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SOME REMARKS ON THE ALMOST SURE CENTRAL LIMIT THEOREM FOR INDEPENDENT RANDOM VARIABLES*

BY

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Abstract. The purpose of this paper is the proof of an almost sure central limit theorem for subsequences. We obtain an almost sure convergence limit theorem for independent nonidentically distributed random variables. The presented results extend, to nonidentically distributed random variables, theorems given by Schatte [13].

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1. INTRODUCTION

Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, such that $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. Let us put $S_0 = 0$, $B_0^2 = 0$, $S_n = X_1 + \ldots + X_n$, $B_n^2 = ES_n^2$, $n \geqslant 1$. Assume that

$$B_n/B_{n+1} \to 1, \quad B_n \to \infty \quad \text{as } n \to \infty. \tag{1.1}$$

Let us observe that (1.1) holds if and only if the following, so-called Feller's condition, holds

$$\big(\max_{1 \leqslant k \leqslant n} \sigma_k^2\big)/B_n^2 \to 0 \quad \text{as } n \to \infty. \tag{1.2}$$

Let us put

$$M(t) = \max\{k \geqslant 0\colon\ B_k^2 \leqslant t\}, \quad M_n(t) = M(tB_n^2).$$

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Then, for every $t > 0$,

$$B^2_{M(t)} \leqslant t < B^2_{M(t)+1} \leqslant B^2_{M(t)} + \max_{1 \leqslant k \leqslant M(t)+1} \sigma_k^2.$$

Thus, by (1.1), for every $t > 0$,

$$\lim_{n\to\infty} B^2_{M_n(t)}/B_n^2 = t. \tag{1.3}$$

We introduce the usual "broken line process" on [0, 1]:

$$Y_n(t) = S_{M_n(t)}/B_n + X_{M_n(t)+1}(tB_n^2 - B^2_{M_n(t)})/(B_n \sigma^2_{M_n(t)+1}), \qquad t \in [0, 1]. \tag{1.4}$$

It is clear that $Y_n(t) = S_k/B_n$ whenever $t = B_k^2/B_n^2$, $0 \leqslant k \leqslant n$, and $Y_n(t)$ is the straight line joining $(B_k^2/B_n^2, S_k/B_n)$ and $(B_{k+1}^2/B_n^2, S_{k+1}/B_n)$ in the interval $[B_k^2/B_n^2, B_{k+1}^2/B_n^2]$, $k = 0, 1, \ldots, n-1$. Thus $Y_n(t)$, $t \in [0, 1]$, is continuous with probability one, so that there is a measure P_n on the space $(C[0, 1], C)$, according to which the stochastic process $\{Y_n(t), 0 \leqslant t \leqslant 1\}$ is distributed. Here and in what follows $C[0, 1]$ denotes the space of real-valued continuous functions on [0, 1] and C means the σ-field of Borel sets generated by the open sets of uniform topology.

It is well known that if $\{X_n, n \geqslant 1\}$ satisfies the Lindeberg condition, i.e., for every $\varepsilon > 0$,

$$\lim_{n\to\infty} B_n^{-2} \sum_{k=1}^{n} EX_k^2 I(|X_k| \geqslant \varepsilon B_n) = 0, \tag{1.5}$$

then by Prokhorov's theorem (Prokhorov [8]; cf. also Billingsley [2], Section 10) we have

$$P_n \Rightarrow W \quad \text{as } n \to \infty, \tag{1.6}$$

where, here and in what follows, W denotes the standard Wiener measure on $(C[0, 1], C)$ with the corresponding standard Wiener process $\{W(t), 0 \leqslant t \leqslant 1\}$ (cf. Billingsley [2], p. 61), and $\Rightarrow$ denotes the weak convergence of measures on the space $(C[0, 1], C)$.

We also note that (1.5) implies (1.2) and, in consequence, (1.1).

Let $\delta(x)$ denote the probability measure which assigns its total mass to $x \in C[0, 1]$. Then, if (1.5) holds, we get

$$(\log B_n^2)^{-1} \sum_{k=1}^{n} (\sigma_k^2/B_k^2)\, \delta(Y_k) \Rightarrow W \;\; P\text{-a.s.} \quad \text{as } n \to \infty. \tag{1.7}$$

The limit relation (1.7) is called an *almost sure version of the functional central limit theorem* and was proved independently by Atlagh [1] (Theorem 1.1) and by Rodzik and Rychlik [9] under the additional assumption $E|X_n|^{2+\delta} < \infty$, $n \geqslant 1$, for some $\delta > 0$. Recently Fazekas and Rychlik [6] (Proposition 2.2) have also shown that (1.7) is a consequence of (1.5). In fact, Proposition 2.2 is a consequence of their more general result, which is presented in Theorem 1.1. Moreover, we would like to mention that from (1.7) we also get the following result:

$$(1.8)\quad \lim_{n\to\infty} \sup_{-\infty<x<\infty} \Big|(\log B_n^2)^{-1} \sum_{k=1}^{n} (\sigma_k^2/B_k^2)\, I(S_k/B_k \leqslant x) - \Phi(x)\Big| = 0 \quad P\text{-a.s.},$$

where $\Phi(x)$ is the standard normal distribution function. The result more general than (1.8) can be found in Rodzik and Rychlik [10].

Let us observe that (1.8) can be viewed as a uniform strong law of large numbers or the Glivenko–Cantelli type theorem (cf., Csörgő and Horváth [4]). On the other hand, (1.7) is a functional version of (1.8), and therefore can be considered as a functional Glivenko–Cantelli type result.

In this paper we present an almost sure version of the functional central limit theorem for subsequences. The presented results generalize, to nonidentically distributed random variables X_n, $n \geqslant 1$, Theorem 1 given by Schatte [14], and extend theorems proved by Brosamler [3], Schatte [12], [13], Lacey and Philipp [7], Atlagh [1], Rodzik and Rychlik [9], [10], and Rychlik and Szuster [11] to an almost sure version of the functional central limit theorem for subsequences.

2. RESULTS

Denote by $BL = BL(C[0, 1], \|\cdot\|_{BL})$, as in Lacey and Philipp [7], the class of functions $f\colon C[0, 1] \to \mathbf{R}$ with $\|f\|_{BL} = \|f\|_L + \|f\|_\infty < \infty$, where

$$(2.1)\qquad \|f\|_L = \sup\{\|f(x)-f(y)\|/\|x-y\|_\infty\colon\ x, y \in C[0, 1],\ x \neq y\}.$$

We will consider subsequences

$$(2.2)\qquad n_k = M(c^{k^\alpha}), \qquad n \geqslant 1,$$

for any $c > 1$ and $\alpha > 0$.

THEOREM 1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent random variables with $EX_n = 0$ and $0 < EX_n^2 = \sigma_n^2 < \infty$, $n \geqslant 1$. Assume that* (1.5) *holds. Then, for every $c > 1$ and $\alpha > 0$,*

$$(2.3)\qquad \frac{1}{N}\sum_{k=1}^{N} \delta(Y_{n_k}) \Rightarrow W \quad P\text{-a.s.} \qquad \text{as } N \to \infty.$$

Taking into account the proof of Dudley [5] (Theorem 11.3.3, $b \Rightarrow c$) and the proof of Theorem 1, we also have

$$P\Big\{\lim_{N\to\infty} \sup\Big\{\Big|\frac{1}{N}\sum_{k=1}^{N} f(Y_{n_k}) - Ef(W)\Big|\colon\ \|f\|_{BL} \leqslant 1\Big\} = 0\Big\} = 1.$$

On the other hand, if h is a measurable mapping from $C[0, 1]$ into another metric space S with metric ϱ and σ-field $\mathbf{S}$ of Borel sets, then each probability measure P on $(C[0, 1], \mathbf{C})$ induces on $(S, \mathbf{S})$ a unique probability measure Ph^{-1}, defined by $Ph^{-1}(A) = P(h^{-1}A)$ for $A \in \mathbf{S}$. Thus, by Theorem 5.1 of Billingsley [2], and by (2.3) of Theorem 1, we get the following assertion: there is a P-null set B such that for all $\omega \in B^c$

$$\frac{1}{N}\sum_{k=1}^{N} \delta\big(h(Y_{n_k}(\omega))\big) \Rightarrow Wh^{-1} \quad \text{as } N \to \infty \tag{2.4}$$

for all measurable h: $C[0, 1] \to S$ which are continuous W-a.e. Here Wh^{-1} is the image measure of W under h. Thus, letting in particular $\varphi(x) = x(1)$ and $\psi(x) = \sup_{0 \leqslant t \leqslant 1} |x(t)|^\varrho$, $\varrho > 0$, from Theorem 1 we easily get the following

THEOREM 2. *Under the assumptions of Theorem* 1

$$\lim_{N\to\infty} \sup_x \left|\frac{1}{N}\sum_{k}^{N} I(S_{n_k}/B_{n_k} \leqslant x) - \Phi(x)\right| = 0 \;\; P\text{-a.s.} \tag{2.5}$$

and, for every $\varrho > 0$,

$$\frac{1}{N}\sum_{k=1}^{N} \delta(\max\{|S_1|^\varrho, \ldots, |S_{n_k}|^\varrho\}/B_{n_k}^\varrho) \Rightarrow W\psi^{-1} \;\; P\text{-a.s.} \quad \text{as } N \to \infty. \tag{2.6}$$

We may also obtain pointwise asymptotic results for functionals such as relative frequency of positive S_i's or the last change of sign in $\{S_i, i \leqslant n_N\}$. Some of the functions which may be considered in this context are:

$$h_1(x) = \sup_{0 \leqslant t \leqslant 1} x(t), \qquad h_2(x) = \sup\{t \in [0, 1]: x(t) = 0\},$$

$$h_3(x) = \lambda\{t \in [0, 1]: x(t) > 0\}, \qquad h_4(x) = \lambda\{t \in [0, h_2(x)]: x(t) > 0\},$$

where λ denotes the Lebesgue measure.

THEOREM 3. *Under the assumptions of Theorem* 1, *for every* $1 \leqslant i \leqslant 4$, *P-a.s.*

$$\frac{1}{N}\sum_{k=1}^{N} \delta\big(h_i(Y_{n_k})\big) \Rightarrow Wh_i^{-1} \quad \textit{as } N \to \infty, \tag{2.7}$$

where

$$Wh_1^{-1}\big((-\infty, x]\big) = \frac{2}{\sqrt{2\pi}}\int_0^x \exp(-u^2/2)\,du, \qquad x \geqslant 0,$$

$$Wh_2^{-1}\big((-\infty, x]\big) = Wh_3^{-1}\big((-\infty, x]\big) = \frac{2}{\pi}\arcsin\sqrt{x}, \qquad 0 < x < 1.$$

3. PROOFS

3.1. Proof of Theorem 1. Taking into account Theorem 7.1 of Billingsley [2], Theorem 11.3.3 ($b \Rightarrow c$) of Dudley [5], and Section 2 of Lacey and Philipp [7] (cf. their (6)), we easily note that (2.3) is equivalent to the following statement:

For every $f \in BL$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(Y_{n_k}) = Ef(W) \quad P\text{-a.s.} \tag{3.1}$$

On the other hand, taking into account (1.6) and Theorem 7.1 of Billingsley [2], we clearly have:

For every $f \in BL$

$$\lim_{n\to\infty} Ef(Y_n) = Ef(W), \tag{3.2}$$

which implies

$$\lim_{k\to\infty} Ef(Y_{n_k}) = Ef(W),$$

and, in consequence, we obtain

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} Ef(Y_{n_k}) = Ef(W). \tag{3.3}$$

Thus, by (3.1) and (3.3), it is enough to prove that for every $f \in BL$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \big(f(Y_{n_k}) - Ef(Y_{n_k})\big) = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} Z_{n_k} = 0 \quad P\text{-a.s.}, \tag{3.4}$$

where, for every $k \geqslant 1$,

$$Z_k = f(Y_k) - Ef(Y_k). \tag{3.5}$$

Let $f \in BL$ be given. At first we prove that there is a constant $C > 0$ depending only on $\|f\|_{BL}$ such that for all $j < k$

$$|EZ_j Z_k| \leqslant C(B_j/B_k). \tag{3.6}$$

We have

$$\begin{aligned} EZ_j Z_k &= Ef(Y_j)f(Y_k) - Ef(Y_j)Ef(Y_k) \\ &= E\big(f(Y_j) - Ef(Y_j)\big)\big(f(Y_k) - f(T_{j,k})\big) \\ &\quad + E\big(f(Y_j) - Ef(Y_j)\big)\big(f(T_{j,k}) - Ef(Y_k)\big), \end{aligned} \tag{3.7}$$

where

$$T_{j,k}(t) = \big(Y_k(t) - S_j/B_k\big) I_{[B_j^2/B_k^2, 1]}(t), \qquad t \in [0, 1].$$

Let us observe that if $t \in [0, B_j^2/B_k^2]$, then $T_{j,k}(t) = 0$. Thus, for every $j < k$, $T_{j,k}$ depends only on $X_{j+1}, \ldots, X_k$, and therefore is independent of Y_j.

Hence, by (3.7) and (2.1), we easily get

$$(3.8)\quad |EZ_j Z_k| = \big|E\big(f(Y_j)-Ef(Y_j)\big)\big(f(Y_k)-f(T_{j,k})\big)\big|$$
$$\leqslant E|f(Y_j)-Ef(Y_j)|\,|f(Y_k)-f(T_{j,k})| \leqslant 2\|f\|_\infty E|f(Y_k)-f(T_{j,k})|$$
$$\leqslant 4\|f\|_\infty \|f\|_L E\|Y_k - T_{j,k}\|_\infty$$
$$= 4\|f\|_\infty \|f\|_L B_k^{-1}\{E \sup_{0\leqslant t\leqslant 1} |Y_k(t)-T_{j,k}(t)|\}$$
$$\leqslant 4\|f_\infty\|\,\|f\|_L B_k^{-1}\{E|S_j| + E\max_{1\leqslant i\leqslant j}|S_i|\}$$
$$\leqslant 8\|f\|_{BL} B_k^{-1} E\{\max_{1\leqslant i\leqslant j}|S_i|\}.$$

Furthermore, taking into account Doob's inequality, we also obtain

$$(3.9)\qquad E\{\max_{1\leqslant i\leqslant j}|S_i|\} \leqslant \{E\max_{1\leqslant i\leqslant j}|S_i|^2\}^{1/2} \leqslant \{ES_j^2\}^{1/2} = B_j.$$

Thus, by (3.8) and (3.9), we get (3.6). Furthermore, we have

$$EZ_k^2 \leqslant (\|f\|_{BL})^2, \qquad k \geqslant 1.$$

On the other hand, taking into account (3.6), (2.2) and the definition of the function $M(t)$, we have

$$(3.10)\quad E\Big(\sum_{k=1}^{N} Z_{n_k}\Big)^2 \leqslant 2\sum_{k=1}^{N}\sum_{j=1}^{k}|EZ_{n_k}Z_{n_j}| \leqslant 2C\sum_{k=2}^{N}\sum_{j=1}^{k-1}(B_{n_j}/B_{n_k}) + C_1 N$$
$$\leqslant (2C+C_1)\Big\{\sum_{k=2}^{N}\sum_{j=1}^{k-1}(c^{j^\alpha/2}/B_{n_k}) + N\Big\},$$

where C and C_1 are absolute constants depending only on $\|f\|_{BL}$. In what follows such constants are denoted by C, and the same symbol may be used for different constants.

We also note that, by (2.2), we get

$$B_{n_k}^2 \leqslant c^{k^\alpha} < B_{n_k+1}^2 \leqslant B_{n_k}^2 + B_{n_k}^2\big(\max_{1\leqslant i\leqslant n_k+1}\sigma_i^2/B_{n_k+1}^2\big)(B_{n_k+1}^2/B_{n_k}^2) \leqslant B_{n_k}^2 + \delta_{n_k+1}c^{k^\alpha},$$

where, here and in what follows,

$$\delta_l = \big(\max_{1\leqslant i\leqslant l}\sigma_i^2/B_l^2\big)(B_{l+1}^2/B_l^2).$$

Clearly, by (1.5), we get (1.1) and (1.2), so that

$$\delta_n \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Hence

$$B_{n_k}^2 \geqslant c^{k^\alpha}(1-\delta_{n_k+1}), \qquad k \geqslant 1. \tag{3.12}$$

Using (3.10)–(3.12), we obtain

$$E\Big(\sum_{k=1}^{N} Z_{n_k}\Big)^2 \leqslant C\Big\{\sum_{k=2}^{N}\sum_{j=1}^{k-1} c^{(j^\alpha-k^\alpha)/2}(1-\delta_{n_k+1})^{-1/2}+N\Big\}. \tag{3.13}$$

On the other hand, taking into account de l'Hospital's rule, we conclude that

$$\sum_{j=1}^{k-1} c^{j^\alpha/2} \leqslant \int_1^k c^{x^\alpha/2}\,dx \sim 2k^{1-\alpha}c^{k^\alpha/2}/(\alpha\log c). \tag{3.14}$$

Using (3.13) and (3.14), we arrive at

$$E\Big\{N^{-1}\sum_{k=1}^{N} Z_{n_k}\Big\}^2 \leqslant C\Big\{\sum_{k=2}^{N} k^{1-\alpha}(1-\delta_{n_k+1})^{-1/2}+N\Big\}N^{-2} \leqslant CN^{-d(\alpha)}, \tag{3.15}$$

where $d(\alpha) = \min(1, \alpha)$.

Define an increasing sequence of integers, for example, putting

$$N_l = [l^{2/d(\alpha)}], \qquad l \geqslant 1.$$

Then, by Chebyshev's inequality, (3.15) and the Borel–Cantelli lemma, we immediately reach the following conclusion:

$$\lim_{l\to\infty}\frac{1}{N_l}\sum_{k=1}^{N_l} Z_{n_k} = 0 \ \ P\text{-a.s.} \tag{3.16}$$

On the other hand, for $N_l < N < N_{l+1}$, we have

$$N^{-1}\Big|\sum_{k=N_l+1}^{N}\big(f(Y_{n_k})-Ef(Y_{n_k})\big)\Big| \leqslant N_l^{-1}\sum_{k=N_l+1}^{N_{l+1}}|f(Y_{n_k})-Ef(Y_{n_k})| \tag{3.17}$$

$$\leqslant 2\|f\|_{BL}(N_{l+1}-N_l)/N_l \to 0 \quad \text{as } l\to\infty.$$

Thus, combining (3.16) and (3.17), we conclude (3.4), which completes the proof of Theorem 1.

3.2. Proof of Theorem 2. Let us define the function ϕ: $C[0, 1] \to \boldsymbol{R}$ as follows: $\phi(x) = x(1)$ for $x \in C[0, 1]$. The function ϕ is continuous on the space $(C[0, 1], \mathcal{C})$. Thus, by (2.4), we have for all $\omega \in B^c$

$$\frac{1}{N}\sum_{k=1}^{N}\delta\big(\phi(Y_{n_k}(\omega))\big) \Rightarrow W\phi^{-1} \quad \text{as } N\to\infty.$$

But, for every $x \in \boldsymbol{R}$,

$$W\phi^{-1}((-\infty, x]) = W\{x \in C[0, 1]: x(1) \leqslant x\} = \Phi(x),$$

and

$$\phi(Y_{n_k}(\omega)) = S_{n_k}(\omega)/B_{n_k}, \quad k \geqslant 1.$$

On the other hand, if $A \in \boldsymbol{C}$ and

$$\mu_N(A) = \frac{1}{N}\sum_{k=1}^{N} \delta(Y_{n_k})(A) = \#\{1 \leqslant i \leqslant N: Y_{n_i} \in A\}/N,$$

then

$$\mu_N \phi^{-1}(A) = \#\{1 \leqslant i \leqslant N: Y_{n_i} \in \phi^{-1}(A)\}/N = \#\{1 \leqslant i \leqslant N: \phi(Y_{n_i}) \in A\}/N.$$

Thus Theorem 2 is a consequence of (2.4).

3.3. Proof of Theorem 3. The function h_1 can be treated as in the proof of Theorem 2. On the other hand, in Billingsley [2] (cf. Appendix II) it is shown that each of the mappings h_2, h_3 and h_4 is measurable and is W-a.e. continuous. Thus Theorem 3 is also a consequence of (2.4).

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