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SHARP INEQUALITIES FOR THE SQUARE FUNCTION OF A NONNEGATIVE MARTINGALE*

BY

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Abstract. We determine the optimal constants C_p and C_p^* such that the following holds: if f is a nonnegative martingale and $S(f)$ and f^* denote its square and maximal functions, respectively, then

$$\|S(f)\|_p \leqslant C_p \|f\|_p, \quad p < 1,$$

and

$$\|S(f)\|_p \leqslant C_p^* \|f^*\|_p, \quad p \leqslant 1.$$

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1. INTRODUCTION

Square-function inequalities play an important role in harmonic analysis, classical and noncommutative probability theory and other areas of mathematics. The reader is referred to, for example, the works of Stein [9], [10], Dellacherie and Meyer [5], Pisier and Xu [7] and Randrianantoanina [8]. The purpose of this paper is to provide some new sharp bounds for the moments of a square function under the assumption that the martingale is nonnegative.

Let us start with some definitions. Throughout the paper, $(\Omega, \mathcal{F}, \mathbb{P})$ will be a nonatomic probability space, filtered by a nondecreasing family $(\mathcal{F}_n)_{n=0}^{\infty}$ of sub-σ-fields of $\mathcal{F}$. Let $f = (f_n)$ be a real-valued martingale adapted to $(\mathcal{F}_n)$ and let $df = (df_n)$ stand for its difference sequence:

$$df_0 = f_0, \quad df_n = f_n - f_{n-1}, \quad n = 1, 2, \ldots$$

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A martingale f is called *simple* if for any $n = 0, 1, 2, \ldots$ the random variable f_n takes only a finite number of values and there exists an integer m such that $f_n = f_m$ almost surely for $n > m$.

For any nonnegative integer n, let $S_n(f)$ and f_n^* be given by

$$S_n(f) = \Big(\sum_{k=0}^{n} |df_k|^2\Big)^{1/2} \quad \text{and} \quad f_n^* = \max_{0\leqslant k\leqslant n} |f_k|.$$

Then one defines the *square function* $S(f)$ and the *maximal function* f^* by

$$S(f) = \lim_{n\to\infty} S_n(f) \quad \text{and} \quad f^* = \lim_{n\to\infty} f_n^*.$$

In the paper we are interested in the inequalities between the moments of $S(f)$, f and f^*. For $p \in \mathbb{R}$, let

$$\|f\|_p = \sup_n \|f_n\|_p = \sup_n \left(\mathbb{E}|f_n|^p\right)^{1/p} \qquad \text{if } p \neq 0,$$

and

$$\|f\|_0 = \sup_n \|f_n\|_0 = \sup_n \exp(\mathbb{E}\log|f_n|)$$

with the convention that if $p \leqslant 0$ and $\mathbb{P}(|X| = 0) > 0$, then $\|X\|_p = 0$.

Let us mention here some related results from the literature. An excellent source of information is the survey [2] by Burkholder (see also the references therein). The inequality

$$c_p\|f\|_p \leqslant \|S(f)\|_p \leqslant C_p\|f\|_p \quad \text{if } 1 < p < \infty, \tag{1.1}$$

valid for all martingales, was proved by Burkholder in [1]. Later, Burkholder refined his proof and showed that (cf. [2]) the inequality holds with $c_p^{-1} = C_p = p^* - 1$, where $p^* = \max\{p, p/(p-1)\}$. Furthermore, the constant c_p is optimal for $p \geqslant 2$, C_p is the best for $1 < p \leqslant 2$ and the proof carries over to the case of martingales taking values in a separable Hilbert space. The right inequality (1.1) does not hold for general martingales if $p \leqslant 1$ and nor does the left one if $p < 1$. It was shown by the author in [6] that $c_1 = 1/2$ is the best. In the remaining cases the optimal constants c_p and C_p are not known.

Let us now turn to a related maximal inequality. If $p > 1$, then the estimate (1.1) and Doob's maximal inequality imply the existence of some finite c_p^*, C_p^* such that, for any martingale f,

$$c_p^*\|f^*\|_p \leqslant \|S(f)\|_p \leqslant C_p^*\|f^*\|_p. \tag{1.2}$$

On the other hand, neither of the inequalities holds for $p < 1$ without additional assumptions on f. The limit case $p = 1$ was studied by Davis [4], who proved the validity of the estimate using a clever decomposition of the martingale f. Then

Burkholder proved in [3] that the optimal choice for the constant C_1^* is $\sqrt{3}$. In the other cases (except for $p = 2$, when $c_2^* = 1/2$ and $C_2^* = 1$) the optimal values of c_p^* and C_p^* are not known.

In the paper we study the square-function inequalities for the case $p < 1$ under the additional assumption that the martingale f is nonnegative. The main results of the paper are summarized in the theorem below. For $p < 1$, let

$$C_p = \left(\int_1^\infty (1+t^2)^{p/2} \frac{dt}{t^2} \right)^{1/p} \quad \text{if } p \neq 0,$$

$$C_0 = \lim_{p\to 0} C_p = \exp\left(\int_1^\infty \frac{1}{2} \log(1+t^2) \frac{dt}{t^2} \right).$$

THEOREM 1.1. *Assume f is a nonnegative martingale.*
(i) *We have*

$$\|f\|_p \leqslant \|S(f)\|_p \leqslant C_p \|f\|_p \quad \textit{if } p < 1, \tag{1.3}$$

and the inequality is sharp.
(ii) *We have*

$$\|S(f)\|_p \leqslant \sqrt{2} \|f^*\|_p \quad \textit{if } p \leqslant 1, \tag{1.4}$$

and the constant $\sqrt{2}$ is the best possible.

The result above can be easily extended to the continuous-time setting, using standard approximation arguments (see, for example, Section 6 in [3] for details). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $(\mathcal{F}_t)_{t\geqslant 0}$ be a continuous-time filtration such that $\mathcal{F}_0$ contains all the events of probability 0. For any adapted right-continuous martingale $M = (M_t)$ which has limits on the left, let $[M, M]$ denote its square bracket (consult e.g. [5]). Let $M^* = \sup_t |M_t|$ and $\|M\|_p = \sup_t \|M_t\|_p$.

THEOREM 1.2. *Let $M \geqslant 0$ be as above.*
(i) *We have*

$$\|M\|_p \leqslant \|[M,M]^{1/2}\|_p \leqslant C_p \|M\|_p \quad \textit{if } p < 1, \tag{1.5}$$

and the inequality is sharp.
(ii) *We have*

$$\|[M,M]^{1/2}\|_p \leqslant \sqrt{2} \|M^*\|_p \quad \textit{if } p \leqslant 1, \tag{1.6}$$

and the constant $\sqrt{2}$ is the best possible.

The paper is organized as follows. In the next section we describe the technique invented by Burkholder to study the inequalities involving a martingale, its square and maximal function and present its extension, which is needed to establish (1.6). Section 3 is devoted to the proofs of the inequalities (1.5) and (1.6), while in Section 4 it is shown that these estimates are sharp. Finally, in the last section we present a different proof of the inequality (1.6) in the case $p = 1$.

2. ON BURKHOLDER'S METHOD

The inequalities (1.5) and (1.6) will be established using Burkholder's technique, which reduces the problem of proving a given martingale inequality to finding a certain special function. Let us state the following version of Theorem 2.1 from [3].

THEOREM 2.1. *Suppose that U and V are functions from $(0,\infty)^2$ into $\mathbb{R}$ satisfying*

$$V(x,y) \leqslant U(x,y) \tag{2.1}$$

and the further condition that if d is a simple $\mathcal{F}$-measurable function with $\mathbb{E}d = 0$ and $\mathbb{P}(x+d>0)=1$, then

$$\mathbb{E}U(x+d, \sqrt{y^2+d^2}) \leqslant U(x,y). \tag{2.2}$$

Under these two conditions, we have

$$\mathbb{E}V\big(f_n, S_n(f)\big) \leqslant \mathbb{E}U(f_0, f_0) \tag{2.3}$$

for all nonnegative integers n and simple positive martingales f.

The condition (2.2) can be immediately obtained from the following inequality, which is a bit easier to check: for any positive x and any *number* $d > -x$,

$$U(x+d, \sqrt{y^2+d^2}) \leqslant U(x,y) + U_x(x,y)d. \tag{2.2$'$}$$

The inequality (1.6) may be proved using a special function involving three variables. However, this function seems to be difficult to construct and we have managed to find it only in the case $p = 1$ (see Section 5 below). To overcome this problem, we need an extension of Burkholder's method allowing to work with other operators: we will establish a stronger result, that is

$$\|T(f)\|_p \leqslant \sqrt{2}\|f^*\|_p \quad \text{if } p \leqslant 1. \tag{2.4}$$

Here, given a martingale f, we define a sequence $\big(T_n(f)\big)$ by

$$T_0(f) = |f_0|, \quad T_{n+1}(f) = \big(T_n^2(f) + df_{n+1}^2\big)^{1/2} \vee f_{n+1}^*, \quad n = 0, 1, 2, \ldots,$$

and $T(f) = \lim_{n\to\infty} T_n(f)$. Observe that $T_n(f) \geqslant S_n(f)$ for all n, which can be easily proved by induction. Thus (2.4) implies (1.6).

THEOREM 2.2. *Suppose that* U *and* V *are functions from* $\{(x,y,z)\in(0,\infty)^3 : y \geqslant x\vee z\}$ *into* $\mathbb{R}$ *satisfying*

$$V(x,y,z) \leqslant U(x,y,z), \tag{2.5}$$

$$U(x,y,z) = U(x,y,x\vee z) \tag{2.6}$$

and the further condition that if $0 < x \leqslant z \leqslant y$ *and* d *is a simple* $\mathcal{F}$*-measurable function with* $\mathbb{E}d = 0$ *and* $\mathbb{P}(x+d>0)=1$, *then*

$$\mathbb{E}U\big(x+d, \sqrt{y^2+d^2}\vee(x+d), z\big) \leqslant U(x,y,z). \tag{2.7}$$

Under these three conditions, *we have*

$$\mathbb{E}V\big(f_n, T_n(f), f_n^*\big) \leqslant \mathbb{E}U(f_0,f_0,f_0) \tag{2.8}$$

for all nonnegative integers n *and simple positive martingales* f.

P r o o f. By (2.5), it suffices to show that

$$\mathbb{E}U\big(f_n, T_n(f), f_n^*\big) \leqslant \mathbb{E}U(f_0,f_0,f_0)$$

for all nonnegative integers n and simple positive martingales f. To this end, we will prove that the process $(X_n)_{n=1}^\infty$, given by $X_n = U\big(f_n, T_n(f), f_n^*\big)$, is a supermartingale. Observe that $T_{n+1}(f) = \big(T_n^2(f) + df_{n+1}^2\big)^{1/2} \vee f_{n+1}$ for any $n = 0, 1, 2, \ldots$ Hence we have, by (2.6),

$$\begin{aligned}&\mathbb{E}\big[U\big(f_{n+1}, T_{n+1}(f), f_{n+1}^*\big)|\mathcal{F}_n\big]\\ &= \mathbb{E}\Big[U\Big(f_n+df_{n+1}, \big(T_n^2(f)+df_{n+1}^2\big)^{1/2}\vee(f_n+df_{n+1}), f_n^*\Big)|\mathcal{F}_n\Big].\end{aligned}$$

Using the inequality (2.7) conditionally on $\mathcal{F}_n$, this can be bounded from above by $U\big(f_n, T_n(f), f_n^*\big)$. ■

As previously, we do not work with the property (2.7), but replace it with the following stronger condition: for any $0 < x \leqslant z \leqslant y$ and any $d > -x$,

$$U\big(x+d, \sqrt{y^2+d^2}\vee(x+d), z\big) \leqslant U(x,y,z) + Ad, \tag{2.7$'$}$$

where

$$A = A(x,y,z) = \begin{cases} U_x(x,y,z) & \text{if } x < z,\\ \lim_{t\uparrow z} U_x(t,y,z) & \text{if } x = z.\end{cases}$$

3. PROOFS OF (1.5) AND (1.6)

Let us start with some reductions. By standard approximation, it is enough to establish the inequalities (1.5) and (1.6) for *simple* and *positive* martingales only. The next observation is that, by Jensen's inequality, we have $\|f\|_p = \|f_0\|_p$. Therefore, all we need is to show the following "local" versions: for $n = 0, 1, 2, \ldots$,

$$\|f_0\|_p \leqslant \|S_n(f)\|_p \leqslant C_p\|f_0\|_p \quad \text{if } p < 1, \tag{3.1}$$

and

$$\|T_n(f)\|_p \leqslant \sqrt{2}\|f_n^*\|_p \quad \text{if } p \leqslant 1. \tag{3.2}$$

Finally, we will be done if we establish the inequalities (3.1) and (3.2) for $p \neq 0$; the case $p = 0$ follows then by passing to the limit. Hence, till the end of this section, we assume $p \neq 0$.

3.1. Proof of (3.1). First note that the left inequality is obvious, since $\|f_0\|_p = \|S_0(f)\|_p \leqslant \|S_n(f)\|_p$. Furthermore, clearly, it is sharp; hence we may restrict ourselves to the right inequality in (3.1). It is equivalent to

$$p\mathbb{E}S_n^p(f) \leqslant pC_p^p\mathbb{E}f_0^p. \tag{3.3}$$

Let us introduce the functions V_p, $U_p : (0,\infty)^2 \to \mathbb{R}$ by

$$V_p(x,y) = py^p$$

and

$$U_p(x,y) = px\int_x^\infty (y^2+t^2)^{p/2}\frac{dt}{t^2}.$$

Now (3.3) can be stated as

$$\mathbb{E}V_p\big(f_n, S_n(f)\big) \leqslant \mathbb{E}U_p(f_0, f_0),$$

that is, the inequality (2.3). Therefore, by Theorem 2.1, we need to check the conditions (2.1) and (2.2′).

The inequality (2.1) follows from the identity

$$U_p(x,y) - V_p(x,y) = px\int_x^\infty [(y^2+t^2)^{p/2} - y^p]\frac{dt}{t^2}.$$

To check (2.2′), note that the integration by parts yields

$$U_p(x,y) = p(y^2+x^2)^{p/2} + p^2x\int_x^\infty (y^2+t^2)^{p/2-1}dt \tag{3.4}$$

and

$$U_{px}(x,y) = p\int_x^\infty (y^2+t^2)^{p/2}\frac{dt}{t} - p\frac{(y^2+x^2)^{p/2}}{x} = p^2\int_x^\infty (y^2+t^2)^{p/2-1}dt.$$

Hence we must prove that

$$p\big(y^2+d^2+(x+d)^2\big)^{p/2} + p^2(x+d)\int_{x+d}^\infty (y^2+d^2+t^2)^{p/2-1}dt$$
$$- p(y^2+x^2)^{p/2} - p^2x\int_x^\infty (y^2+t^2)^{p/2-1}dt - p^2d\int_x^\infty (y^2+t^2)^{p/2-1} \leqslant 0$$

or, equivalently,

$$F(x) := p\frac{\big(y^2+d^2+(x+d)^2\big)^{p/2} - (y^2+x^2)^{p/2}}{x+d}$$
$$- p^2\Big[\int_x^\infty (y^2+t^2)^{p/2-1}dt - \int_{x+d}^\infty (y^2+d^2+t^2)^{p/2-1}dt\Big] \leqslant 0.$$

We have

$$F'(x)(x+d)^2 = p^2(y^2+x^2)^{p/2-1}(x+d)d - p\big[\big(y^2+d^2+(x+d)^2\big)^{p/2} - (y^2+x^2)^{p/2}\big], \tag{3.5}$$

which is nonnegative due to the mean value property of the function $t \mapsto t^{p/2}$. Hence

$$F(x) \leqslant \lim_{s\to\infty} F(s) = 0$$

and the proof is complete.

3.2. Proof of the inequality (3.2). We start with an auxiliary technical result.

LEMMA 3.1. (i) *If* $z \geqslant d > 0$ *and* $y > 0$, *then*

$$p\big[(y^2+d^2+z^2)^{p/2} - \big(y^2+(z-d)^2\big)^{p/2}\big] - p^2z\int_{z-d}^{z} (y^2+t^2)^{p/2-1}dt \leqslant 0. \tag{3.6}$$

(ii) *If* $-z < d \leqslant 0$ *and* $Y > 0$, *then*

$$p\frac{\big(Y+(z+d)^2\big)^{p/2} - (Y^2-d^2+z^2)^{p/2}}{z+d} + p^2\int_{z+d}^{z} (Y+t^2)^{p/2-1}dt \leqslant 0. \tag{3.7}$$

(iii) *If* $y \geqslant z \geqslant x > 0$, *then*

$$p[(y^2+x^2)^{p/2} - 2^{p/2}z^p] + p^2\frac{x^2+y^2}{2x}\int_x^z (y^2+t^2)^{p/2-1}dt \geqslant 0. \tag{3.8}$$

(iv) *If* $D \geqslant z \geqslant x > 0$ *and* $y \geqslant z$, *then*

$$p\big[\big(y^2+(D-x)^2+D^2\big)^{p/2}-(y^2+x^2)^{p/2}+2^{p/2}(z^p-D^p)\big] - p^2D\int_x^z (y^2+t^2)^{p/2-1}dt \leqslant 0. \tag{3.9}$$

Proof. Denote the left-hand sides of (3.6)–(3.9) by $F_1(d)$, $F_2(d)$, $F_3(x)$ and $F_4(x)$, respectively. The inequalities will follow by simple analysis of the derivatives.

(i) We have

$$F_1'(d)=p^2d\big[(y^2+d^2+z^2)^{p/2-1}-\big(y^2+(z-d)^2\big)^{p/2-1}\big]\leqslant 0,$$

as $(z-d)^2\leqslant d^2+z^2$. Hence $F_1(d)\leqslant F_1(0+)=0$.

(ii) The expression $F_2'(d)(z+d)^2$ equals

$$p\left[(Y-d^2+z^2)^{p/2}-\big(Y+(z+d)^2\big)^{p/2}+\frac{p}{2}(Y-d^2+z^2)^{p/2-1}\cdot 2d(z+d)\right] \geqslant 0,$$

due to the mean value property. This yields $F_2(d)\leqslant F_2(0)=0$.

(iii) We have

$$F_3'(x)=\frac{p^2}{2}\left(1-\frac{y^2}{x^2}\right)\big[(y^2+x^2)^{p/2-1}x+\int_x^z(y^2+t^2)^{p/2-1}dt\big]\leqslant 0$$

and $F_3(x)\geqslant F_3(z)=p[(y^2+z^2)^{p/2}-2^{p/2}z^p]\geqslant 0$.

(iv) Finally,

$$F_4'(x)=p^2(D-x)\big[-\big(y^2+(D-x)^2+D^2\big)^{p/2-1}+(y^2+x^2)^{p/2-1}\big]\geqslant 0,$$

and hence

$$\begin{aligned}F_4(x)&\leqslant F_4(z)\\&=p\big[\big(y^2+(D-z)^2+D^2\big)^{p/2}-(y^2+z^2)^{p/2}\big]-p2^{p/2}(D^p-z^p).\end{aligned}$$

The right-hand side decreases as y increases. Therefore

$$F_4(z)\leqslant p\big[\big(z^2+(D-z)^2+D^2\big)^{p/2}-2^{p/2}D^p\big]\leqslant 0,$$

as $z^2+(D-z)^2+D^2\leqslant 2D^2$. ■

Now we reduce the inequality (3.2) to (2.8). Let

$$V_p(x,y,z) = p\big(y^p - 2^{p/2}(x\vee z)^p\big)$$

and

$$\text{(3.10)}\quad U_p(x,y,z) = p^2 x \int_x^{x\vee z} (y^2+t^2)^{p/2-1}dt + p(y^2+x^2)^{p/2} - p2^{p/2}(x\vee z)^p.$$

Now we see that (3.2) is equivalent to

$$\mathbb{E}V_p\big(f_n, T_n(f), f_n^*\big) \leqslant \mathbb{E}U_p(f_0,f_0,f_0),$$

which is (2.8). Hence we need to check (2.5), (2.6) and (2.7′).

The property (2.5) is a consequence of the identity

$$U_p(x,y,z) - V_p(x,y,z) = p[(y^2+x^2)^{p/2} - y^p] + p^2 x \int_x^{x\vee z} (y^2+t^2)^{p/2-1}dt.$$

The equation (2.6) follows directly from the definition of U_p. All that is left is to prove the last condition. We consider two cases.

1. The case $x+d \leqslant z$. Then (2.7′) reads

$$p\big(y^2+d^2+(x+d)^2\big)^{p/2} + p^2(x+d)\int_{x+d}^{z}(y^2+d^2+t^2)^{p/2-1}dt$$
$$\leqslant p(y^2+x^2)^{p/2} + p^2(x+d)\int_x^z (y^2+t^2)^{p/2-1}dt$$

or, in the equivalent form,

$$p\frac{\big(y^2+d^2+(x+d)^2\big)^{p/2} - (y^2+x^2)^{p/2}}{x+d}$$
$$- p^2\Big[\int_x^z (y^2+t^2)^{p/2-1}dt - \int_{x+d}^{z}(y^2+d^2+t^2)^{p/2-1}dt\Big] \leqslant 0.$$

Denote the left-hand side by $F(x)$ and observe that (3.5) is valid; this implies $F(x) \leqslant F\big((z-d)\wedge z\big)$. If $z-d<z$, then $F(z-d)\leqslant 0$, which follows from (3.6). If, conversely, $z \leqslant z-d$, then $F(z)\leqslant 0$, which is a consequence of (3.7) (with $Y = y^2+d^2$).

2. The case $x+d>z$. If $x+d \geqslant \sqrt{y^2+d^2}$, then (2.7′) takes the form

$$p[(y^2+x^2)^{p/2} - 2^{p/2}z^p] + p^2(x+d)\int_x^z (y^2+t^2)^{p/2-1}dt \geqslant 0.$$

The left-hand side is an increasing function of d, hence, if we fix all the other parameters, it suffices to show the inequality for the least d, which is determined by the condition $x + d = \sqrt{y^2 + d^2}$, that is, $d = (y^2 - x^2)/(2x)$; however, then the estimate is exactly (3.8). Finally, assume $x + d < \sqrt{y^2 + d^2}$. Then (2.7′) becomes

$$p\big(y^2 + d^2 + (x+d)^2\big)^{p/2} - p2^{p/2}(x+d)^p \\ \leqslant p(y^2 + x^2)^{p/2} + p^2(x+d)\int_x^z (y^2 + t^2)^{p/2-1}dt - p2^{p/2}z^p,$$

which is (3.9) with $D = x + d$.

4. SHARPNESS

Now we will prove that the constants C_p and $\sqrt{2}$ in (1.5) and (1.6) cannot be replaced by smaller ones. We will construct the appropriate examples on the probability space $\big([0,1], \mathcal{B}([0,1]), |\cdot|\big)$, a unit interval equipped with its Borel subsets and the Lebesgue measure. We will identify a set $A \in \mathcal{B}([0,1])$ with its indicator function.

4.1. Sharpness of (1.5). Fix $\varepsilon > 0$ and define f by

$$f_n = (1 + n\varepsilon)\big(0, (1+n\varepsilon)^{-1}\big], \qquad n = 0, 1, 2, \ldots$$

Then it is easy to check that f is a nonnegative martingale, $df_0 = (0, 1]$,

$$df_n = \varepsilon\big(0, (1+n\varepsilon)^{-1}\big] - \big(1 + (n-1)\varepsilon\big)\Big((1+n\varepsilon)^{-1}, \big(1 + (n-1)\varepsilon\big)^{-1}\Big]$$

for $n = 1, 2, \ldots$, and

$$S(f) = \sum_{n=0}^{\infty} \big(1 + n\varepsilon^2 + (1+n\varepsilon)^2\big)^{1/2} \Big(\big(1 + (n+1)\varepsilon\big)^{-1}, (1+n\varepsilon)^{-1}\Big].$$

Furthermore, for $p < 1$ we have $\|f\|_p = 1$ and, if $p \neq 0$,

$$\|S(f)\|_p^p = \varepsilon \sum_{n=0}^{\infty} \frac{\big(1 + n\varepsilon^2 + (1+n\varepsilon)^2\big)^{p/2}}{\big(1 + (n+1)\varepsilon\big)(1+n\varepsilon)},$$

which is a Riemann sum for C_p^p. Finally, the case $p = 0$ is dealt with by passing to the limit; this is straightforward, as the martingale f does not depend on p.

4.2. Sharpness of (1.6). Fix $M > 1$, an integer $N \geqslant 1$ and let $f = f^{(N,M)}$ be given by

$$f_n = M^n(0, M^{-n}], \;\; n = 0, 1, 2, \ldots, N, \qquad \text{and} \qquad f_N = f_{N+1} = f_{N+2} = \ldots$$

Then f is a nonnegative martingale,

$$f^* = M^N(0, M^{-N}] + \sum_{n=1}^{N} M^{n-1}(M^{-n}, N^{-n+1}],$$

$$df_0 = (0,1], \quad df_n = (M^n - M^{n-1})(0, M^{-n}] - M^{n-1}(M^n, M^{-n+1}],$$

for $n = 1, 2, \ldots, N$, and $df_n = 0$ for $n > N$. Hence the square function equals

$$\Big(1 + \sum_{k=1}^{N} (M^k - M^{k-1})^2\Big)^{1/2} = \Big(1 + \frac{M-1}{M+1}(M^{2N} - 1)\Big)^{1/2}$$

on the interval $(0, M^{-N}]$, and is given by

$$\Big(1 + \sum_{k=1}^{n-1} (M^k - M^{k-1})^2 + M^{2n-2}\Big)^{1/2}$$
$$= \Big(1 + \frac{M-1}{M+1}(M^{2n-2} - 1) + M^{2n-2}\Big)^{1/2}$$

on the set $(M^{-n}, M^{-n+1}]$ for $n = 1, 2, \ldots, N$.

Now, if $M \to \infty$, then $\|S(f)\|_1 \to 1 + \sqrt{2}N$ and $\|f\|_1 \to 1 + N$; therefore, for M and N sufficiently large, the ratio $\|S(f)\|_1 / \|f\|_1$ can be made arbitrarily close to $\sqrt{2}$. Similarly, for $p < 1$, $\|S(f)\|_p / \|f\|_p \to \sqrt{2}$ as $M \to \infty$ (here we may keep N fixed). Thus the constant $\sqrt{2}$ is the best possible.

5. ON AN ALTERNATIVE PROOF OF (1.6)

Let us present here (the sketch of) the direct proof of the inequality (1.6) in the case $p = 1$, without using the operators $\big(T_n(f)\big)$. As previously, it is based on a construction of the special function; here is a modification of Theorem 2.1 from [3] for the case of positive martingales.

THEOREM 5.1. *Suppose that U and V are functions from $(0, \infty)^3$ into $\mathbb{R}$ satisfying*

(5.1) $$V(x, y, z) \leqslant U(x, y, z),$$

(5.2) $$U(x, y, z) = U(x, y, x \vee z)$$

and the further condition that if $0 < x \leqslant z$ and d is a simple $\mathcal{F}$-measurable function with $\mathbb{E}d = 0$ and $\mathbb{P}(x + d > 0) = 1$, then

(5.3) $$\mathbb{E}U(x + d, \sqrt{y^2 + d^2}, z) \leqslant U(x, y, z).$$

Under these three conditions, we have

(5.4) $$\mathbb{E}V\big(f_n, S_n(f), f_n\big) \leqslant \mathbb{E}U(f_0, f_0, f_0)$$

for all nonnegative integers n and simple positive martingales f.

To show (1.6), take $V(x,y,z)=y-\sqrt{2}(x\vee z)$ and introduce the function

$$U(x,y,z)=\frac{1}{2\sqrt{2}}\frac{y^2-x^2-(x\vee z)^2}{x\vee z}.$$

These functions satisfy (5.1), (5.2) and (5.3). Indeed, the inequality (5.1) is equivalent to

$$\frac{\big(y-\sqrt{2}(x\vee z)\big)^2}{2\sqrt{2}(x\vee z)}\geqslant 0,$$

and the equation (5.2) follows immediately from the definition of U. The condition (5.3) is a consequence of the stronger estimate

$$U(x+d,\sqrt{y^2+d^2},z)\leqslant U(x,y,z)+U_x(x,y,z)d,$$

valid for $x,\,y,\,z>0$ and $d>-x$. The final observation is that $U(x,x,x)\leqslant 0$ for all positive x. By the theorem above and the approximation argument (leading from simple to general martingales), (1.6) follows. The proof is complete.

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