PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 24, Fasc. 2 (2004), pp. 339–353

MARKOV PROCESSES CONDITIONED TO NEVER EXIT A SUBSPACE OF THE STATE SPACE

BY

ZBIGNIEW PALMOWSKI (WROCŁAW AND UTRECHT)
AND TOMASZ ROLSKI* (WROCŁAW)

Abstract. In this paper we study Markov processes never exiting (NE) a subspace A of the state space E or, in other words, Markov processes conditioned to stay in the subspace A. We show how the knowledge of the exact asymptotics of the tail distribution of the exit time helps to find the suitable exponential martingale, which, in turn, serves for the change of measure. Under the new probability measure the process is the sought for never exiting one the subspace A. We also find its extended generator and study relationships between the invariant measure (INE) and the quasi-stationary (QS) distribution. We analyze in detail the PDMP processes.

1991 AMS Subject Classification: Primary: 60J25, 60J35.

Key words and phrases: Markov process, extended generator, exponential change of measure, piecewise deterministic Markov process, workload conditioned to stay positive, NE process, INE measure, quasi-stationary distribution.

1. INTRODUCTION

In this paper we study Markov processes never exiting a subspace A of the state space E or, in other words, Markov processes conditioned to stay always in A. Classical examples concern real-valued Markov processes conditioned to stay positive. We use the abbreviation NE for never exiting. Typically, the event that the process never exits the subspace has probability zero, and thus we cannot define NE processes by the conditional probability. In this paper we propose to define NE processes either by a limiting probability or by the change of probability measure. First studies of such a concept, under the name of the *taboo process*, were done by Knight (1969). He showed that the Brownian process NE the positive axis is the Bessel^3 process. Knight also studied Brownian

* This work was partially supported by KBN Grant No 2 P03A 020 23 (2002–2004).

processes NE an interval. Diffusion processes NE a bounded open subset of R^d were studied by Pinsky (1985). The case of spectrally negative Lévy processes conditioned to stay positive was considered by Bertoin (1996) and Chaumont (1996); see also Lambert (2000) for the Lévy process conditioned to stay in an interval. Other examples of such processes are so-called non-colliding Brownian motions, which are k-dimensional Brownian motions never exiting the Weyl chamber $\{\boldsymbol{x}: x_1 \leqslant x_2 \leqslant \ldots < x_k\}$. A limit of such a process turns out to be the eigenvalue process associated with the Hermitian Brownian motion; see Grabiner (1999), König et al. (2002). Jacka and Roberts (1995) study the notion of NE processes for continuous time Markov chains (CTMC).

In this paper, in contrast to Pinsky (1985), we present the theory for general Markov processes and we allow an unbounded subspace A. We present a detailed account of the exponential change of measure approach to define NE processes. Among others, we show how to adapt some results from the paper of Palmowski and Rolski (2002) to make it possible to study NE processes. In particular, we show how to choose a *good function* (the terminology of Palmowski and Rolski (2002)) to define the NE process via change of measure. It turns out that this good function can be learned from the asymptotic behavior of the tail probability of the exit time from the subspace A. This part complements the study of Glynn and Thorisson (2001), (2002) explaining the role of the change of measure technique. The case of piecewise deterministic Markov processes (PDMP) is worked out. We also give a relationship between invariant measures of NE Markov processes (provided they exist), abbreviated here as INE measures, and quasi-stationary (QS) distributions. Note that if the NE process is recurrent, then INE measure is just its stationary measure. The concept of quasi-stationary distributions attracted attention in many papers, in particular see Kyprianou (1971), Keener (1992), Nair and Pollett (1993), Ferrari et al. (1995) and references therein.

2. FINITE CTMC PROTOTYPE

Consider a CTMC $\{X(t)\}$ with $E = \{0, 1, \ldots, k\}$ and $A = \{1, 2, \ldots, k\}$ with transition matrix $\boldsymbol{Q} = (q_{ij})_{i,j=0,1,\ldots,k}$. We assume that realizations are right continuous and with left-hand limits. Let $\tau = \min\{t \geqslant 0{:}\ X(t) = 0\}$ be the exit time from A and

$$X^{\dagger}(t) = \begin{cases} X(t), & t < \tau, \\ \dagger, & t \geqslant \tau, \end{cases} \tag{1}$$

be the process $\{X(t)\}$ killed at the exit from A. The transition matrix of the process $\{X^{\dagger}(t)\}$ on $\{1, \ldots, k\}$ is as follows. Take the subintensity matrix $\boldsymbol{Q}^{\dagger} = (q_{ij})_{i,j=1,\ldots,k}$ of the intensity matrix of the process $\{X(t)\}$ and set

$$\boldsymbol{P}^{\dagger}(t) = (p^{\dagger}_{i,j}(t))_{i,j=1,\ldots,k} = \exp(\boldsymbol{Q}^{\dagger} t).$$

Then $\mathbf{P}_i(X^\dagger(t)=j)=p^\dagger_{ij}(t)$ for $i,j=1,\ldots,k$. Notice that $\sum_{j=1}^k p^\dagger_{ij}(t)\leqslant 1$, $i=1,\ldots,k$. We also remark that the operator $A^\dagger f=Q^\dagger f'$, working on vectors $f=(f_1,\ldots,f_k,f_\dagger)$ such that $f_\dagger=0$ and $f_i=f_i'$ $(i\neq\dagger)$, is a full generator of $\{X^\dagger(t)\}$. Assume the irreducibility on the subspace $1,\ldots,k$ and let $-\gamma$ be the Perron–Frobenius eigenvalue of $Q^\dagger$ with μ and h to be the left (row vector normalized to have sum 1) or the right (column vector) eigenvectors (respectively). Without loss of generality we may assume that $\mu h=1$. We assume $\gamma>0$, which means that τ is proper a.s. (we always have $\gamma\geqslant 0$). Thus

$$P^\dagger(t)h=e^{-\gamma t}h,\qquad \mu P^\dagger(t)=e^{-\gamma t}\mu$$

for all $t\geqslant 0$ or, equivalently,

$$Q^\dagger h=-\gamma h,\qquad \mu Q^\dagger=-\gamma\mu.$$

We will call μ the *quasi-stationary* (QS) distribution of $\{X(t)\}$ on A. From the Perron–Frobenius theorem we know that for $i,j=1,\ldots,k$

$$\mathbf{P}_i(X^\dagger(t)=j)=h_i\mu_j e^{-\gamma t}+o(e^{-\gamma t})\quad \text{as } t\to\infty.$$

Conversely, the knowledge of the limit

$$\lim_{t\to\infty}\frac{\mathbf{P}_i(\tau>t)}{\mathbf{P}_j(\tau>s+t)}$$

gives us the needed spectral information. Namely, we have, as $s\to\infty$,

$$\frac{\mathbf{P}_i(\tau>s)}{\mathbf{P}_j(\tau>s+t)}=\frac{\sum_{l=1}^k\mathbf{P}_i(\tau>s,\,X(s)=l)}{\sum_{l=1}^k\mathbf{P}_j(\tau>s+t,\,X(t+s)=l)}$$

$$\to\frac{\sum_{l=1}^k e^{-\gamma s}h_i\mu_l}{\sum_{l=1}^k e^{-\gamma(t+s)}h_j\mu_l}=e^{\gamma t}\frac{h_i}{h_j}.$$

Similarly we can show that for $i,j=1,\ldots,k$ and $t>0$ the limit

$$p^\dagger_{ij}(t)=\lim_{s\to\infty}\mathbf{P}_i(X^\dagger(t)=j\mid \tau>t+s)$$

exists and defines a probability transition function for a CTMC on $\{1,\ldots,k\}$. Markov processes like $\{X^\dagger(t)\}$ with transition function $P^\dagger(t)=(p^\dagger_{ij}(t))_{i,j=1,\ldots,k}$ are to be studied in this paper. Can we find its generator? The starting point is the observation that h is a $(-\gamma)$-harmonic function on $\{1,\ldots,k\}$, thus $\{\mathscr{E}^\dagger(t)=e^{\gamma t}h_{X^\dagger(t)}\mathbf{1}(t<\tau)\}$ is an $\mathscr{F}_t$-martingale. As we show later, this is an exponential martingale, and we may use the theory from Palmowski and Rolski (2002) to define Markov process $\{X^\uparrow(t)\}$ on $\{1,\ldots,k\}$ via exponential change of measure using the density process $\{\mathscr{E}^\dagger(t)\}$. We shall call $\{X^\uparrow(t)\}$ the process $X(t)$ conditioned to never exit (NE) a set A. This new process is a true Markov process with intensity matrix (check that rows sum to 0):

$$Q^\uparrow=(\Delta^{-1}Q^\dagger\Delta+\gamma I),$$

where $\Delta = \operatorname{diag}(h)$. Thus, indeed, the process lives on $A = \{1, \ldots, k\}$. We may compute the stationary distribution π for $Q^{\dagger}$, that is, the distribution fulfilling $\pi Q^{\dagger} = 0$, which is the INE distribution. Now writing $\pi \Delta^{-1} Q^{\dagger} \Delta = -\gamma\pi$, we obtain $(\pi\Delta^{-1}) Q^{\dagger} = -\gamma(\pi\Delta^{-1})$. This means that $\mu = b\pi\Delta^{-1}$ is QS for some normalizing constant b. Our goal is to generalize above considerations for more general Markov processes on general state spaces.

3. GENERAL FRAMEWORK

Let a state space E be Borel and consider a continuous time Markov process $\{X(t)\}$ on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P}_x)$, $x \in E$. We assume that all considered processes are cádlág. Thus without loss of generality we may suppose that $\Omega \subset E^{[0,\infty)}$ is the space $D_E[0, \infty)$ of cádlág functions from $[0, \infty)$ into E (in some cases we can also use $\mathscr{C}_E[0, \infty)$, the space of continuous functions into E) and consider the canonical process $\{X(t)\}$ on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P}_x)$ defined by $X(\omega, t) = \omega(t)$. We denote the natural filtration by $\{\mathscr{F}_t\}$, where $\mathscr{F}_t = \sigma\{X(s), s \leqslant t\}$ and $\mathscr{F} = \bigvee_{t \geqslant 0} \mathscr{F}_t$.

Let $A \subset E$ be an open subset of E and $\tau = \min\{t \geqslant 0\colon X(t) \notin A\}$ be an exit time. Note that τ is a Markov time. Throughout this note, A always denotes the subspace and we assume that

$$\mathbf{P}_x(\tau < \infty) = 1 \quad \text{for all } x \in A.$$

We also need the killed process $\{X^{\dagger}(t)\}$ defined in (1).

3.1. Never exiting process. The concept of *Never Exiting* $A \subset E$ by a CTMP $\{X(t)\}$ (or, in short, NE) can be defined as follows. For $t \geqslant 0$ and $x \in A$ define

$$\lim_{s\to\infty} \mathbf{P}_x(\Lambda \mid \tau > t+s) = P^{\updownarrow}_{x,t}(\Lambda), \quad \Lambda \in \mathscr{F}_t, \tag{2}$$

provided the limits exist for all t, Λ, and they define the proper probability measures. Let $x \in A$. Then the family $\{P^{\updownarrow}_{x,t}(\cdot), t \geqslant 0\}$ defines one probability measure $\mathbf{P}^{\updownarrow}_x$ on $D_E[0, \infty)$ such that $P^{\updownarrow}_{x,t}(\Lambda) = \mathbf{P}^{\updownarrow}_x(\Lambda)$ for $\Lambda \in \mathscr{F}_t$; see e.g. for more details Palmowski and Rolski (2002). Thus, provided these limits exist, we may consider the process $\{X(t)\}$ on the space $D_E[0, \infty)$ (and in view of Proposition 3.3 on $D_A[0, \infty)$) under $P^{\updownarrow}_x$, which we call *NE the subspace A*. With this approach we face the following problems. The first question is to determine when the limit has sense and the second one is whether we can define $\mathbf{P}^{\updownarrow}_x$ from $\mathbf{P}_x$ by the exponential change of the probability measure argument. Then it turns out that the process $X(t)$ under $\mathbf{P}^{\updownarrow}_x$ is Markovian.

The following general scheme can be found in some papers, like Lambert (2000) and Jacka and Roberts (1995).

LEMMA 3.1. *For all $s, t \geqslant 0$ and $x \in A$*

$$\mathbf{P}_x(\Lambda \mid \tau > t+s) = \mathbf{E}_x\big[D\big(X^{\dagger}(t), x, t, s\big); \Lambda\big], \qquad \Lambda \in \mathcal{F}_t,$$

where

$$D(y, x, t, s) = \frac{\mathbf{P}_y(\tau > s)}{\mathbf{P}_x(\tau > t+s)}.$$

Proof. Let $\Lambda \in \mathcal{F}_t$. We write

$$\begin{aligned}
\mathbf{P}_x(\Lambda \mid \tau > t+s) &= \frac{\mathbf{P}_x(\Lambda, \tau > t+s)}{\mathbf{P}_x(\tau > t+s)} \\
&= \frac{\mathbf{E}_x\big(\mathbf{P}_x(\Lambda, \tau > t, \tau > t+s \mid \mathcal{F}_t)\big)}{\mathbf{P}_x(\tau > t+s)} = \frac{\mathbf{E}_x[\mathbf{P}_{X(t)}(\Lambda, \tau > s)\mathbf{1}(\tau > t)]}{\mathbf{P}_x(\tau > t+s)} \\
&= \frac{\mathbf{E}_x[\mathbf{P}_{X(t)}(\Lambda, \tau > s)]}{\mathbf{P}_x(\tau > t+s)} = \mathbf{E}_x\left[\frac{\mathbf{P}_{X(t)}(\tau > s)\mathbf{1}(\tau > t)}{\mathbf{P}_x(\tau > t+s)}; \Lambda\right] \\
&= \mathbf{E}_x\big[D\big(X(t), x, t, s\big)\mathbf{1}(\tau > t); \Lambda\big].
\end{aligned}$$

We now look for the limiting function $h^*(x, t)$ such that

$$\lim_{s \to \infty} D(y, x, t, s) = \frac{h^*(y, t)}{h^*(x, 0)}. \tag{3}$$

Let τ be the exit time. We now introduce a hypothesis, which yields the required spectral information.

(PF) We say that the *Perron–Frobenius hypothesis* is fulfilled if the limit

$$\lim_{t \to \infty} \frac{\mathbf{P}_x(\tau > t)}{\mathbf{P}_y(\tau > t)} = z(x, y) \tag{4}$$

exists for all $x, y \in A$, $0 < z(x, y) < \infty$, and

$$\lim_{s \to \infty} \mathbf{E}_x\left[\frac{\mathbf{P}_{X(t)}(\tau > s)}{\mathbf{P}_y(\tau > s)}\mathbf{1}(t < \tau)\right] = \mathbf{E}_y z\big(X(t), y\big)\mathbf{1}(t < \tau) \quad \text{for all } y \in A. \tag{5}$$

Conditions under which (PF) holds in the case of $E = \{0, 1, \ldots\}$ and $A = \{1, 2, \ldots\}$ are given e.g. in Kesten (1995) and references therein.

PROPOSITION 3.1. *If the* (PF) *hypothesis is fulfilled, then there exists a function $h\colon A \cup \{\dagger\} \to \mathbf{R}$ such that $h(x) > 0$, $h(\dagger) = 0$ and a constant $\gamma' \geqslant 0$ such that*

$$\lim_{s \to \infty} \frac{\mathbf{P}_x(\tau > s)}{\mathbf{P}_y(\tau > s+t)} = e^{\gamma' t}\frac{h(x)}{h(y)} \quad \textit{for all } x, y \in A.$$

Proof. Fix $x_0 \in A$ and define

$$h(x) = \begin{cases} \lim_{s\to\infty} \mathbf{P}_x(\tau > s)/\mathbf{P}_{x_0}(\tau > s), & x \in A, \\ 0, & x = \dagger. \end{cases}$$

Then

$$\lim_{s\to\infty} \frac{\mathbf{P}_y(\tau > s)}{\mathbf{P}_x(\tau > s)} = \frac{h(y)}{h(x)}, \qquad x, y \in A.$$

Since

$$\mathbf{P}_y(\tau > t+s) = \mathbf{E}_y[\mathbf{P}_{X^\dagger(t)}(\tau > s);\, t < \tau],$$

we infer by (5) that

$$\lim_{s\to\infty} \frac{\mathbf{P}_y(\tau > t+s)}{\mathbf{P}_x(\tau > s)} = \lim_{s\to\infty} \frac{\mathbf{E}_y[\mathbf{P}_{X^\dagger(t)}(\tau > s);\, t < \tau]}{\mathbf{P}_x(\tau > s)}$$

$$= \mathbf{E}_y\left[\frac{h(X^\dagger(t))}{h(x)};\, t < \tau\right] = \frac{h(y)}{h(x)} \mathbf{E}_y\left[\frac{h(X^\dagger(t))}{h(y)};\, t < \tau\right].$$

Define now

$$f(t) = \mathbf{E}_x\left[\frac{h(X^\dagger(t))}{h(x)};\, t < \tau\right].$$

We first show that the above definition does not depend on $x \in A$. We write

$$\frac{\mathbf{E}_x[h(X^\dagger(t))/h(x);\, t < \tau]}{\mathbf{E}_y[h(X^\dagger(t))/h(y);\, t < \tau]} = \frac{\lim_{s\to\infty} \mathbf{P}_x(\tau > s+t)/\mathbf{P}_x(\tau > s)}{\lim_{s\to\infty} \mathbf{P}_y(\tau > s+t)/\mathbf{P}_y(\tau > s)}$$

$$= \frac{\lim_{s\to\infty} \mathbf{P}_x(\tau > s+t)/\mathbf{P}_y(\tau > s+t)}{\lim_{s\to\infty} \mathbf{P}_y(\tau > s)/\mathbf{P}_x(\tau > s)} = 1.$$

Recall that $X^\dagger(t) = \dagger$ for $t \geqslant \tau$. We now have, by the Markov property,

$$f(t+t') = \mathbf{E}_x\left[\frac{h(X^\dagger(t))}{h(x)} \frac{h(X^\dagger(t+t'))}{h(X^\dagger(t))};\, t+t' < \tau\right]$$

$$= \mathbf{E}_x\left[\mathbf{E}_x\left[\frac{h(X^\dagger(t))}{h(x)} \frac{h(X^\dagger(t+t'))}{h(X^\dagger(t))} \mathbf{1}(\tau > t+t') \,|\, \mathcal{F}_t\right]\right]$$

$$= \mathbf{E}_x\left[\frac{h(X^\dagger(t))}{h(x)} \mathbf{E}_{X^\dagger(t)}\left[\frac{h(X^\dagger(t'))}{h(X^\dagger(t))};\, t' < \tau\right];\, t < \tau\right].$$

Since $\mathbf{E}_x[h(X^\dagger(t'))]/h(x)$ does not depend on $x \in A$, we infer that the above equals $f(t+t') = f(t)f(t')$. Clearly, the function $f(t)$ is nonincreasing, so it must be of the form $f(t) = \exp(-\gamma' t)$ for some $\gamma' \geqslant 0$. ∎

The following case will be assumed from now on. Suppose that for all $x \in A$ we have

$$\mathbf{P}_x(\tau > t) \sim Ch(x)e^{-\gamma t}L(t) \quad \text{as } t \to \infty, \tag{6}$$

where $\gamma \geqslant 0$, $C > 0$ and $0 < h(x) < \infty$, and $L(t)$ belongs to the class $\mathscr{L}$ of functions such that

$$\frac{L(s+t)}{L(s)} \to 1 \quad \text{as } s \to \infty$$

and that (5) holds. Then

$$D(y, x, t, s) \to \frac{e^{\gamma t} h(y)}{h(x)} \quad \text{as } s \to \infty \tag{7}$$

for all $t \geqslant 0$, $x, y \in A$. Hence $h^*(x, t) = e^{\gamma t} h(x)$, where $h^*(x, t)$ was defined in (3). This means that $\gamma = \gamma'$ from Proposition 3.1.

We have the following proposition.

PROPOSITION 3.2. *Suppose that* (7) *holds for some positive function* $h(x)$ *on* A *and* $\gamma \geqslant 0$, *and*

$$\mathbf{E}_x[h(X^\dagger(t)); t < \tau] = e^{-\gamma t} h(x), \quad t \geqslant 0, x \in A. \tag{8}$$

Then for all $x \in A$, $t \geqslant 0$

$$\lim_{s\to\infty} \mathbf{P}_x(\Lambda \mid \tau > t+s) = \mathbf{E}_x\left[e^{\gamma t}\frac{h(X^\dagger(t))}{h(x)}\mathbf{1}(\tau > t); \Lambda\right], \quad \Lambda \in \mathscr{F}_t. \tag{9}$$

Proof. Thus under the assumption (8) applying the Markov property, we see that the right-hand side of (9),

$$\mathbf{P}^\updownarrow_{x,t}(\Lambda) = \mathbf{E}_x\left[e^{\gamma t}\frac{h(X^\dagger(t))}{h(x)}\mathbf{1}(\tau > t); \Lambda\right], \quad \Lambda \in \mathscr{F}_t, \tag{10}$$

defines a consistent family of distributions. By the Kolmogorov consistency theorem, for each $x \in A$, (10) defines the unique probability measure $\mathbf{P}^\updownarrow_x$ on $(D_E[0, \infty), \mathscr{F}, \{\mathscr{F}_t\})$ such that its restriction to $\mathscr{F}_t$ is $\mathbf{P}^\updownarrow_{x,t}$ for all $t \geqslant 0$. The proof will follow the proof of Theorem 3.1 in Lambert (2000) and the proof of Theorem 1 in Bertoin and Doney (1994). Suppose that $0 \leqslant f \leqslant 1$ for f living on $D[0, t]$. It follows from Lemma 3.1 and Fatou's lemma that

$$\liminf_{s\to\infty} \mathbf{E}_x[f(X) \mid \tau > t+s] \geqslant \mathbf{E}^\updownarrow_x f(X). \tag{11}$$

Replacing f by $1-f$, we get

$$\limsup_{s\to\infty} \mathbf{E}_x[f(X) \mid \tau > t+s]$$
$$= 1 - \liminf_{s\to\infty} \mathbf{E}_x[(1-f)(X) \mid \tau > t+s] \leqslant 1 - \mathbf{E}^\updownarrow_x[(1-f)(X)] = \mathbf{E}^\updownarrow_x f(X),$$

where the last equality comes from the fact that $\mathbf{P}^{\dagger}$ is conservative (see equation (8)) and would be false otherwise. ■

Note that if (5) and (7) are satisfied, then condition (8) holds. In other words, by Proposition 3.1, if condition (PF) holds, then the assumptions of Proposition 3.2 are fulfilled.

PROPOSITION 3.3. *For all* $x \in A$

$$\mathbf{P}_x^{\dagger}(\tau < \infty) = 0 \quad \textit{and} \quad \mathbf{P}_x^{\dagger}(D_A[0, \infty)) = 1.$$

Proof. For each $t \geqslant 0$

$$\mathbf{P}_x^{\dagger}(\tau < \infty) = e^{\gamma t}\mathbf{E}_x\left[\frac{h(X(t))}{h(x)}\mathbf{1}(\tau < t);\ t \geqslant \tau\right] = 0.$$

Hence $\mathbf{P}_x^{\dagger}(\tau < \infty) = \lim_{t\to\infty}\mathbf{P}_x^{\dagger}(\tau \leqslant t) = 0$. ■

We may conclude that the conditioned process $\{X(t)\}$ can be considered on $(D_A[0, \infty), \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P}_x^{\dagger})$ and it is a Markov process with the state space A. This Markov process is a process NE a subspace A defined by $(h(x), \gamma)$. Note that then

$$\mathscr{E}^{\dagger}(t) = e^{\gamma t}\frac{h(X^{\dagger}(t))}{h(x)}\mathbf{1}(\tau > t) \tag{12}$$

is a $(\mathbf{P}_x, \mathscr{F}_t)$-martingale and (10) can be read

$$\frac{d\mathbf{P}_{x,t}^{\dagger}}{d\mathbf{P}_{x,t}} = \mathscr{E}^{\dagger}(t), \qquad t \geqslant 0. \tag{13}$$

3.2. Exponential change of measure. Let $\{X(t)\}$ be a Markov process on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P}_x)$ with values in E. We define the *full* (*extended*) *generator* A of the process $\{X(t)\}$ by

$$A = \{(f, f^*) \in \mathscr{M}(E) \times \mathscr{M}(E)\colon D_f(t) \text{ is a (local) martingale}\},$$

where

$$D_f(t) = f(X(t)) - \int_0^t f^*(X(s))\,ds \tag{14}$$

is called *Dynkin's* (*local*) *martingale* and the function $s \to f^*(X(s))$ is integrable on $[0, t]$ a.s. for all $t \geqslant 0$. Further on we will identify all versions of functions f^* up to the sets of potential zero and we denote all these versions by Af if $(f, f^*) \in A$. The family of the functions f for which (14) is a (local) martingale forms the domain denoted by $\mathscr{D}(A)$. For details see Ethier and Kurtz (1986).

For a function $f' \in \mathscr{D}(A)$ such that $f'(x) = 0$, $x \notin A$, define

$$f(x) = \begin{cases} f'(x), & x \in A, \\ 0, & x = \dagger, \end{cases}$$

and formally

$$(A^{\dagger} f)(x) = \begin{cases} (Af')(x), & x \in A, \\ 0, & x = \dagger. \end{cases}$$

We denote by $\mathscr{D}(A^{\dagger})$ the set of such f functions.

LEMMA 3.2. *For* $f \in \mathscr{D}(A^{\dagger})$

$$f(X^{\dagger}(t)) - \int_0^t (A^{\dagger} f)(X^{\dagger}(s))\, ds$$

is a local $\mathscr{F}_t$*-martingale.*

Proof. Let $f' \in \mathscr{D}(A)$. Thus $D_{f'}(t)$ and $D_{f'}(t \wedge \tau)$ are local $\mathscr{F}_t$-martingales. Notice however that

$$f'(X(t \wedge \tau)) - \int_0^{t \wedge \tau} (Af')(X(s))\, ds = f(X^{\dagger}(t)) - \int_0^t (A^{\dagger} f)(X^{\dagger}(s))\, ds,$$

which completes the proof. ■

By the above lemma we see that $A^{\dagger}$ is an extended generator of the Markov process $\{X^{\dagger}(t)\}$ and $\mathscr{D}(A^{\dagger})$ is the domain of $A^{\dagger}$.

By $f \to g \cdot f$ we mean the multiplication operator by function g. If $g(x) \neq 0$, then $f \to g^{-1} \cdot f$ denotes multiplication by $g^{-1}(x)$. Define now formally the operator $A^{\uparrow}$ by

$$(A^{\uparrow} f)(x) = \frac{(A^{\dagger} f \cdot h)(x) + \gamma f \cdot h(x)}{h(x)} = h^{-1}(x)(A^{\dagger} f \cdot h)(x) + \gamma f(x), \qquad x \in A,$$

where the function h and the constant γ are given in Proposition 3.2.

Consider the exponential martingale (12). We set $h(\dagger) = 0$, and then $\mathscr{E}^{\uparrow}(t) = e^{\gamma t} h(X^{\dagger}(t))/h(x)$ is a $(\mathbf{P}_x, \mathscr{F}_t)$-martingale for all $x \in A$. Repeating the proof of Lemma 3.1 from Palmowski and Rolski (2002) for $\{\mathscr{E}^{\uparrow}(t)\}$ we see that

$$h(X^{\dagger}(t)) + \gamma \int_0^t h(X^{\dagger}(s))\, ds$$

is a local martingale. Thus directly from definition (14) of the extended generator we can conclude $h \in \mathscr{D}(A^{\dagger})$ and $A^{\dagger} h = -\gamma h$. Therefore $h\colon A \cup \{\dagger\} \to \boldsymbol{R}$ is a good function according to the terminology of Palmowski and Rolski (2002) and $\{\mathscr{E}^{\uparrow}(t)\}$ is an exponential martingale. Since the NE process is obtained by the exponential change of measure with respect to exponential martingale $\{\mathscr{E}^{\uparrow}(t)\}$, the NE process is a *Markov process* with extended generator $A^{\uparrow}$ (see Theorem 4.2 of Palmowski and Rolski (2002)). We summarize the above considerations in the following proposition.

PROPOSITION 3.4. *We have* $h \in \mathscr{D}(A^{\dagger})$ *and* $A^{\dagger} h = -\gamma h$, *and hence the NE process defined by* $(h(x), \gamma)$ *is a Markov process with extended generator* $A^{\uparrow}$.

EXAMPLE 3.1. Let $X(t) = x+B(t)-\theta t$ be a Brownian motion with drift and $A = (0, \infty)$. We have to assume that $\theta \geqslant 0$. Following Borodin and Salminen (1996), for $x > 0$

$$\mathbf{P}_x(\tau > s) = \frac{x}{\sqrt{2\pi}} \int_s^\infty t^{-3/2} \exp\left(-\frac{(x-\theta t)^2}{2t}\right) dt.$$

Hence

$$\lim_{s\to\infty} \frac{\mathbf{P}_y(\tau > s)}{\mathbf{P}_x(\tau > s+t)} = \frac{ye^{\theta y} \exp(\theta^2 t/2)}{xe^{\theta x}}.$$

Therefore $h^*(x, t) = xe^{\theta x} \exp(\theta^2 t/2)$ and $h(x) = xe^{\theta x}$. For $\theta = 0$, the NE process is the Bessel[3] process, which is known to be transient (see Knight (1969)), having invariant measure $\pi(dx) = 2x^2\,dx$ (see Karatzas and Shreve (1991), p. 362). Moreover, for $\theta > 0$ we have

$$\int_0^\infty \frac{1}{h(x)} \pi(dx) = 2\int_0^\infty xe^{-\theta x}\,dx < \infty$$

and there exists a QS distribution

$$\mu(dx) = \frac{\theta^2}{2h(x)} \pi(dx) = \theta^2 xe^{-\theta x}$$

(see Martinez and Martin (1994) and Proposition 4.1 for a general Markov process). Consider now k independent Brownian motions $X_i(t) = x_i+B_i(t)$, where $x_1 < x_2 < \ldots < x_k$, running until the collision, that is up to time

$$\tau = \min\left\{t \geqslant 0\colon \prod_{i=1}^{k-1} \left(X_{i+1}(t)-X_i(t)\right) = 0\right\}.$$

In this case $E = \mathbf{R}^k$ and $A = \{\boldsymbol{x}\in\mathbf{R}^k\colon x_1 < x_2 < \ldots < x_k\}$. Then $\mathbf{P}_{\boldsymbol{x}}(\tau < \infty) = 1$ for $\boldsymbol{x}\in A$. Using the asymptotics of the tail of the exit time we have

$$\lim_{s\to\infty} \frac{\mathbf{P}_{\boldsymbol{x}}(\tau > s)}{\mathbf{P}_{\boldsymbol{y}}(\tau > s+t)} = \frac{h(\boldsymbol{x})}{h(\boldsymbol{y})}, \qquad \boldsymbol{x}, \boldsymbol{y}\in A,$$

where

$$h(\boldsymbol{x}) = \prod_{i<j} (x_j - x_i)$$

is the Vandermond determinant (see Grabiner (1999)). In this case the NE process defined by $h(\boldsymbol{x})$ is the so-called non-colliding Brownian motion; see also König et al. (2002).

3.3. Application to PDMPs. We now show how the Palmowski and Rolski (2002) result for PDMPs can be adapted for the case of NE PDMPs. We refer to Palmowski and Rolski (2002), based on Davis (1993), for the description of PDMP $\{X(t)\}$. We recall only that E is a state space consisting of pairs $x = (v, z)$, where v assumes a finite number of values from a finite set $\mathcal{I}$ and z belongs to an open subset $\mathcal{O}_v$ of $\mathbf{R}^{d(v)}$ (note that E is a Borel space). Moreover, the PDMP process $\{X(t)\}$ is determined by a differential operator $\mathcal{X}$ describing the deterministic vector field between jumps, a jump intensity $\lambda(x)$ and a transition kernel $Q(x, dy)$. Let $\{T_n\}$ be jump epochs of $\{X(t)\}$. We assume that $T_n \to \infty$ a.s.

Consider now an open subspace $A \subset E$ consisting of all pairs $x = (v, z)$, where v assumes a finite number of values from a set $\mathcal{I}^A$, z belongs to an open subset $\mathcal{O}_v^A \subset \mathcal{O}_v$ of $\mathbf{R}^{d(v)}$ and τ is the exit time of the PDMP $\{X(t)\}$ from A. We denote by Γ_A the active boundary defined for the subspace A and $t_A^*(v, z)$ the time needed to reach the boundary from (v, z). We assume that $\phi_v(t_A^*(v, z), z) \in \Gamma_A$ if $t_A^*(v, z) < \infty$, where $\phi_v(t, z)$ is the integral curve defined by $\mathcal{X}$. Then $\{X^\dagger(t)\}$ is also PDMP with parameters $(\mathcal{X}^\dagger, \lambda^\dagger, Q^\dagger) = (\mathcal{X}, \lambda, Q)$ on A and on the cemetery state $\lambda^\dagger(\dagger) = \mathcal{X}^\dagger f(\dagger) = 0$, $Q^\dagger(x, \dagger) = Q(x, A^c)$ for $x \in A$. We have the following lemma.

LEMMA 3.3. *The formula for the extended generator is*

$$A^\dagger f(x) = \mathcal{X} f(x) + \lambda(x) \int_A \big(f(y) - f(x)\big) Q(x, dy), \tag{15}$$

where $x \in A$ and $A^\dagger f(\dagger) = 0$. The domain $\mathcal{D}(A^\dagger)$ consists of every function f that is the restriction to A of a measurable function $\bar{f}$: $A \cup \Gamma_A \cup \{\dagger\} \to \mathbf{R}$ such that $\bar{f}(\dagger) = 0$ and satisfying the following three conditions:

(i) *for each $(v, z) \in E$ the function $t \to \bar{f}(v, \phi_v(t, z))$ is absolutely continuous on $(0, t_A^*(v, z))$*;

(ii) *for each $x \in \Gamma_A$,*

$$\bar{f}(x) = \int_A \bar{f}(y) Q(x, dy); \tag{16}$$

(iii) *for $n = 1, 2, \ldots$,*

$$\mathbf{E}\Big(\sum_{i=1}^{n} |\bar{f}(X(\tau \wedge T_i)) - \bar{f}(\tau \wedge X(T_i -))|\Big) < \infty. \tag{17}$$

If moreover, for all $t \geq 0$,

$$\mathbf{E}\Big(\sum_{i \leq t} |\bar{f}(X(\tau \wedge T_i)) - \bar{f}(X(\tau \wedge T_i -))|\Big) < \infty, \tag{18}$$

then f is in the domain of the full generator.

Following Theorem 5.3 of Palmowski and Rolski (2002), we now compute the extended generator of the NE process defined by $(h(x), \gamma)$.

THEOREM 3.1. *Assume that* $h(x)$ *is such that* $h \in \mathscr{D}(A^\dagger)$ *and* $H(x) = \int_A h(y) Q(x, dy) < \infty$ *for all* $x \in A$. *Then on the new probability space* $(\mathscr{D}_A[0, \infty), \mathscr{F}, \{\mathscr{F}_t\}, \mathbf{P}_x^\dagger)$, *the process* $X(t)$ *is a PDMP with the unchanged differential operator* $\mathscr{X}$ *and the following jump intensity and transition kernel*:

$$\lambda^\dagger(x) = \frac{\lambda(x) H(x)}{h(x)}, \quad Q^\dagger(x, dy) = \frac{h(y)}{H(x)} Q(x, dy). \tag{19}$$

4. INE AND QS DISTRIBUTIONS

Define for $B \in \mathscr{B}(A)$

$$p_t^\dagger(x, B) = \mathbf{P}_x^\dagger(X^\dagger(t) \in B) = \mathbf{E}_x\left[\frac{h(X^\dagger(t)) e^{\gamma t}}{h(x)} \mathbb{1}(\tau > t);\ X^\dagger(t) \in B\right]. \tag{20}$$

The family $\{p_t^\dagger(x, \cdot)\}$ is a family of Markov transition functions on A. The Markov process with the family of transition functions $\{p_t^\dagger(x, B)\}$ is NE space A. If there exists an invariant measure π on A for $p_t^\dagger(x, B)$, that is, fulfilling $\pi(B) = \int_A \pi(dx) p^\dagger(x, B)$ for all B, we call it *Invariant Never Exiting* (INE) measure. If additionally the NE process $X^\dagger(t)$ is recurrent, then π is a stationary measure.

In the literature there is another related concept, which we now recall. We say that a distribution μ on $A \subset E$ is *quasi-stationary* (QS) if

$$\mathbf{P}_\mu(X(t) \in B \mid \tau > t) = \mu(B), \quad B \in \mathscr{B}(E). \tag{21}$$

We have immediately from the definition that $\mathbf{P}_\mu(\tau > t+s) = \mathbf{P}_\mu(\tau > s)\mathbf{P}_\mu(\tau > t)$, which yields $\mathbf{P}_\mu(\tau > t) = e^{-\gamma' t}$ for some $\gamma' > 0$. A vast number of results exist giving conditions for the existence of QS distributions. A special case with $E = \boldsymbol{R}_+$ or $E = \boldsymbol{Z}_+$ and $A = E \setminus \{0\}$ attracted a special attention; see e.g. Ferrari et al. (1995) and references therein.

The question is whether INE and QS distributions are related. We saw the answer in the Introduction for the case of a finite state space. The result can be generalized as follows.

PROPOSITION 4.1. *Suppose that for* $\{X(t)\}$ *there exists an NE process defined by* $(h(x), \gamma)$. *If there exists an INE* σ-*finite measure* π (*not necessarily finite*) *such that*

$$\int_A \frac{1}{h(x)} \pi(dx) < \infty, \tag{22}$$

then the probability measure μ *defined by*

$$\mu(dx) = \frac{b}{h(x)} \pi(dx),$$

where b is a normalizing constant, is a QS distribution, and $\mathbf{P}_\mu(\tau > t) = \exp(-\gamma t)$. *Conversely, suppose that for $\{X(t)\}$ there exists a QS distribution μ with $\gamma' > 0$ and there exists an NE process defined by $(h(x), \gamma)$. Then π defined by*

$$\pi(dx) = h(x)\,\mu(dx)$$

is an INE measure, and $\gamma = \gamma'$.

Proof. Suppose there exists an NE process defined by $(h(x), \gamma)$ and π is an INE measure fulfilling (22). It means that

$$\pi(B) = \int_A \pi(dx)\,h^{-1}(x)\,e^{\gamma t}\,\mathbf{E}_x\left[h(X(t));\ \tau > t,\ X(t)\in B\right]$$

for all $B \subset A$ and $t \geqslant 0$. Hence

$$e^{-\gamma t}\int_B h^{-1}(y)\,d\pi(dy) = \int_A \pi(dx)\,h^{-1}(x)\int_B \mathbf{E}_x\left[h(X(t));\ \tau > t,\ X(t)\in dy\right]h^{-1}(y)$$

and

$$e^{-\gamma t}\int_B h^{-1}(y)\,d\pi(dy) = \int_A \pi(dx)\,h^{-1}(x)\,\mathbf{P}_x\big(X(t)\in B,\ \tau > t\big).$$

Now define $\mu(dy) = h^{-1}(y)\,\pi(dy)$ and suppose it is a probability measure (otherwise we may make normalizations by b). Then we see first that substituting $A = B$ we have

$$e^{-\gamma t} = \mathbf{P}_\mu(\tau > t),$$

and hence μ is QS. For the converse implication, suppose there exists a QS distribution μ with $\gamma' > 0$ and an NE process defined by $(h(x), \gamma)$. Then

$$\begin{aligned} e^{-\gamma' t}\int_B h(x)\,\mu(dx) &= \int_A \mu(dx)\int_B \mathbf{P}_x\big(X(t)\in dy,\ \tau > t\big)\,h(y) \\ &= \int_A \mu(dx)\,h(x)\,\frac{\mathbf{E}_x\left[h(X(t)),\ \tau > t,\ X(t)\in B\right]}{h(x)}. \end{aligned}$$

Hence $\gamma = \gamma'$ and $\pi(dx) = h(x)\,\mu(dx)$ is INE. ∎

We remark that, in view of Example 3.1, condition (22) is essential.

Remark 4.1. The same assertion under a stronger assumption of γ-recurrence can be found in Theorem 7 in Touminen and Tweedie (1979) and Theorem 7 in Arjas et al. (1980) (in fact, they additionally prove the existence of a so-called limiting quasi-stationary distribution $\mu(B) = \lim_{t\to\infty}\mathbf{P}_x\big(X(t)\in B \mid \tau > t\big)$ which implies (21)).

Acknowledgments. Both authors appreciate discussions with David McDonald on the previous version of this paper. The second author is grateful to Masakiyo Miyazawa for attracting his attention to the concept of quasi-stationary distributions. Part of this collaborative work was carried out when the first author was a researcher at EURANDOM and the second author was visiting EURANDOM. Both authors would like to express respective thanks to this institution for its hospitality and support. In addition, the first author gratefully acknowledges grant nr 613.000.310 from Nederlandse Organisatie voor Wetenschappelijk Onderzoek.

REFERENCES

[1] E. Arjas, E. Nummelin and R. L. Tweedie, *Semi-Markov processes on a general state space: α-theory and quasi-stationarity*, J. Austral. Math. Soc. Ser. A 30 (1980), pp. 187–200.

[2] S. Asmussen, *Conditioned limit theorems relating a random walk to its associate, with applications to risk reserve process and the $GI/G/1$ queue*, Adv. in Appl. Probab. 14 (1982), pp. 143–170.

[3] S. Asmussen, *Ruin Probabilities*, World Scientific, Singapore 2000.

[4] S. Asmussen, *Applied Probability and Queues*, second ed., Springer, New York 2003.

[5] J. Bertoin, *Lévy Processes*, Cambridge University Press, Cambridge 1996.

[6] J. Bertoin and R. A. Doney, *On conditioning a random walk to stay nonnegative*, Ann. Probab. 22 (1994), pp. 2152–2167.

[7] J. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge 1987.

[8] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion — Facts and Formulae*, Birkhäuser, Basel 1996.

[9] L. Chaumont, *Conditionings and path decompositions for Lévy processes*, Stochastic Process. Appl. 64 (1) (1996), pp. 39–54.

[10] M. H. A. Davis, *Markov Models and Optimization*, Chapman and Hall, London 1993.

[11] R. A. Doney, *On the asymptotics of first passage times for transient random walks*, Z. Wahrsch. Verw. Gebiete 81 (1989), pp. 239–246.

[12] H. J. Ethier and T. G. Kurtz, *Markov Processes. Characterization and Convergence*, Wiley, New York 1986.

[13] W. Feller, *An Introduction to Probability Theory and Its Applications*, Wiley, New York 1971.

[14] P. A. Ferrari, H. Kesten, S. Martinez and P. Picco, *Existence of quasi-stationary distributions. A renewal dynamical approach*, Ann. Probab. 23 (1995), pp. 501–521.

[15] P. Glynn and H. Thorisson, *Two-sided taboo limits for Markov processes and associated perfect simulation*, Stochastic Proc. Appl. 91 (2001), pp. 1–20.

[16] P. Glynn and H. Thorisson, *Structural characterization of taboo-stationarity for general processes in two-sided time*, Stochastic Proc. Appl. 102 (2002), pp. 311–318.

[17] D. J. Grabiner, *Brownian motion in a Wayl chamber, non-colliding particles, and random matrices*, Ann. Inst. H. Poincaré 35 (1999), pp. 177–204.

[18] S. D. Jacka and G. O. Roberts, *Weak convergence of conditioned processes on a countable state space*, J. Appl. Probab. 32 (1995), pp. 902–916.

[19] I. Karatzas and E. S. Shreve, *Brownian motion and stochastic calculus*, second ed., Springer, New York 1991.

[20] R. W. Keener, *Limit theorems for random walks conditioned to stay positive*, Ann. Probab. 20 (1992), pp. 801–824.

[21] H. Kesten, *A ratio limit theorem for (sub)Markov chains on $\{1, 2, \ldots\}$ with bounded jumps*, Adv. in Appl. Probab. 27 (1995), pp. 652–691.

[22] F. B. Knight, *Brownian local times and taboo processes*, Trans. Amer. Math. Soc. 143 (1969), pp. 173–185.

[23] W. König, N. O'Connell and S. Roch, *Non-colliding random walks, tandem queues and discrete ensembles*, Elect. J. Probab. 7 (2002), pp. 1–24.

[24] E. K. Kyprianou, *On the quasi-stationary distribution of the virtual waiting time in queues with Poisson arrivals*, J. Appl. Probab. 8 (1971), pp. 494–507.

[25] A. Lambert, *Completely asymmetric Lévy processes confined in a finite interval*, Ann. Inst. H. Poincaré. Probab. Statist. 36 (2000), pp. 251–274.

[26] S. Martinez and J. S. Martin, *Quasi-stationary distributions for a Brownian motion with drift and associated limit laws*, J. Appl. Probab. 31 (1994), pp. 911–920.

[27] M. G. Nair and P. K. Pollett, *On the relationship between μ-invariant measures and quasi-stationary distributions for continuous-time Markov chains*, Adv. in Appl. Probab. 25 (1) (1993), pp. 82–102.

[28] Z. Palmowski and T. Rolski, *A technique for exponential change of measure for Markov processes*, Bernoulli 8 (2002), pp. 767–785.

[29] R. G. Pinsky, *On the convergence of diffusion processes conditioned to remain in a bounded region for a large time to limiting positive recurrent diffusion processes*, Ann. Probab. 13 (1985), pp. 363–378.

[30] N. U. Prabhu, *Stochastic Storage Processes*, Springer, New York 1980.

[31] T. Rolski, H. Schmidli, V. Schmidt and J. Teugels, *Stochastic Processes for Insurance and Finance*, Wiley, Chichester 1999.

[32] P. Touminen and R. L. Tweedie, *Exponential decay and ergodicity of general Markov processes and their discrete skeletons*, Adv. in Appl. Probab. 11 (1979), pp. 784–803.

Zbigniew Palmowski
Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
and
Mathematical Institute
Utrecht University
P.O. Box 80.010
3508 TA Utrecht, The Netherlands

Tomasz Rolski
Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland

Received on 15.9.2004