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LIMIT THEOREMS FOR ARRAYS OF MAXIMAL ORDER STATISTICS

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Abstract. Let $\{X, X_{nj}, 1 \leqslant j \leqslant m_n, n \geqslant 1\}$ be independent and identically distributed random variables with the Pareto distribution. Let $X_{n(k)}$ be the k-th largest order statistic from the n-th row of our array. This paper establishes unusual limit theorems involving weighted sums for the sequence $\{X_{n(k)}, n \geqslant 1\}$.

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Consider independent and identically distributed random variables $\{X, X_{nj}, 1 \leqslant j \leqslant m_n, n \geqslant 1\}$ with common density $f_X(x) = px^{-p-1} I(x \geqslant 1)$, where $p > 0$. Let $X_{n(k)}$ be the k-th largest order statistic from each row of our array. Hence $k \leqslant m_n \to \infty$. Therefore the density of $X_{n(k)}$ is

$$f_{X_{n(k)}}(x) = \frac{p \cdot m_n!}{(m_n-k)!\,(k-1)!}(1-x^{-p})^{m_n-k} x^{-pk-1} I(x \geqslant 1).$$

In this paper we will study limit theorems involving weighted sums of $\{X_{n(k)}, n \geqslant 1\}$. If $pk > 1$, then $EX_{n(k)}$ is finite and the associated theorems are straightforward and unremarkable. If $pk < 1$, then these limit theorems fail to exist, see Theorem 4. The only interesting case is when $pk = 1$. Strange and unusual limit theorems occur when examining random variables that barely do or do not have a first moment. While this problem can be traced back to the St. Petersburg Game, some of the techniques in this paper can be traced back to Klass and Teicher [4] and Adler [1].

Our first theorem establishes a Weak Law, while our second result is a Generalized Law of the Iterated Logarithm and our third theorem is a closely related Strong Law. As usual, we define $\lg x = \log(\max\{e, x\})$ and $\lg_2 x = \lg(\lg x)$. Also we use the constant C to denote a generic real number that is not necessarily the same in each appearance.

THEOREM 1. *If $pk = 1$ and $\alpha > -1$, then*

$$\frac{\sum_{n=1}^{N} (n^\alpha/m_n^k) X_{n(k)}}{N^{\alpha+1}\lg N} \xrightarrow{P} \frac{1}{(\alpha+1)k!} \quad \text{as } N \to \infty.$$

Proof. Set $a_n = n^\alpha/m_n^k$ and $b_N = N^{\alpha+1}\lg N$. From the Degenerate Convergence Theorem, which can be found on page 356 of Chow and Teicher [2], we have for all $\varepsilon > 0$

$$\begin{aligned}\sum_{n=1}^{N} P\{X_{n(k)} > \varepsilon b_N/a_n\} &< C\sum_{n=1}^{N} \frac{m_n!}{(m_n-k)!} \int_{\varepsilon b_N/a_n}^{\infty} (1-x^{-p})^{m_n-k} x^{-2}\,dx \\ &< C\sum_{n=1}^{N} m_n^k \int_{\varepsilon b_N/a_n}^{\infty} x^{-2}\,dx < \frac{C}{b_N}\sum_{n=1}^{N} a_n m_n^k \\ &= \frac{C\sum_{n=1}^{N} n^\alpha}{N^{\alpha+1}\lg N} < \frac{CN^{\alpha+1}}{N^{\alpha+1}\lg N} = \frac{C}{\lg N} \to 0.\end{aligned}$$

Similarly, the variance term in the Degenerate Convergence Theorem is bounded above by

$$\begin{aligned}\sum_{n=1}^{N} \frac{a_n^2}{b_N^2} EX_{n(k)}^2 I(1 \leqslant X_{n(k)} \leqslant b_N/a_n) &< \frac{C}{b_N^2}\sum_{n=1}^{N} a_n^2 m_n^k \int_{1}^{b_N/a_n} dx \\ &< \frac{C\sum_{n=1}^{N} a_n m_n^k}{b_N} = \frac{C\sum_{n=1}^{N} n^\alpha}{N^{\alpha+1}\lg N} < \frac{CN^{\alpha+1}}{N^{\alpha+1}\lg N} = \frac{C}{\lg N} \to 0.\end{aligned}$$

Next we observe where our truncated first moment is heading. From Gradshteyn and Ryzhik [3], page 4, we have

$$\sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^j}{j} = -\sum_{j=1}^{m_n-k} \frac{1}{j}.$$

Thus

$$\begin{aligned}&\sum_{n=1}^{N} E\left(\frac{a_n X_{n(k)}}{b_N} I(1 \leqslant X_{n(k)} \leqslant b_N/a_n)\right) \\ &= \frac{1}{b_N k!}\sum_{n=1}^{N} \frac{a_n m_n!}{(m_n-k)!} \int_{1}^{b_N/a_n} (1-x^{-p})^{m_n-k} x^{-1}\,dx \\ &= \frac{1}{b_N k!}\sum_{n=1}^{N} \frac{a_n m_n!}{(m_n-k)!} \int_{1}^{b_N/a_n} \left[\frac{1}{x} + \sum_{j=1}^{m_n-k} \binom{m_n-k}{j}(-1)^j x^{-pj-1}\right] dx \\ &= \frac{1}{b_N k!}\sum_{n=1}^{N} \frac{a_n m_n!}{(m_n-k)!}\left[\lg b_N - \lg a_n + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^j}{pj}\right.\end{aligned}$$

$$+\sum_{j=1}^{m_n-k}\frac{\binom{m_n-k}{j}(-1)^{j+1}a_n^{pj}}{pjb_N^{pj}}\Bigg]$$

$$=\frac{1}{k!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha m_n!}{m_n^k(m_n-k)!}\Bigg[(\alpha+1)\lg N+\lg_2 N-\alpha\lg n+k\lg m_n$$

$$-k\sum_{j=1}^{m_n-k}\frac{1}{j}+\sum_{j=1}^{m_n-k}\frac{\binom{m_n-k}{j}(-1)^{j+1}(n^\alpha/m_n^k)^{pj}}{pj(N^{\alpha+1}\lg N)^{pj}}\Bigg]$$

$$=\frac{(\alpha+1)\lg N+\lg_2 N}{k!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha m_n!}{m_n^k(m_n-k)!}-\frac{\alpha}{k!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha m_n!\lg n}{m_n^k(m_n-k)!}$$

$$+\frac{1}{(k-1)!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha m_n!}{m_n^k(m_n-k)!}\Bigg[\lg m_n-\sum_{j=1}^{m_n-k}\frac{1}{j}\Bigg]$$

$$+\frac{1}{k!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha m_n!}{m_n^k(m_n-k)!}\sum_{j=1}^{m_n-k}\frac{\binom{m_n-k}{j}(-1)^{j+1}(n^\alpha/m_n^k)^{pj}}{pj(N^{\alpha+1}\lg N)^{pj}}.$$

The first term is

$$\frac{(\alpha+1)\lg N+\lg_2 N}{k!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha m_n!}{m_n^k(m_n-k)!}\sim\frac{\alpha+1}{k!\,N^{\alpha+1}}\sum_{n=1}^{N}n^\alpha\to\frac{1}{k!}.$$

The second term is

$$\frac{-\alpha}{k!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha m_n!\lg n}{m_n^k(m_n-k)!}\sim\left(\frac{-\alpha}{k!\,N^{\alpha+1}\lg N}\right)\cdot\left(\sum_{n=1}^{N}n^\alpha\lg n\right)$$

$$\sim\left(\frac{-\alpha}{k!\,N^{\alpha+1}\lg N}\right)\cdot\left(\frac{N^{\alpha+1}\lg N}{\alpha+1}\right)=\frac{-\alpha}{(\alpha+1)\,k!}.$$

The third term is bounded above by

$$\left|\frac{1}{(k-1)!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^\alpha m_n!}{m_n^k(m_n-k)!}\Bigg[\lg m_n-\sum_{j=1}^{m_n-k}\frac{1}{j}\Bigg]\right|$$

$$<\frac{C}{N^{\alpha+1}\lg N}\sum_{n=1}^{N}n^\alpha<\frac{C}{\lg N}\to 0.$$

Finally, if we choose N large enough so that

$$\max_{1\leqslant n\leqslant N}\frac{n^\alpha}{N^{\alpha+1}\lg N}<\frac{1}{2^{1/p}},$$

then

$$\left|\frac{1}{k!\,N^{\alpha+1}\lg N}\sum_{n=1}^{N}\frac{n^{\alpha}m_n!}{m_n^k(m_n-k)!}\sum_{j=1}^{m_n-k}\frac{\binom{m_n-k}{j}(-1)^{j+1}(n^{\alpha}/m_n^k)^{pj}}{pj\,(N^{\alpha+1}\lg N)^{pj}}\right|$$

$$< \frac{C}{N^{\alpha+1}\lg N}\sum_{n=1}^{N} n^{\alpha}\sum_{j=1}^{m_n}\frac{m_n^j(n^{\alpha})^{pj}}{(m_n^k N^{\alpha+1}\lg N)^{pj}}$$

$$= \frac{C}{N^{\alpha+1}\lg N}\sum_{n=1}^{N} n^{\alpha}\sum_{j=1}^{m_n}\left(\frac{n^{\alpha}}{N^{\alpha+1}\lg N}\right)^{pj}$$

$$< \frac{C}{N^{\alpha+1}\lg N}\sum_{n=1}^{N} n^{\alpha}\sum_{j=1}^{\infty}\left(\frac{1}{2}\right)^{j} = \frac{C}{N^{\alpha+1}\lg N}\sum_{n=1}^{N} n^{\alpha} < \frac{C}{\lg N}\to 0.$$

Therefore

$$\sum_{n=1}^{N} E\left(\frac{a_n X_{n(k)}}{b_N} I(1\leqslant X_{n(k)}\leqslant b_N/a_n)\right)\to \frac{1}{k!}-\frac{\alpha}{(\alpha+1)k!}=\frac{1}{(\alpha+1)k!},$$

which completes the proof. ■

It is important to note that under the assumptions of Theorem 1 a Strong Law fails to hold. The ensuing result shows us the almost sure behaviour of the normalized partial sums observed in Theorem 1. This type of result is generally called a *Generalized Law of the Iterated Logarithm.*

THEOREM 2. *If* $pk = 1$ *and* $\alpha > -1$, *then*

$$\liminf_{N\to\infty}\frac{\sum_{n=1}^{N}(n^{\alpha}/m_n^k)X_{n(k)}}{N^{\alpha+1}\lg N}=\frac{1}{(\alpha+1)k!}\quad \textit{almost surely}$$

and

$$\limsup_{N\to\infty}\frac{\sum_{n=1}^{N}(n^{\alpha}/m_n^k)X_{n(k)}}{N^{\alpha+1}\lg N}=\infty\quad \textit{almost surely.}$$

Proof. Let $a_n = n^{\alpha}/m_n^k$, $b_n = n^{\alpha+1}\lg n$ and $c_n = b_n/a_n = nm_n^k\lg n$. From Theorem 1 we can conclude that

$$\liminf_{N\to\infty}\frac{\sum_{n=1}^{N}(n^{\alpha}/m_n^k)X_{n(k)}}{N^{\alpha+1}\lg N}\leqslant\frac{1}{(\alpha+1)k!}\quad \text{almost surely.}$$

In order to establish the opposite inequality we note that

$$\frac{\sum_{n=1}^{N}a_n X_{n(k)}}{b_N}\geqslant\frac{\sum_{n=1}^{N}a_n X_{n(k)} I(1\leqslant X_{n(k)}\leqslant nm_n^k)}{b_N}$$

$$= \frac{\sum_{n=1}^{N} a_n \left[X_{n(k)} I(1 \leqslant X_{n(k)} \leqslant nm_n^k) - EX_{n(k)} I(1 \leqslant X_{n(k)} \leqslant nm_n^k)\right]}{b_N}$$

$$+ \frac{\sum_{n=1}^{N} a_n EX_{n(k)} I(1 \leqslant X_{n(k)} \leqslant nm_n^k)}{b_N}.$$

The first term vanishes almost surely by the usual Khintchine–Kolmogorov Convergence Theorem (see Chow and Teicher [2]) and Kronecker's lemma since

$$\sum_{n=1}^{\infty} c_n^{-2} EX_{n(k)}^2 I(1 \leqslant X_{n(k)} \leqslant nm_n^k) < C \sum_{n=1}^{\infty} \frac{m_n^k}{[nm_n^k \lg n]^2} \int_1^{nm_n^k} dx$$

$$< C \sum_{n=1}^{\infty} \frac{nm_n^{2k}}{n^2 m_n^{2k} (\lg n)^2} = C \sum_{n=1}^{\infty} \frac{1}{n (\lg n)^2} < \infty.$$

As for the second term

$$EX_{n(k)} I(1 \leqslant X_{n(k)} \leqslant nm_n^k) \sim \frac{m_n^k}{k!} \int_1^{nm_n^k} (1 - x^{-p})^{m_n - k} x^{-1}\, dx$$

$$= \frac{m_n^k}{k!} \int_1^{nm_n^k} \left[\frac{1}{x} + \sum_{j=1}^{m_n - k} \binom{m_n - k}{j} (-1)^j x^{-pj-1}\right] dx$$

$$= \frac{m_n^k}{k!} \left[\lg(nm_n^k) + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^j}{pj} + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^{j+1}}{pj\,(nm_n^k)^{pj}}\right]$$

$$= \frac{m_n^k}{k!} \left[\lg n + k \lg m_n - k \sum_{j=1}^{m_n-k} \frac{1}{j} + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^{j+1}}{pj\,(nm_n^k)^{pj}}\right]$$

$$= \frac{m_n^k}{k!} \left[\lg n + k\left[\lg m_n - \sum_{j=1}^{m_n-k} \frac{1}{j}\right] + k \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^{j+1}}{j\,(nm_n^k)^{pj}}\right] \sim \frac{m_n^k \lg n}{k!}$$

since

$$\lg m_n - \sum_{j=1}^{m_n-k} \frac{1}{j} = O(1)$$

and

$$\left| \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^{j+1}}{j\,(nm_n^k)^{pj}} \right| < \sum_{j=1}^{m_n} \frac{m_n^j}{(nm_n^k)^{pj}} = \sum_{j=1}^{m_n} \left(\frac{1}{n^p}\right)^j = O(1).$$

Thus

$$\frac{\sum_{n=1}^{N} a_n EX_{n(k)} I(1 \leqslant X_{n(k)} \leqslant nm_n^k)}{b_N} \sim \frac{\sum_{n=1}^{N} (n^\alpha/m_n^k)\cdot((m_n^k \lg n)/k!)}{N^{\alpha+1}\lg N}$$

$$= \frac{\sum_{n=1}^{N} n^\alpha \lg n}{k!\, N^{\alpha+1}\lg N} \to \frac{1}{(\alpha+1)k!},$$

whence

$$\liminf_{N\to\infty} \frac{\sum_{n=1}^{N} (n^\alpha/m_n^k) X_{n(k)}}{N^{\alpha+1}\lg N} \geqslant \frac{1}{(\alpha+1)k!} \quad \text{almost surely,}$$

which leads us to equality for the lower limit.

As for the upper limit, we let $M > 0$. Then

$$\sum_{n=1}^{\infty} P\{X_{n(k)} > Mc_n\} = \sum_{n=1}^{\infty} \frac{m_n!}{(m_n-k)!\,k!} \int_{Mc_n}^{\infty} (1-x^{-p})^{m_n-k} x^{-2}\,dx$$

$$= \frac{1}{k!} \sum_{n=1}^{\infty} \frac{m_n!}{(m_n-k)!} \sum_{j=0}^{m_n-k} \binom{m_n-k}{j} (-1)^j \int_{Mc_n}^{\infty} x^{-pj-2}\,dx$$

$$= \frac{1}{k!} \sum_{n=1}^{\infty} \frac{m_n!}{(m_n-k)!} \sum_{j=0}^{m_n-k} \binom{m_n-k}{j} \frac{(-1)^j}{(pj+1)(Mc_n)^{pj+1}}$$

$$= \frac{1}{M(k-1)!} \sum_{n=1}^{\infty} \frac{m_n!}{c_n(m_n-k)!} \left[\frac{1}{k} + \sum_{j=1}^{m_n-k} \binom{m_n-k}{j} \frac{(-1)^j}{(j+k)(Mc_n)^{pj}}\right]$$

$$> C \sum_{n=1}^{\infty} \frac{m_n!}{c_n(m_n-k)!}$$

since

$$\left| \sum_{j=1}^{m_n-k} \binom{m_n-k}{j} \frac{(-1)^j}{(j+k)(Mc_n)^{pj}} \right| < \sum_{j=1}^{m_n} \frac{m_n^j}{(Mc_n)^{pj}} = \sum_{j=1}^{m_n} \frac{m_n^j}{M^{pj}[nm_n^k \lg n]^{pj}}$$

$$< \sum_{j=1}^{\infty} \left(\frac{1}{[Mn\lg n]^p}\right)^j = \frac{1}{[Mn\lg n]^p - 1} \to 0.$$

Hence

$$\sum_{n=1}^{\infty} P\{X_{n(k)} > Mc_n\} > C \sum_{n=1}^{\infty} \frac{m_n!}{c_n(m_n-k)!}$$

$$= C \sum_{n=1}^{\infty} \frac{m_n!}{nm_n^k \lg n\,(m_n-k)!} > C \sum_{n=1}^{\infty} \frac{1}{n\lg n} = \infty,$$

which implies that

$$\limsup_{n\to\infty} \frac{a_n X_{n(k)}}{b_n} = \infty \text{ almost surely,}$$

which in turn allows us to conclude that

$$\limsup_{N\to\infty} \frac{\sum_{n=1}^{N} a_n X_{n(k)}}{b_N} = \infty \text{ almost surely,}$$

completing this proof. ∎

What proves to be quite interesting is that if we let $\alpha = -1$ in Theorems 1 and 2, then not only does a Weak Law, but also a Strong Law exists. This next result is our *Strong Law of Large Numbers*. Notice that the norming sequence is different, we need an extra logarithm term.

THEOREM 3. *If $pk = 1$ and $\alpha > -2$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \left((\lg n)^{\alpha}/nm_n^k\right) X_{n(k)}}{(\lg N)^{\alpha+2}} = \frac{1}{(\alpha+2)\,k!} \text{ almost surely.}$$

Proof. Let $a_n = (\lg n)^{\alpha}/(nm_n^k)$, $b_n = (\lg n)^{\alpha+2}$ and $c_n = b_n/a_n = nm_n^k (\lg n)^2$. We partition our sum into the following three terms:

$$\frac{\sum_{n=1}^{N} a_n X_{n(k)}}{b_N}$$

$$= \frac{\sum_{n=1}^{N} a_n \left[X_{n(k)} I(1 \leqslant X_{n(k)} \leqslant c_n) - EX_{n(k)} I(1 \leqslant X_{n(k)} \leqslant c_n)\right]}{b_N}$$

$$+ \frac{\sum_{n=1}^{N} a_n X_{n(k)} I(X_{n(k)} > c_n)}{b_N} + \frac{\sum_{n=1}^{N} a_n EX_{n(k)} I(1 \leqslant X_{n(k)} \leqslant c_n)}{b_N}.$$

The first term vanishes almost surely since

$$\sum_{n=1}^{\infty} c_n^{-2} EX_{n(k)}^2 I(1 \leqslant X_{n(k)} \leqslant c_n) < C \sum_{n=1}^{\infty} \frac{m_n^k}{c_n^2} \int_1^{c_n} dx$$

$$< C \sum_{n=1}^{\infty} \frac{m_n^k}{c_n} = C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

The second term vanishes almost surely since

$$\sum_{n=1}^{\infty} P\{X_{n(k)} > c_n\} < C \sum_{n=1}^{\infty} m_n^k \int_{c_n}^{\infty} x^{-2}\,dx = C \sum_{n=1}^{\infty} \frac{m_n^k}{c_n} = C \sum_{n=1}^{\infty} \frac{1}{n(\lg n)^2} < \infty.$$

As for the third term

$$EX_{n(k)}I(1 \leqslant X_{n(k)} \leqslant c_n) = \frac{p \cdot m_n!}{(m_n-k)!(k-1)!} \int_1^{c_n} (1-x^{-p})^{m_n-k} x^{-1}\,dx$$

$$= \frac{p \cdot m_n!}{(m_n-k)!(k-1)!} \int_1^{c_n} \left[\frac{1}{x} + \sum_{j=1}^{m_n-k} \binom{m_n-k}{j} (-1)^j x^{-pj-1}\right] dx$$

$$= \frac{p \cdot m_n!}{(m_n-k)!(k-1)!} \left[\lg c_n + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^j}{pj} + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^{j+1}}{pjc_n^{pj}}\right]$$

$$= \frac{m_n!}{(m_n-k)!\,k!} \left[k \lg m_n + \lg n + 2\lg_2 n - k\sum_{j=1}^{m_n-k} \frac{1}{j} + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^{j+1}}{pjc_n^{pj}}\right]$$

$$= \frac{m_n!}{(m_n-k)!\,k!} \left[\lg n + 2\lg_2 n + k\left[\lg m_n - \sum_{j=1}^{m_n-k} \frac{1}{j}\right] + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^{j+1}}{pjc_n^{pj}}\right]$$

$$\sim \frac{m_n! \lg n}{(m_n-k)!\,k!} \sim \frac{m_n^k \lg n}{k!}$$

since $\lg m_n - \sum_{j=1}^{m_n} j^{-1} = O(1)$, and if we let $n(\lg n)^2 > 2^{1/p}$, then

$$\left|\sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j}(-1)^{j+1}}{pjc_n^{pj}}\right| < \sum_{j=1}^{m_n-k} \frac{m_n^j}{c_n^{pj}} < \sum_{j=1}^{\infty} \frac{m_n^j}{[nm_n^k(\lg n)^2]^{pj}}$$

$$= \sum_{j=1}^{\infty} \left[\frac{1}{[n(\lg n)^2]^p}\right]^j < \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 1.$$

Thus we are able to show that the final term converges to the desired result, i.e.,

$$\frac{\sum_{n=1}^N a_n EX_{n(k)} I(1 \leqslant X_{n(k)} \leqslant c_n)}{b_N} \sim \frac{\sum_{n=1}^N ((\lg n)^\alpha / nm_n^k) \cdot ((m_n^k \lg n)/k!)}{(\lg N)^{\alpha+2}}$$

$$= \frac{\sum_{n=1}^N (\lg n)^{\alpha+1}/n}{k!(\lg N)^{\alpha+2}} \to \frac{1}{(\alpha+2)\,k!},$$

which completes the proof. ∎

If $pk < 1$, then under rather mild conditions no interesting results can occur.

THEOREM 4. *If $pk < 1$ and a_n and b_N are positive constants, where*

(1) $$\max_{1 \leqslant n \leqslant N} m_n a_n^p = o(b_N^p),$$

then the only finite limit of our normalized sums is zero, i.e.,

$$\frac{\sum_{n=1}^{N} a_n X_{n(k)}}{b_N} \xrightarrow{P} 0 \quad \text{as } N \to \infty .$$

Proof. Let w_n be the median from the density of $X_{n(k)}$. Thus

$$\frac{1}{2} = \frac{p \cdot m_n!}{(m_n-k)!\,(k-1)!} \int_{w_n}^{\infty} (1-x^{-p})^{m_n-k} x^{-pk-1}\, dx$$

$$< \frac{p \cdot m_n^k}{(k-1)!} \int_{w_n}^{\infty} x^{-pk-1}\, dx = \frac{m_n^k}{k!\, w_n^{pk}}.$$

Hence, we can conclude that $w_n < C m_n^{1/p}$ and from (1) we have

$$\frac{\max_{1 \leqslant n \leqslant N} a_n w_n}{b_N} < \frac{C \max_{1 \leqslant n \leqslant N} a_n m_n^{1/p}}{b_N} \to 0.$$

Assuming that a Weak Law holds we have, by the Degenerate Convergence Theorem,

$$0 \leftarrow \sum_{n=1}^{N} P\{X_{n(k)} > b_N/a_n\} = \frac{p}{(k-1)!} \sum_{n=1}^{N} \frac{m_n!}{(m_n-k)!} \int_{b_N/a_n}^{\infty} (1-x^{-p})^{m_n-k} x^{-pk-1}\, dx$$

$$= \frac{p}{(k-1)!} \sum_{n=1}^{N} \frac{m_n!}{(m_n-k)!} \sum_{j=0}^{m_n-k} \binom{m_n-k}{j} (-1)^j \int_{b_N/a_n}^{\infty} x^{-p(j+k)-1}\, dx$$

$$= \frac{p}{(k-1)!} \sum_{n=1}^{N} \frac{m_n!}{(m_n-k)!} \sum_{j=0}^{m_n-k} \frac{\binom{m_n-k}{j} (-1)^j (a_n/b_N)^{p(j+k)}}{p(j+k)}$$

$$= \frac{1}{(k-1)!} \sum_{n=1}^{N} \frac{m_n!}{(m_n-k)!} \left(\frac{a_n}{b_N}\right)^{pk} \left[\frac{1}{k} + \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j} (-1)^j}{j+k} \left(\frac{a_n}{b_N}\right)^{pj}\right]$$

$$> C \sum_{n=1}^{N} \frac{m_n!}{(m_n-k)!} \left(\frac{a_n}{b_N}\right)^{pk}$$

since, for if we select N large enough so that $m_n a_n^p < \varepsilon b_N^p$ for all $1 \leqslant n \leqslant N$ and $0 < \varepsilon < 1/2$, it follows that

$$\left| \sum_{j=1}^{m_n-k} \frac{\binom{m_n-k}{j} (-1)^j}{j+k} \left(\frac{a_n}{b_N}\right)^{pj} \right| < \sum_{j=1}^{m_n-k} \binom{m_n-k}{j} \left(\frac{a_n}{b_N}\right)^{pj}$$

$$< \sum_{j=1}^{m_n-k} (m_n-k)^j \left(\frac{a_n}{b_N}\right)^{pj} < \sum_{j=1}^{\infty} m_n^j \left(\frac{a_n}{b_N}\right)^{pj} = \sum_{j=1}^{\infty} \left(\frac{m_n a_n^p}{b_N^p}\right)^j < \sum_{j=1}^{\infty} \varepsilon^j < 2\varepsilon.$$

Therefore

$$\sum_{n=1}^{N} \frac{m_n!}{(m_n-k)!} \left(\frac{a_n}{b_N}\right)^{pk} \to 0.$$

Then, by once again utilizing the Degenerate Convergence Theorem, the limit of our normalized partial sum is zero since

$$\sum_{n=1}^{N} \frac{a_n}{b_N} EX_{n(k)} I(1 \leqslant X_{n(k)} \leqslant b_N/a_n) < \frac{C}{b_N} \sum_{n=1}^{N} \frac{a_n m_n!}{(m_n-k)!} \int_1^{b_N/a_n} x^{-pk} dx$$

$$< \frac{C}{b_N} \sum_{n=1}^{N} \frac{a_n m_n!}{(m_n-k)!} \left(\frac{b_N}{a_n}\right)^{-pk+1} = C \sum_{n=1}^{N} \frac{m_n!}{(m_n-k)!} \left(\frac{a_n}{b_N}\right)^{pk} \to 0,$$

which completes the proof. ∎

In conclusion there are two final comments. The first is that (1) is quite mild. Note that (1) holds for all the selected constants, a_n and b_N, in our first three theorems. The other comment is that if $pk > 1$, then a Strong Laws exists since $EX_{n(k)}$ is finite.

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