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ASSOCIATION CRITERIA FOR M-INFINITELY-DIVISIBLE AND U-INFINITELY-DIVISIBLE RANDOM SETS

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Abstract. We study random convex compact sets infinitely divisible with respect to the Minkowski addition and establish a sufficient condition for their association as well as a necessary and sufficient condition for the so-called infinite association. Further, we show also that every union infinitely-divisible random closed set and every convex compact set infinitely divisible for convex hulls of unions are associated.

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1. INTRODUCTION

The notion and basic properties of association were described by Esary, Proschan and Walkup in 1967 [4] in the context of random vectors, but the definition of the concept admits a natural and straightforward generalization to a general measurable space $(\mathscr{S}, \mathscr{F}_{\mathscr{S}})$ endowed with a partial order $\preceq$. A random element $X \in \mathscr{S}$ is called *associated* if

$$\mathrm{Cov}\big(f(X), g(X)\big) \geqslant 0$$

for all $\preceq$-non-decreasing functions $f, g : \mathscr{S} \to \mathbb{R}$ for which the covariance is defined. In the case of d-dimensional random vectors $\preceq$ stands for the coordinatewise ordering whereas in our set-valued setting below $\preceq$ is the usual set-theoretic inclusion $\subseteq$.

Association is a much deeper property than the positivity of covariance matrix entries and it found its wide-ranging applications in mathematical theory of reliability (see e.g. Barlow and Proschan [1], Burton and Waymire [3], Kwieciński and Szekli [8], Lindqvist [9] and the references therein) as well as in financial mathematics (see e.g. Rachev and Xin [14]). For stochastic processes the problem of association of increments has been discussed for instance in contexts of Poisson processes or birth-death processes (see e.g. Glasserman [6]).

A crucial example of associated processes are Poisson point processes on general spaces; see Proposition 5.31 in [18]. We shall use this result in our paper, and hence we quote it here as a proposition; see *ibidem* for details.

PROPOSITION 1.1. *Each Poisson point process on a locally compact and separable space is associated with respect to the natural ordering on point measures.*

Effective association criteria were studied for various interesting classes of random vectors, a prominent example being infinitely divisible random vectors for which a particularly elegant characterization of association was found, arising as the sum of results by Pitt [13], Resnick [18] and Samorodnitsky [19] and further described in detail by Houdré et al. [7]. For interesting new developments we refer the reader also to Bäuerle et al. [2]. Let $X \sim \mathcal{ID}(a, \Sigma, \nu)$ be an infinitely divisible random d-dimensional vector with the Lévy–Khinchin representation for characteristic function $\varphi(t) = \exp\big(\psi(t; a, \Sigma, \nu)\big)$, with

$$\psi(t; a, \Sigma, \nu) = i\langle t, a\rangle - \frac{1}{2}\langle \Sigma t, t\rangle + \int_{\mathbb{R}^d} \big(e^{i\langle t,u\rangle} - 1 - i\langle t, u\rangle \cdot \mathbb{1}_{\{|u|<1\}}(u)\big)\nu(du),$$

where $a \in \mathbb{R}^d$ is a vector, $\Sigma \in \mathbb{R}^d \otimes \mathbb{R}^d$ is the covariance matrix of the Gaussian component, whereas ν stands for the Lévy measure. We have then

PROPOSITION 1.2. *A vector* $X \sim \mathcal{ID}(a, \Sigma, \nu)$ *is associated if all coefficients* σ_{ij} *of the matrix* Σ *are nonnegative and* $\operatorname{supp}(\nu) \subseteq (\mathbb{R}_+)^d \cup (\mathbb{R}_-)^d$, *where* $\operatorname{supp}(\nu)$ *denotes the support of the measure* ν.

The converse is not true. Samorodnitsky [19] carried out a construction of an associated infinitely divisible random vector $(\xi_1, \xi_2) \in \mathbb{R}^2$ with Lévy measure ν such that $\nu\big(\{x = (x_1, x_2) \in \mathbb{R}^2;\ x_1 x_2 < 0\}\big) > 0$. However, to make the sufficient conditions of Proposition 1.2 necessary as well it is enough to strengthen slightly the notion of association. If $\varphi(t)$ is the characteristic function of a vector $X \sim \mathcal{ID}(a, \Sigma, \nu)$, then, for all $s \in \mathbb{R}_+$, $\exp\big(s\psi(t; a, \Sigma, \nu)\big) = \big(\varphi(t)\big)^s$ is the characteristic function of some infinitely divisible vector X_s. Moreover, $X_s \sim \mathcal{ID}(sa, s\Sigma, s\nu)$. With this notation, a vector $X \sim \mathcal{ID}(a, \Sigma, \nu)$ is *infinitely associated* iff X_s is associated for all $0 < s \leqslant 1$. Observe that by definition this is the same as the association of the corresponding Lévy process. We have then (see *ibidem*)

PROPOSITION 1.3. $X \sim \mathcal{ID}(a, \Sigma, \nu)$ *is infinitely associated if and only if* $\sigma_{ij} \geqslant 0$ *for all* $1 \leqslant i, j \leqslant d$ *and* $\operatorname{supp}(\nu) \subseteq (\mathbb{R}_+)^d \cup (\mathbb{R}_-)^d$.

The concept of association in context of max-infinitely-divisible random vectors explains positivity of various dependency measures between components of extreme value vectors. Recall that a random vector $X \in \mathbb{R}^d$ is *max-infinitely-*

divisible if for every $n \in \mathbb{N}$ there exist i.i.d. random vectors $X_{n1}, X_{n2}, \ldots, X_{nn}$ such that $X =_d \max\{X_{n1}, X_{n2}, \ldots, X_{nn}\}$. Following Resnick [17], Proposition 5.29, we quote

PROPOSITION 1.4. *Every max-infinitely-divisible random vector $Y \in \mathbb{R}^d$ is associated.*

The purpose of this paper is to extend the above association criteria making them applicable in a broad context of set-valued random elements, namely for the so-called random closed sets. Write $\mathcal{F}$ for the family of all closed subsets of $\mathbb{R}^d$ endowed with the usual Effros σ-algebra generated by families $\mathcal{F}_K := \{F \in \mathcal{F},\ F \cap K = \emptyset\}$ for K ranging through the family $\mathcal{K}$ of all compact subsets of $\mathbb{R}^d$. The Effros σ-field is known to be the Borel σ-field for the so-called Fell topology which is compact on $\mathcal{F}$; see [11], [12]. By a random closed set (in $\mathbb{R}^d$) we shall mean each $\mathcal{F}$-valued random element; see [11] and references therein. By analogy with those for random vectors, different concepts of infinite divisibility have been considered for random sets, including M-infinite-divisibility (infinite divisibility with respect to the so-called Minkowski addition) and U-infinite-divisibility (infinite divisibility with respect to set-theoretic unions); see [11], [12] for extensive reference as well as for various applications of these notions. Our goal is to establish association criteria for infinitely divisible random sets. This is done in separate sections for various concepts of infinite divisibility. First, M-infinite-divisibility is treated in Section 2 below. Next, in Section 3 we study U-infinitely-divisible random closed sets as well as random convex compact sets infinitely divisible for convex hulls of unions.

2. ASSOCIATION FOR M-INFINITELY-DIVISIBLE RANDOM SETS

The present section is devoted to random sets infinitely divisible with respect to the Minkowski addition $\oplus$, defined for $A, B \subseteq \mathbb{R}^d$ by

$$A \oplus B = \{x + y;\ x \in A,\ y \in B\}.$$

A deterministic compact set K is *M-infinitely-divisible* if for all $n \geqslant 2$ there exists a convex set L_n such that

$$K = \underbrace{L_n \oplus \ldots \oplus L_n}_{n}.$$

It turns out that a compact set is M-infinitely-divisible if and only if it is convex; see Theorem 3.1.3 in [12]. Consequently, to avoid unnecessary technicalities the theory of M-infinite-divisibility for random sets is predominantly restricted to the family of random convex compact sets which enjoys the property of being closed with respect to Minkowski sums. In formal terms, by a *random convex compact set* we mean here a random element taking values in the family $\mathrm{co}\mathscr{K}$ of convex compact sets in $\mathbb{R}^d$, endowed with the restriction of the Effros σ-field induced by the inclusion $\mathrm{co}\mathscr{K} \subseteq \mathscr{F}$, which is also easily verified to coincide with the Borel σ-field generated by the usual Hausdorff distance between compacts; moreover, the corresponding topology is locally compact and separable; see [12], Appendix C.

DEFINITION 2.1 (3.2.16 in [12]). A random set $X \in \mathrm{co}\mathscr{K}$ is *M-infinitely-divisible* if for all $n \geqslant 1$ there exist random i.i.d. sets $Z_{n1}, \ldots, Z_{nn} \in \mathrm{co}\mathscr{K}$ such that $X =_d Z_{n1} \oplus \ldots \oplus Z_{nn}$.

Not unexpectedly, M-infinitely-divisible random convex compact sets admit a characterization analogous to the Lévy–Khinchin representation. We follow [12], Subsection 3.2.4, in our brief presentation below. Let Λ be a finite measure on the space $\mathrm{co}\mathscr{K}$ and let $\Pi_\Lambda = \{X_1, \ldots, X_N\}$, where $N \sim \mathrm{Po}\big(\Lambda(\mathrm{co}\mathscr{K})\big)$, be the Poisson point process on $\mathrm{co}\mathscr{K}$ with intensity measure Λ. Then the random convex compact set $Z = X_1 \oplus \ldots \oplus X_N$ is called the *compound Poisson set with intensity measure* Λ and we write $Z \in \mathrm{Pois}_\oplus(\Lambda)$. Now, a σ-finite measure Λ on $\mathrm{co}\mathscr{K}$ is called a *Lévy measure* iff there exist finite measures Λ_n on $\mathrm{co}\mathscr{K}$ and convex compact sets K_n such that $\Lambda_n \uparrow \Lambda$ and for $Z_n \in \mathrm{Pois}_\oplus(\Lambda_n)$ the sequence of random convex compact sets $\{K_n \oplus Z_n\}$ converges weakly with respect to the Hausdorff metric topology to a random convex compact set Z. The distribution of the set Z will be also written as $Z \in \mathrm{Pois}_\oplus(\Lambda)$. The presence of the inclusion sign $\in$ here instead of a more natural equality (in law) is due to technical reasons. The deterministic sets K_n play the role of compensating constants so that the compound M-sums do not diverge – the same shows up in the context of random vectors as well, but there the compensating sequences are standardized, and thus uniquely determined by the Lévy measures. Here an additional degree of freedom is present due to the choice of K_n, and thus $\mathrm{Pois}_\oplus(\Lambda)$ is in fact the entire family of possible limits obtained in the above procedure corresponding to various compensation schemes, whence the $\in$ notation above. Therefore, to avoid possible ambiguities a standardized compensating scheme will be given in Proposition 2.1 below.

The following proposition gives the characterization of M-infinitely-divisible random set X. Note that $s(L)$ stands here for the Steiner point of the convex compact set L defined as the vector-valued integral $(1/\upsilon_d) \int_{\mathbb{S}^{d-1}} h(L,u)u\, \mathcal{H}^{d-1}(du)$ with respect to the $(d-1)$-dimensional Hausdorff measure over the sphere in $\mathbb{R}^d$ with υ_d being the volume of the unit ball (see Appendix F in [12]). The proof can be found in [12], Subsection 3.2.4.

PROPOSITION 2.1. *A random convex compact set X is M-infinitely-divisible if and only if there exist*

(i) *a deterministic set $K \in \mathrm{co}\mathscr{K}$,*
(ii) *a centered Gaussian random vector $\xi \in \mathbb{R}^d$,*
(iii) *a Lévy measure Λ on $\mathrm{co}\mathscr{K}$ satisfying*

$$\int_{\mathrm{co}\mathscr{K}} \min\big(1, \|s(L)\|^2\big)\Lambda(dL) < \infty, \qquad \int_{\mathrm{co}\mathscr{K}} \min\big(1, \|L - s(L)\|\big)\Lambda(dL) < \infty$$

such that $X =_d \xi \oplus K \oplus Z$, *where* Z *is the weak limit of the sequence of random convex compact sets*

$$Z_n = \int\limits_{n^{-1}<\|L\|\leqslant 1} s(L)\Lambda(dL) \oplus Z_n', \quad n \geqslant 1,$$

with $Z_n' \in \mathrm{Pois}_{\oplus}(\Lambda_n)$ *and* Λ_n *being the restriction of* Λ *onto the family of convex compact sets with the norm greater than* n^{-1}.

Roughly speaking, a random convex compact set X is M-infinitely-divisible iff it arises as the M-sum of a deterministic set, a Gaussian random vector and a compound Poisson set with intensity measure given by the Lévy measure. In the sequel we shall write $X \sim \mathcal{IDM}(K, \Sigma, \Lambda)$ for X as in Proposition 2.1, with Σ standing for the covariance matrix of ξ.

In the following theorem we provide a sufficient condition for an M-infinitely-divisible random convex compact set to be associated. To some extent this may be regarded as an equivalent of Proposition 1.2, but an important difference is present, which is the lack of the Gaussian component.

THEOREM 2.1. *If M-infinitely-divisible convex compact set* X *has no Gaussian summand and its Lévy measure concentrates on the family of sets containing the origin*, *then* X *is associated.*

Proof. From the Lévy–Kchinchin-type representation in Proposition 2.1 we know that the set X which satisfies our assumptions arises as the Minkowski sum of a deterministic set K and a Poisson compound set $Z \in \mathrm{Pois}_{\oplus}(\Lambda)$. If the measure Λ is finite, then Z is the Minkowski sum of the Poisson point process Π_Λ and its summands all contain the origin. When restricted to such sets containing the origin, the Minkowski addition becomes a non-decreasing function from the family of subsets of $\mathrm{co}\mathscr{K}$ ordered by inclusion to $\mathrm{co}\mathscr{K}$. Since the Poisson point process Π_Λ is associated when regarded as a random subcollection of $\mathrm{co}\mathscr{K}$ (see Proposition 1.1 above), Z is also associated as obtained from Π_Λ by application of a non-decreasing $\mathrm{co}\mathscr{K}$-valued mapping, because a non-decreasing function of an associated set is again associated (see [4]). Of course, $Z \mapsto K \oplus Z$ is again non-decreasing, so X is associated as well.

For a σ-finite Lévy measure Λ we get our result from the fact that association is preserved under the week limit; see [4]. ■

By analogy with Proposition 1.2, bearing in mind the vector found by Samorodnitsky [19], we do not expect that the converse is true. However, to overcome this nuisance, following [19], for an M-infinitely-divisible $X \in \mathcal{IDM}(K, \Sigma, \Lambda)$ and $t \in (0, 1]$ we consider the random sets $X(t) \in \mathcal{IDM}(tK, t\Sigma, t\Lambda)$, obtainable by putting $X(t) =_d tK \oplus \sqrt{t}\xi \oplus Z_t$ and $Z_t \in \mathrm{Pois}_{\oplus}(t\Lambda)$. Note that this is a natural way to construct a set valued M-infinitely-divisible stochastic process, but we do not discuss such construction here as falling beyond the scope of the present article. Here we only note that, should we choose to speak in terms of this construction, the infinite association concept defined below would coincide with the usual association of the set valued stochastic process $X(t)$; see, e.g., Definition 2.5 in [2].

DEFINITION 2.2. X is *infinitely associated* if $X(t)$ is associated for every $t \in (0,1]$.

In Theorem 2.2 below we show that for infinitely associated random sets the sufficient conditions from Theorem 2.1 become also necessary, by analogy with Proposition 1.3 above. Before proceeding to this result we shall need some additional notation and auxiliary lemmas. An important instrument for our argument is the *support function* of a closed set F, defined on $\mathbb{R}^d$ as

$$h(F,u) = \sup\{\langle x,u\rangle;\ x \in F\};$$

see Subsection 3.1.2 in [12]. Clearly, this function is uniquely determined by its restriction to the unit sphere $\mathbb{S}^{d-1}$. Roughly speaking, for $u \in \mathbb{S}^{d-1}$ the value of $h(F,u)$ tells us how far F extends in the direction u (possibly with the negative sign). A useful and easily verified feature of the functional $h(\cdot,\cdot)$ is that

$$h(F_1 \oplus F_2, u) = h(F_1,u) + h(F_2,u).$$

The next lemma follows directly by a straightforward check.

LEMMA 2.1. *If $X \sim \mathcal{IDM}(K,\Sigma,\Lambda)$ is an M-infinitely-divisible random convex compact set in $\mathbb{R}^d$, then for every $n \in \mathbb{N}$ and $\bar{u} := (u_1,u_2,\ldots,u_n) \in (\mathbb{R}^d)^n$ the random vector*

$$\big(h(X,u_1), h(X,u_2), \ldots, h(X,u_n)\big)^T$$

is infinitely divisible $\mathcal{ID}(a_{\bar{u}},\Sigma_{\bar{u}},\Lambda_{\bar{u}})$, where

$$(a_{\bar{u}})_i = h(K,u_i), \quad i = 1,\ldots,n,$$

$$(\Sigma_{\bar{u}})_{ij} = u_i \Sigma u_j^T, \quad i,j = 1,\ldots,n,$$

$$\Lambda_{\bar{u}}(A) = \Lambda\big(\phi_{\bar{u}}^{-1}(A)\big), \quad A \subseteq \mathbb{R}^n,$$

and $\phi_{\bar{u}}(K) = \big(h(K,u_1),\ldots,h(K,u_n)\big)^T,\ K \in \mathrm{co}\mathcal{K}$.

We are now in a position to establish the main result of this section.

THEOREM 2.2. *An M-infinitely-divisible convex compact set X is infinitely associated if and only if it has no Gaussian summand and its Lévy measure concentrates on the family of sets containing the origin.*

P r o o f. The "if" part may be proved in much the same way as Theorem 2.1. For the "only if" part we first prove that an infinitely associated M-infinitely-divisible X cannot have a Gaussian summand. Let $X =_d K \oplus \xi \oplus Z$ and $X(t) =_d tK \oplus \sqrt{t}\xi \oplus Z_t$ as in Definition 2.2. For $t \in (0,1]$ define functions

$$h_1\big(X(t)\big) = h(tK \oplus \sqrt{t}\xi \oplus Z_t, e_1) = th(K, e_1) + \sqrt{t}\langle \xi, e_1\rangle + h(Z_t, e_1),$$
$$h_2\big(X(t)\big) = h(tK \oplus \sqrt{t}\xi \oplus Z_t, -e_1) = th(K, -e_1) - \sqrt{t}\langle \xi, e_1\rangle + h(Z_t, -e_1),$$

where e_1 is a basis vector in $\mathbb{R}^d$. Both h_1 and h_2 are non-decreasing functions of the set X. If X is infinitely associated, then for all non-decreasing functions $f, g : \mathbb{R}^2 \to \mathbb{R}$

$$\mathrm{Cov}\Big(f\Big(\big(h_1(X(t)), h_2(X(t))\big)^T\Big), g\Big(\big(h_1(X(t)), h_2(X(t))\big)^T\Big)\Big) \geqslant 0,$$

because $f\big(h_1(\cdot), h_2(\cdot)\big)$ and $g\big(h_1(\cdot), h_2(\cdot)\big)$ are non-decreasing. This yields infinite association of the vector $\Big(h_1\big(X(t)\big), h_2\big(X(t)\big)\Big)^T$. Since this vector is infinitely divisible as in Lemma 2.1 and

$$\begin{pmatrix} h_1\big(X(t)\big) \\ h_2\big(X(t)\big) \end{pmatrix} = t\begin{pmatrix} h(K, e_1) \\ h(K, -e_1) \end{pmatrix} + \sqrt{t}\begin{pmatrix} \langle \xi, e_1\rangle \\ -\langle \xi, e_1\rangle \end{pmatrix} + \begin{pmatrix} h(Z_t, e_1) \\ h(Z_t, -e_1) \end{pmatrix},$$

by Proposition 1.3 the matrix Σ in its Lévy–Kchinchin representation must have nonnegative coefficients. Thus $\mathrm{Cov}(\langle \xi, e_1\rangle, -\langle \xi, e_1\rangle) \geqslant 0$, and hence we have $\mathrm{Var}(\langle \xi, e_1\rangle) = 0$, which implies that $\langle \xi, e_1\rangle$ is a constant random variable. A similar result can be obtained for the remaining basis vectors $e_2, \dots, e_n$. Consequently, ξ cannot be a non-degenerate Gaussian vector, which proves our assertion.

What is left to show is that the Lévy measure Λ of the infinitely associated M-infinitely-divisible random set X concentrates on the family of sets in $\mathrm{co}\mathcal{K}$ containing the origin. We proceed by contradiction: suppose the Lévy measure does not have this property. It means that for some $u \in \mathbb{R}^d$ and $c > 0$ we get $\Lambda\big(\{K,\ h(K, u) < -c\}\big) > 0$. However, the fact that $h(K, u) < -c < 0$ implies that $h(K, -u) > c > 0$. Consequently, putting $\bar{u} := (u, -u)$ and letting $\Lambda_{\bar{u}} = \Lambda_{(u,-u)}$ be as in Lemma 2.1 above we see that $\Lambda_{\bar{u}}$ assigns non-zero mass to $\mathbb{R}_- \times \mathbb{R}_+$. Now, consider the random vector $\big(h(X, u), h(X, -u)\big)^T$. On the one hand, by Lemma 2.1 it is infinitely divisible with Lévy measure $\Lambda_{\bar{u}}$. On the other hand, it is also infinitely associated because so is X and the mapping $K \mapsto \big(h(K, u), h(K, -u)\big)^T$ is non-decreasing on $\mathrm{co}\mathcal{K}$ with respect to the inclusion ordering. Putting these conclusions together yields a contradiction with Proposition 1.3 stating that $\Lambda_{\bar{u}}$ concentrates on $(\mathbb{R}_+)^2 \cup (\mathbb{R}_-)^2$. This completes the proof. ■

3. ASSOCIATION FOR U-INFINITELY-DIVISIBLE RANDOM SETS

The notion of max-infinitely-divisible random vector can be transferred onto the ground of the theory of random sets in two different ways. First, following Matheron [10], we can say that a random closed set $X \subseteq \mathbb{R}^d$ is *infinitely divisible for unions* (or *union-infinitely-divisible*) if for all $n \geqslant 1$

$$X =_d X_{n1} \cup X_{n2} \cup \ldots \cup X_{nn},$$

where $X_{n1}, X_{n2}, \ldots, X_{nn}$ are i.i.d. random closed sets. Note that no convexity assumptions are present here.

As it was shown in Matheron [10] (see also [12], Subsection 4.1.2), every union-infinitely-divisible random closed set can by represented as the set-theoretic union of a deterministic closed set and a collection of sets constituting a Poisson point process on the family $\mathcal{F}$ of closed subsets of $\mathbb{R}^d$. When combined with Proposition 1.1 stating the association of Poisson point processes and with the fact that the operation of taking unions is a non-decreasing operation from $2^{\mathcal{F}}$ to $\mathcal{F}$, this yields the following theorem:

THEOREM 3.1. *Every union-infinitely-divisible random closed set is associated.*

If we wish to stay within the family of convex compact random sets, we need to make the sum of sets $X_{n1}, X_{n2}, \ldots, X_{nn}$ convex. So a random convex compact set $X \subseteq \mathbb{R}^d$ is called *infinitely divisible for convex hulls of unions* if for all $n \geqslant 1$

$$X =_d \overline{\text{co}}(X_{n1} \cup X_{n2} \cup \ldots \cup X_{nn}),$$

where $X_{n1}, X_{n2}, \ldots, X_{nn}$ are i.i.d. random convex compact sets (Definition 4.4.1 in [12]).

It is worth to notice that the support function of $\overline{\text{co}}(X_{n1} \cup X_{n2} \cup \ldots \cup X_{nn})$ is equal to the maximum of support functions of sets $X_{n1}, X_{n2}, \ldots, X_{nn}$, which confirms the correspondence of infinite divisibility for convex hulls of unions for random sets and max-infinite-divisibility of random vectors.

The next proposition, due to Giné et al. [5], characterizes the structure of non-empty convex compact sets infinitely divisible for convex hulls of unions.

PROPOSITION 3.1. *X is a non-empty convex compact set infinitely divisible for convex hulls of unions iff there exist a non-empty convex compact set $H \in \mathbb{R}^d$ and a locally finite Borel measure ν on*

$$\mathscr{K}^H = \left\{K \in \text{co}\mathscr{K} \backslash \{\emptyset\};\ K \supseteq H, K \neq H\right\}$$

fulfilling the condition

$$\nu\{K \in \mathscr{K}^H;\ K \nsubseteq D\} < \infty, \quad D \in \mathscr{K}^H,$$

such that

$$X =_d \overline{\mathrm{co}}\Big(H \cup \big(\bigcup_{i=1}^{\infty} K_i\big)\Big),$$

where $\{K_i\}$ *are the points of the Poisson point process on* $\mathrm{co}\mathscr{K} \setminus \{\emptyset\}$ *with the intensity measure* ν.

We conclude the following

THEOREM 3.2. *Every non-empty convex compact set infinitely divisible for convex hulls of unions is associated.*

Proof. It is enough to notice that a set fulfilling the assumptions of our theorem arises as a non-decreasing mapping $\{K_1, K_2, \ldots\} \mapsto \overline{\mathrm{co}}\big(H \cup \bigcup_i K_i\big)$ of a Poisson point process on $\mathrm{co}\mathscr{K} \setminus \{\emptyset\}$, which yields our assertion as a conclusion of Proposition 1.1. ■

It should be noted that the above theorem applies in particular to convex hulls of finite Poisson point processes, which are an example of convex compact sets infinitely divisible for convex hulls of unions and which have attracted considerable interest in stochastic geometry since the seminal work of Rényi and Sulanke; see e.g. [20] for a classical survey and [15], [22] and references therein for some newest developments. For definiteness consider the following set-up: write C_t for the convex hull of a homogeneous Poisson point process in a unit ball in $\mathbb{R}^d$. A rich supply of limit theorems is known for various functionals of C_t as $t \to \infty$ (see *ibidem*), but much less is known about the finite t behavior of the polytope C_t. In this context it may be useful to use Theorem 3.2 to conclude for all t the association of the vector of the following natural increasing functionals of C_t:

1. volume,
2. surface area,
3. mean width (normalized integral of the support function), which is proportional to perimeter for $d = 2$ (see [21], p. 210),
4. more generally, intrinsic volumes of all orders of C_t (see *ibidem*, Chapter 4).

Note that these observations do not apply to the number of vertices of C_t which is another functional of considerable interest but whose dependence on C_t is in general not monotone.

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