that under our assumptions

$$\limsup_{t\to\infty}\frac{tq_1(t)}{Q_1(t)} \leqslant \limsup_{t\to\infty}\frac{tq(t)-2}{R_F(t)-2\log t}$$
$$\leqslant \limsup_{t\to\infty}\frac{r_\varepsilon\big(R_F(t)-2\log t\big)+2(r_\varepsilon\log t-1)}{R_F(t)-2\log t} < 1.$$

We have

$$\frac{R_F(t)}{t} = \frac{R_F(y)-2\log y}{y} + \frac{2\log y}{y} + \frac{R_F(t)}{t} - \frac{R_F(y)}{y}$$
$$\leqslant s_1(y) + \frac{2\log y}{y} - (1-r_\varepsilon)\frac{R_F(y)}{yt}(t-y) \leqslant s_1(y),$$

where $s_1 := s_1(y) = Q_1(y)/y$. Therefore, from Lemma 3.3 it follows that

$$s \int_{1/s}^{y} u^2 \exp(su - R_F(u))\, du \leqslant s_1 \int_{1/s}^{y} \exp(s_1 u - Q_1(u)) du < \infty. \tag{3.7}$$

From (3.6) it follows that under our assumptions

$$\mathbf{P}(S_n \geqslant u,\ \xi = 0) \leqslant \exp(c^* ns^2)\exp(-su)\mathbf{P}\big(\sum_{j=1}^{n} \delta_j \geqslant u\big). \tag{3.8}$$

We have

$$\mathbf{E}\delta_1^2 \leqslant \big(\mathbf{E}(\exp(sV_1))\big)^{-1}\big(\int_{-\infty}^{1/s} u^2 e^{su}\, dF(u) + \int_{1/s}^{y} u^2 e^{su}\, dF(u)\big).$$

Since $\gamma > 2$, we obtain

$$\int_{-\infty}^{1/s} u^2 e^{su}\, dF(u) < \infty.$$

Note that

$$\int_{1/s}^{y} u^2 e^{su}\, dF(u) \leqslant es^{-2}\,\bar{F}(1/s) + s\int_{1/s}^{y} u^2 e^{su}\,\bar{F}(u)\, du + 2\int_{1/s}^{y} ue^{su}\,\bar{F}(u)\, du$$

$$\leqslant es^{-2}\,\bar{F}(1/s) + s\int_{1/s}^{y} u^2 e^{su}\,\bar{F}(u)\, du + 2s\int_{1/s}^{y} u^2 e^{su}\,\bar{F}(u)\, du.$$

Using (3.7), we obtain

$$\int_{1/s}^{y} u^2 e^{su}\, dF(u) < \infty.$$

Hence $\mathbf{E}\delta_1^2 < \infty$. From this it follows that

$$\mathbf{P}\big(\sum_{i=1}^{n} \delta_i > t\big) \leqslant n\mathbf{E}\,\delta_1^2/t^2 = O(1)\, n/t^2. \tag{3.9}$$

Application of (3.9) now shows that

$$\mathrm{I} = O(1)\,\mathbf{P}(M_n > t)\exp(c^* ns^2)/t^2.$$

To complete the proof, it remains to estimate II. For $\sqrt{n \log n}\, s < 1$ we have

$$\begin{aligned}
\mathrm{II} &= \mathbf{P}(S_n > t,\ t \geqslant M_n > y,\ X_{n-1:n} \leqslant y)\\
&= \mathbf{P}(S_n > t,\ t - 1/s \geqslant M_n > y,\ X_{n-1:n} \leqslant y)\\
&\quad + \mathbf{P}(S_n > t,\ t - \sqrt{n \log n} \geqslant M_n > t - 1/s,\ X_{n-1:n} \leqslant y)\\
&\quad + \mathbf{P}(S_n > t,\ t \geqslant M_n > t - \sqrt{n \log n},\ X_{n-1:n} \leqslant y) := A + B + C.
\end{aligned}$$

Using (3.4), (3.8) and (3.9) we obtain

$$A = O(1)n \int_{y}^{t-1/s} \mathbf{P}(S_{n-1} \geqslant t-u, \max_{k \leqslant n-1} X_k < y)\, dF(u)$$

$$= O(1)n \int_{y}^{t-1/s} \mathbf{P}(\sum_{i=1}^{n} \delta_i > t-u) \exp(-s(t-u))\, dF(u)$$

$$= O(1)n\mathbf{P}(\sum_{i=1}^{n} \delta_i \geqslant 1/s) \exp(-st) \int_{y}^{t-1/s} \exp(su)\, dF(u)$$

$$= O(1)\mathbf{P}(M_n > t)\mathbf{P}(\sum_{i=1}^{n} \delta_i \geqslant 1/s) = O(1)\mathbf{P}(M_n > t)ns^2.$$

Now, we use the next result of [5]: let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables such that $\mathbf{E}Y_1 = 0$, $\mathbf{E}|Y_1|^\beta < \infty$, where $\beta \geqslant 2$. Let us put $B_n = \sum_{k=1}^{n} \mathbf{E}Y_k^2$, $M_{\beta,n} = \sum_{k=1}^{n} \mathbf{E}|Y_k|^\beta$. Then

$$\mathbf{P}(\sum_{k=1}^{n} Y_k \geqslant x) \leqslant (1+2/\beta)^\beta M_{\beta,n} x^{-\beta} + \exp(-c_0 x^2 B_n^{-1}).$$

Moreover, we have

$$B \leqslant n \int_{t-1/s}^{t-\sqrt{n\log n}} \mathbf{P}(S_{n-1} \geqslant t-u, \max_{k \leqslant n-1} X_k < y)\, dF(u)$$

$$= O(1)n\bar{F}(t)\mathbf{P}(S_{n-1} \geqslant \sqrt{n\log n})$$

$$= O(1)\mathbf{P}(M_n > t)\mathbf{P}(S_{n-1} \geqslant \sqrt{n\log n}) = O(1)\mathbf{P}(M_n > t)n^{1-\gamma/2}.$$

For $C$, we have

$$C = \mathbf{P}(S_n > t, t \geqslant M_n > t-\sqrt{n\log n}, X_{n-1:n} \leqslant y) = O(1)\mathbf{P}(t \geqslant M_n > t-\sqrt{n\log n})$$

$$= O(1)(\mathbf{P}(M_n > t-\sqrt{n\log n}) - \mathbf{P}(M_n > t))$$

$$= O(1)\mathbf{P}(M_n > t)(\exp( \int_{t-\sqrt{n\log n}}^{t} q_F(u)\, du) - 1)$$

$$= O(1)\mathbf{P}(M_n > t)( \int_{t-\sqrt{n\log n}}^{t} q_F(u)\, du) = O(1)\mathbf{P}(M_n > t)\sqrt{n\log n}s.$$

If $\sqrt{n\log n}s \geqslant 1$, then

$$\mathrm{II} = \mathbf{P}(S_n > t, t \geqslant M_n > y, X_{n-1:n} \leqslant y)$$

$$= O(1)n\int_{y}^{t} \mathbf{P}(S_{n-1} \geqslant t-u, \max_{k \leqslant n-1} X_k < y)\, dF(u)$$

$$= O(1)n\int_y^t \mathbf{P}\Big(\sum_{i=1}^n \delta_i > t-u\Big)\exp\big(s-(t-u)\big)\,dF(u)$$

$$= O(1)n\exp(-st)\int_y^t \exp(su)\,dF(u) = O(1)\mathbf{P}(M_n > t).$$

Hence

$$\mathrm{II} = O(1)\mathbf{P}(M_n > t)\big(\sqrt{n\log n}s + ns^2 + n^{1-\gamma/2}\big).$$

The lower bound of $\Delta_n(t)$ follows from Lemma 3.1 with $z = 1/\sqrt{n\log n}$. Thus Theorem 3.2 is proved.

## 4. EXAMPLE

We say that d.f. $F$ belongs to the *class* $\mathfrak{D}$ *of dominated-variation distributions* if its tail $\bar{F}$ satisfies

$$\limsup_{t\to\infty} \bar{F}(t)/\bar{F}(2t) < \infty.$$

It follows from this definition that the class of distributions with regularly varying right tails is contained in $\mathfrak{D}\cap\mathfrak{L}$.

It is well known (see e.g. [6]) that if $F\in\mathfrak{D}\cap\mathfrak{L}$, then $F\in S$.

It is also known ([7], Theorem 3.3) that if $\limsup_{t\to\infty} tq(t) < \infty$, then $F\in\mathfrak{D}\cap\mathfrak{L}$. On the other hand, if the hazard rate $q$ is non-increasing, then the statements $F\in\mathfrak{D}\cap\mathfrak{L}$ and $\limsup_{t\to\infty} tq(t) < \infty$ are equivalent (see [7], Corollary 3.4).

The next result is true.

COROLLARY 4.1. *Assume that*
(i) $\mathbf{E}X_1 = 0$;
(ii) $A := \limsup_{t\to\infty} tq_F(t) < \infty$;
(iii) $\gamma > 2$.
*Then for some* $c_0 > 0$, $c^* > 0$, $C_1 > 0$, $C_2 > 0$

$$-\mathbf{P}(M_n > t)\big(c_0 n^{1-\gamma/2} + A\sqrt{n\log n}/t\big) \leqslant \Delta_n(t)$$
$$\leqslant \mathbf{P}(M_n > t)\big(\exp(c^* ns^2)/t^2 + C_1 n^{1-\gamma/2} + C_2\sqrt{n\log n}/t\big).$$

Proof. We restrict ourselves only to indicating the changes which are necessary to make in the proof of Theorem 3.2. The basic change is in the estimates of the term II.

For $t > \sqrt{n\log n}$ we have

$$\mathrm{II} = \mathbf{P}(S_n > t,\, t \geqslant M_n > y,\, X_{n-1:n} \leqslant y)$$
$$= \mathbf{P}(S_n > t,\, t-\sqrt{n\log n} \geqslant M_n > y,\, X_{n-1:n} \leqslant y)$$
$$+\mathbf{P}(S_n > t,\, t \geqslant M_n > t-\sqrt{n\log n},\, X_{n-1:n} \leqslant y) := A + B.$$

For $t$ large enough we have $y > \delta t$, where $\delta > 0$. We obtain

$$\begin{aligned} A &\leqslant n \int_{y}^{t-\sqrt{n\log n}} \mathbf{P}(S_{n-1} \geqslant t-u, \max_{k\leqslant n-1} X_k < y)\, dF(u) \\ &= O(1)\, n\bar{F}(t)\,\mathbf{P}(S_{n-1} \geqslant \sqrt{n\log n}) \\ &= O(1)\,\mathbf{P}(M_n > t)\,\mathbf{P}(S_{n-1} \geqslant \sqrt{n\log n}) = O(1)\,\mathbf{P}(M_n > t)\, n^{1-\gamma/2}. \end{aligned}$$

For $t > \sqrt{n\log n}$ and $n$ large enough we have

$$\begin{aligned} &\mathbf{P}(S_n > t,\, t \geqslant M_n > t-\sqrt{n\log n},\, X_{n-1:n} \leqslant y) = O(1)\,\mathbf{P}(t \geqslant M_n > t-\sqrt{n\log n}) \\ &= O(1)\big(\mathbf{P}(M_n > t-\sqrt{n\log n}) - \mathbf{P}(M_n > t)\big) \\ &= O(1)\,\mathbf{P}(M_n > t)\Big(\exp\Big(\int_{t-\sqrt{n\log n}}^{t} q_F(u)\, du\Big) - 1\Big) \\ &= O(1)\,\mathbf{P}(M_n > t)\big((1-\sqrt{n\log n}/t)^{-A} - 1\big) = O(1)\,\mathbf{P}(M_n > t)\sqrt{n\log n}/t. \end{aligned}$$

The proof is complete.

Remark. Let $t_n$, $n \in N$, be a sequence such that

$$\limsup_{n\to\infty} \sqrt{nR_F(t_n)}/t_n \leqslant \varepsilon (dc^*)^{-1/2} < \infty,$$

where $c^*$ is the same as in Corollary 4.1 and $\infty > d > \alpha$. Then we have

$$\exp(c^* ns^2)/t^2 \leqslant t/t^2 = o(1) \quad \text{as } n \to \infty$$

uniformly with respect to $t \in (t_n, \infty)$. Hence under the conditions of Corollary 4.1 we obtain

$$\Delta_n(t) = o(1)\,\mathbf{P}(M_n > t) \quad \text{as } n \to \infty$$

uniformly with respect to $t \in (t_n, \infty)$.

## REFERENCES


[1] A. Baltrūnas, *On the asymptotics of one-sided large deviation probabilities*, Lithuanian Math. J. 35 (1995), pp. 11–17.

[2] A. A. Borovkov, *Large deviation probabilities for random walk with semiexponential distributions* (in Russian), Sibirsk. Math. Zh. 41 (6) (2000), pp. 1290–1324.

[3] V. P. Chistyakov, *A theorem on sums of independent positive random variables and its applications to branching random processes*, Theory Probab. Appl. 9 (1964), pp. 640–648 (= English translation of: Teor. Veroyatnost. i Primenen. 9 (1964), pp. 710–718).

[4] D. B. H. Cline and T. Hsing, *Large deviation probabilities for sums and maxima of random variables with heavy or subexponential tails*, preprint, Texas A&M University, 1991.

[5] D. H. Fuc and S. V. Nagaev, *Probability inequalities for the sums of independent random variables* (in Russian), Teor. Veroyatnost. i Primenen. 16 (1971), pp. 660–675.
[6] C. M. Goldie, *Subexponential distributions and dominated-variation tail*, J. Appl. Probab. 15 (1978), pp. 440–442.
[7] C. Klüppelberg, *Subexponential distributions and integrated tails*, J. Appl. Probab. 25 (1988), pp. 132–141.
[8] T. Mikosch and A. V. Nagaev, *Large deviations of heavy-tailed sums with applications in insurance*, Extremes 1 (1998), pp. 81–110.
[9] A. V. Nagaev, *On a property of sums of independent random variables* (in Russian), Teor. Veroyatnost. i Primenen. 22 (1977), pp. 335–346.
[10] S. V. Nagaev, *On the asymptotic behaviour of one-sided large deviation probabilities* (in Russian), Teor. Veroyatnost. i Primenen. 26 (1981), pp. 369–372.
[11] I. F. Pinelis, *Asymptotic equivalence of the probabilities of large deviations for sums and maxima of independent random variables* (in Russian): *Limit theorems of the probability theory*, Trudy Inst. Mat. (Novosibirsk) 5 (1985), pp. 144–173.
[12] L. V. Rozovskii, *Probabilities of large deviations on the whole axis* (in Russian), Teor. Veroyatnost. i Primenen. 38 (1993), pp. 79–109.

Institute of Mathematics and Informatics
Akademijos 4
2021 Vilnius, Lithuania
*E-mail*: baltrunas@takas.lt

*Received on 11.12.2001;*
*revised version on 14.10.2002*