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WEAK-TYPE INEQUALITY FOR THE MARTINGALE SQUARE FUNCTION AND A RELATED CHARACTERIZATION OF HILBERT SPACES*

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Abstract. Let f be a martingale taking values in a Banach space $\mathcal{B}$ and let $S(f)$ be its square function. We show that if $\mathcal{B}$ is a Hilbert space, then

$$\mathbb{P}(S(f) \geqslant 1) \leqslant \sqrt{e}\|f\|_1$$

and the constant $\sqrt{e}$ is the best possible. This extends the result of Cox, who established this bound in the real case. Next, we show that this inequality characterizes Hilbert spaces in the following sense: if $\mathcal{B}$ is not a Hilbert space, then there is a martingale f for which the above weak-type estimate does not hold.

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, filtered by $(\mathcal{F}_n)_{n\geqslant 0}$, a non-decreasing sequence of sub-σ-fields of $\mathcal{F}$. Let $f = (f_n)_{n\geqslant 0}$ and $g = (g_n)_{n\geqslant 0}$ be adapted martingales taking values in a certain separable Banach space $(\mathcal{B}, \|\cdot\|)$. The difference sequences $df = (df_n)_{n\geqslant 0}$ and $dg = (dg_n)_{n\geqslant 0}$ of the martingales f and g are defined by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n \geqslant 1$, and similarly for dg_n. We say that g is a ± 1-*transform* of f if there is a deterministic sequence $\varepsilon = (\varepsilon_n)_{n\geqslant 0}$ of signs such that $dg_n = \varepsilon_n df_n$ for each n.

It is well-known that martingale inequalities reflect the geometry of Banach spaces in which the martingales take values: see e.g. [1]–[4] and [7]. We shall mention here only one fact, closely related to the result studied in the present paper. As proved by Burkholder in [2], if f takes values in a separable Hilbert space and

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g is its ± 1-transform, then

$$\mathbb{P}(\sup_n \|g_n\| \geqslant 1) \leqslant 2\|f\|_1 \tag{1.1}$$

and the constant 2 is the best possible (here, as usual, $\|f\|_1 = \sup_n \|f_n\|_1$). In fact, the implication can be reversed: if $\mathcal{B}$ is a separable Banach space with the property that (1.1) holds for any $\mathcal{B}$-valued martingale f and its ± 1-transform g, then $\mathcal{B}$ is a Hilbert space. For details, see Burkholder [2] and Lee [6].

In this paper we shall study a related problem and characterize the class of Hilbert spaces by another weak-type estimate. Let us introduce the *square function* of f by the formula

$$S(f) = \Big(\sum_{k=0}^{\infty} \|df_k\|^2\Big)^{1/2}.$$

We shall also use the notation

$$S_n(f) = \Big(\sum_{k=0}^{n} \|df_k\|^2\Big)^{1/2}$$

for the truncated square function, $n = 0, 1, 2, \ldots$ Suppose that $\mathcal{B}$ is a given and fixed separable Banach space and let $\beta(\mathcal{B})$ denote the least extended real number β such that, for any martingale f taking values in $\mathcal{B}$,

$$\mathbb{P}\big(S(f) \geqslant 1\big) \leqslant \beta(\mathcal{B})\|f\|_1.$$

Using the method of moments, Cox [5] showed that $\beta(\mathbb{R}) = \sqrt{e}$: consequently, $\beta(\mathcal{B}) \geqslant \sqrt{e}$ for any non-degenerate $\mathcal{B}$. We will extend this result to the following.

THEOREM 1.1. *We have $\beta(\mathcal{B}) = \sqrt{e}$ if and only if $\mathcal{B}$ is a Hilbert space.*

Let us sketch the proof. To show that for any martingale f taking values in a Hilbert space $(\mathcal{H}, |\cdot|)$ we have

$$\mathbb{P}\big(S(f) \geqslant 1\big) \leqslant \sqrt{e}\|f\|_1, \tag{1.2}$$

we may restrict ourselves to the class of simple martingales. Recall that f is *simple* if for any n the random variable f_n takes only a finite number of values and there is a deterministic N such that $f_N = f_{N+1} = f_{N+2} = \ldots$ We must prove that

$$\mathbb{E}V\big(f_n, S_n(f)\big) \leqslant 0, \quad n = 0, 1, 2, \ldots,$$

where $V(x, y) = 1_{\{y \geqslant 1\}} - \sqrt{e}|x|$ for $x \in \mathcal{H}$ and $y \in [0, \infty)$.

To do this, we will use Burkholder's method and construct a function U: $\mathcal{H} \times [0, \infty) \to \mathbb{R}$, which satisfies the following three conditions:

1° We have the majorization $U \geqslant V$.

2° For any $x \in \mathcal{H}$, $y \geqslant 0$ and any simple mean-zero random variable T taking values in $\mathcal{H}$ we have $\mathbb{E}U(x+T, \sqrt{y^2+|T|^2}) \leqslant U(x,y)$.

3° For any $x \in \mathcal{H}$ we have $U(x,|x|) \leqslant 0$.

Then (1.2) follows.

To see this, apply 2° conditionally on $\mathcal{F}_n$, with $x = f_n$, $y = S_n(f)$ and $T = df_{n+1}$. As the result, we obtain the inequality

$$\mathbb{E}\big[U\big(f_{n+1}, S_{n+1}(f)\big)|\mathcal{F}_n\big] \leqslant U\big(f_n, S_n(f)\big),$$

so, in other words, the process $\big(U\big(f_n, S_n(f)\big)\big)_{n\geqslant 0}$ is a supermartingale. Hence, by 1° and 3°,

$$\mathbb{E}V\big(f_n, S_n(f)\big) \leqslant \mathbb{E}U\big(f_n, S_n(f)\big) \leqslant \mathbb{E}U\big(f_0, S_0(f)\big) = \mathbb{E}U(f_0, |f_0|) \leqslant 0$$

and we are done.

The special function U is constructed and studied in the next section. In Section 3 we prove the remaining part of Theorem 1.1: we shall show that the validity of (1.2) for all $\mathcal{B}$-valued martingales implies the parallelogram identity.

2. A SPECIAL FUNCTION

Let $\mathcal{H}$ be a separable Hilbert space: in fact, we may and do assume that $\mathcal{H} = \ell^2$. The corresponding norm and scalar product will be denoted by $|\cdot|$ and $\cdot$, respectively. Introduce $U : \mathcal{H} \times [0,\infty) \to \mathbb{R}$ by the formula

$$U(x,y) = \begin{cases} 1-(1-y^2)^{1/2}\exp\big(|x|^2/[2(1-y^2)]\big) & \text{if } |x|^2+y^2<1, \\ 1-\sqrt{e}|x| & \text{if } |x|^2+y^2 \geqslant 1. \end{cases} \tag{2.1}$$

In the lemma below, we study the properties of U and V.

LEMMA 2.1. *The function U satisfies the conditions* 1°, 2° *and* 3°.

P r o o f. To show the majorization, we may assume that $|x|^2+y^2<1$. Then the inequality takes the form

$$\exp\bigg(\frac{|x|^2}{2(1-y^2)}\bigg) \leqslant \sqrt{e}\frac{|x|}{\sqrt{1-y^2}} + \frac{1}{\sqrt{1-y^2}}$$

and follows immediately from an elementary bound $\exp(s^2/2) \leqslant \sqrt{e}s+1$, $s \in [0,1]$, applied to $s = |x|/\sqrt{1-y^2}$. To check 2°, we introduce an auxiliary function

$$A(x,y) = \begin{cases} -x(1-y^2)^{-1/2}\exp\big(|x|^2/[2(1-y^2)]\big) & \text{if } |x|^2+y^2<1, \\ -\sqrt{e}x' & \text{if } |x|^2+y^2 \geqslant 1, \end{cases}$$

where $x' = x/|x|$ for $x \neq 0$, and $x' = 0$ otherwise. We shall establish a pointwise estimate

$$U(x+d, \sqrt{y^2+|d|^2}) \leqslant U(x,y) + A(x,y)\cdot d \tag{2.2}$$

for all x, $d \in \mathcal{H}$ and $y \geqslant 0$. Observe that this inequality immediately yields 2°, simply by putting $d = T$ and taking expectation of both sides.

To prove (2.2), note first that $U(x,y) \leqslant 1 - \sqrt{e}|x|$ for all $x \in \mathcal{H}$ and $y \geqslant 0$. This is trivial for $|x|^2 + y^2 \geqslant 1$, while for the remaining pairs (x,y) it can be transformed into the equivalent inequality:

$$\frac{|x|^2}{1-y^2} \leqslant \exp\left(\frac{|x|^2}{1-y^2} - 1\right),$$

which is obvious. Consequently, when $|x|^2 + y^2 \geqslant 1$, we have

$$\begin{aligned} U(x+d, \sqrt{y^2+|d|^2}) \leqslant 1 - \sqrt{e}|x+d| \leqslant 1 - \sqrt{e}|x| + A(x,y)\cdot d \\ = U(x,y) + A(x,y)\cdot d. \end{aligned}$$

Now suppose that $|x|^2 + y^2 < 1$ and $|x+d|^2 + y^2 + |d|^2 \leqslant 1$. Observe that for X, $D \in \mathcal{H}$ with $|D| < 1$ we have

$$\begin{aligned} \exp\left(\frac{|D|^2|X|^2 + 2X\cdot D + |D|^2}{1-|D|^2}\right) &\geqslant \exp\left(\frac{(X\cdot D)^2 + 2X\cdot D + |D|^2}{1-|D|^2}\right) \\ &\geqslant \frac{(X\cdot D)^2 + 2X\cdot D + |D|^2}{1-|D|^2} + 1 \\ &= \frac{(1+X\cdot D)^2}{1-|D|^2}. \end{aligned}$$

It suffices to plug $X = x/\sqrt{1-y^2}$ and $D = d/\sqrt{1-y^2}$ to obtain (2.2). Finally, if $|x|^2 + y^2 < 1 < |x+d|^2 + y^2 + |d|^2$, then substituting X and D as previously, we have $|X| < 1$, $|X+D|^2 + |D|^2 > 1$ and (2.2) can be written in the form

$$\exp\left(\frac{|X|^2-1}{2}\right)(1 + X\cdot D) \leqslant |X+D|,$$

or

$$\exp\left(\frac{|X|^2-1}{2}\right)\left(1 + \frac{|X+D|^2 - |X|^2 - |D|^2}{2}\right) \leqslant |X+D|.$$

Now we fix $|X|$, $|X+D|$ and maximize the left-hand side over D. Let us consider two cases. If $|X+D|^2 + (|X+D| - |X|)^2 < 1$, then there is $D' \in \mathcal{H}$ satisfying

$|X+D| = |X+D'|$ and $|X+D'|^2 + |D'|^2 = 1$. Consequently,

$$\exp\left(\frac{|X|^2-1}{2}\right)\left(1+\frac{|X+D|^2-|X|^2-|D|^2}{2}\right)$$
$$\leqslant \exp\left(\frac{|X|^2-1}{2}\right)\left(1+\frac{|X+D'|^2-|X|^2-|D'|^2}{2}\right) \leqslant |X+D'| = |X+D|.$$

Here the first passage is due to $|D'| < |D|$, while in the second we have applied (2.2) to $x = X, y = 0$ and $d = D'$ (for these $x,\ y$ and d we have already established the bound). Suppose, then, that $|X+D|^2 + (|X+D| - |X|)^2 \geqslant 1$. This inequality is equivalent to

$$|X+D| \geqslant \frac{1-|X|^2}{\sqrt{2-|X|^2}-|X|},$$

and hence

$$\exp\left(\frac{|X|^2-1}{2}\right)\left(1+\frac{|X+D|^2-|X|^2-|D|^2}{2}\right) - |X+D|$$
$$\leqslant \exp\left(\frac{|X|^2-1}{2}\right)\left(1+\frac{|X+D|^2-|X|^2-(|X+D|-|X|)^2}{2}\right) - |X+D|$$
$$= \exp\left(\frac{|X|^2-1}{2}\right)(1-|X|^2) + \left\{\exp\left(\frac{|X|^2-1}{2}\right)|X| - 1\right\}|X+D|$$
$$\leqslant \frac{1-|X|^2}{\sqrt{2-|X|^2}-|X|}\left[\exp\left(\frac{|X|^2-1}{2}\right)\sqrt{2-|X|^2}-1\right].$$

It suffices to observe that the expression in the square brackets is nonpositive, which follows from the estimate $\exp(1-|X|^2) \geqslant 2-|X|^2$. This completes the proof of 2°. Finally, 3° is a consequence of the inequality (2.2): $U(x,|x|) \leqslant U(0,0) + A(0,0)\cdot x = 0$. ∎

Thus, by the reasoning presented in the Introduction, the inequality (1.2) holds true. The constant $\sqrt{e}$ is optimal even in the real case: see Cox [5]. In fact, we shall reprove this in the next section: see Remark 3.1 below.

3. CHARACTERIZATION OF HILBERT SPACES

Let $(\mathcal{B}, \|\cdot\|)$ be a separable Banach space and recall the number $\beta(\mathcal{B})$ defined in the first section. Thus, for any $\mathcal{B}$-valued martingale f we have

$$\mathbb{P}\big(S(f) \geqslant 1\big) \leqslant \beta(\mathcal{B})\|f\|_1. \tag{3.1}$$

For $x \in \mathcal{B}$ and $y \geqslant 0$, let $M(x,y)$ denote the class of all simple martingales f given on the probability space $\big([0,1], \mathbb{B}(0,1), |\cdot|\big)$, such that f is $\mathcal{B}$-valued, $f_0 \equiv x$ and

$$y^2 - \|x\|^2 + S^2(f) \geqslant 1 \text{ almost surely.} \tag{3.2}$$

Here the filtration may vary. The key object in our further considerations is the function $U^0 : \mathcal{B} \times [0, \infty) \to \mathbb{R}$ given by

$$U^0(x, y) = \inf\{\mathbb{E}\|f_n\|\},$$

where the infimum is taken over all n and all $f \in M(x, y)$. We will prove that U^0 satisfies appropriate versions of the conditions 1°–3°.

LEMMA 3.1. *The function U^0 satisfies the following conditions*:

$1^{o\prime}$ *For any $x \in \mathcal{B}$ and $y \geqslant 0$ we have $U^0(x, y) \geqslant \|x\|$.*

$2^{o\prime}$ *For any $x \in \mathcal{B}$, $y \geqslant 0$ and any simple centered $\mathcal{B}$-valued random variable T,*

$$\mathbb{E}U^0(x + T, \sqrt{y^2 + \|T\|^2}) \geqslant U^0(x, y).$$

$3^{o\prime}$ *For any $x \in \mathcal{B}$ we have $U^0(x, \|x\|) \geqslant \beta(\mathcal{B})^{-1}$.*

P r o o f. The property $1^{o\prime}$ is obvious: when $f \in M(x, y)$, then it follows that $\|f_n\|_1 \geqslant \|f_0\|_1 = \|x\|$ for all n. To establish $2^{o\prime}$, we use a modification of the so-called "splicing argument": see e.g. [1]. Let T be as in the statement and let $\{x_1, x_2, \ldots, x_k\}$ be the set of its values: $\mathbb{P}(T = x_j) = p_j > 0$, $\sum_{j=1}^k p_j = 1$. For any $1 \leqslant j \leqslant k$, pick a martingale f^j from the class $M(x + x_j, \sqrt{y^2 + \|x_j\|^2})$. Let $a_0 = 0$ and $a_j = \sum_{\ell=1}^j p_\ell$, $j = 1, 2, \ldots, k$. Define a simple sequence f on $\big([0, 1], \mathbb{B}(0, 1), |\cdot|\big)$ by $f_0 \equiv x$ and

$$f_n(\omega) = f_{n-1}^j\big((\omega - a_{j-1})/(a_j - a_{j-1})\big), \qquad n \geqslant 1,$$

when $\omega \in (a_{j-1}, a_j]$. Then f is a martingale with respect to its natural filtration and, when $\omega \in (a_{j-1}, a_j]$,

$$\begin{aligned} &y^2 - \|x\|^2 + S^2(f)(\omega) \\ &\quad = y^2 + \|x_j\|^2 - \|x + x_j\|^2 + S^2(f^j)\big((\omega - a_{j-1})/(a_j - a_{j-1})\big) \geqslant 1, \end{aligned}$$

unless ω belongs to a set of measure zero. Therefore (3.2) holds, so by the definition of U^0 we get

$$\|f_n\|_1 \geqslant U^0(x, y).$$

However, the left-hand side equals

$$\sum_{j=1}^k \int_{a_{j-1}}^{a_j} |f_n(\omega)| d\omega = \sum_{j=1}^k p_j \int_0^1 |f_{n-1}^j(\omega)| d\omega,$$

which, by the proper choice of n and f^j, $j = 1, 2, \ldots, k$, can be made arbitrarily close to $\sum_{j=1}^k p_j U^0(x + x_j, \sqrt{y^2 + \|x_j\|^2}) = \mathbb{E}U^0(x + T, \sqrt{y^2 + \|T\|^2})$. This gives $2^{o\prime}$. Finally, the condition $3^{o\prime}$ follows immediately from (3.1) and the definition of U^0. ■

The further properties of U^0 are described in the next lemma.

LEMMA 3.2. (i) *The function U^0 satisfies the symmetry condition*

$$U^0(x,y) = U^0(-x,y)$$

for all $x \in \mathcal{B}$ and $y \geqslant 0$.

(ii) *The function U^0 has the homogeneity-type property*

$$U^0(x,y) = \sqrt{1-y^2}\,U^0\left(\frac{x}{\sqrt{1-y^2}},0\right)$$

for all $x \in \mathcal{B}$ and $y \in [0,1)$.

(iii) *If $z \in \mathcal{B}$ satisfies $\|z\| = 1$ and $0 \leqslant s < t \leqslant 1$, then*

$$U^0(sz,0) \leqslant U^0(tz,0)\exp\big((s^2-t^2)\|z\|^2/2\big). \tag{3.3}$$

P r o o f. (i) It is sufficient to use the equivalence $f \in M(x,y)$ if and only if $-f \in M(-x,y)$.

(ii) This follows immediately from the fact that $f \in M(x,y)$ if and only if $f/\sqrt{1-y^2} \in M(x/\sqrt{1-y^2},0)$.

(iii) Fix $x \in \mathcal{B}$ with $0 < \|x\| < 1$ and $\delta > 0$ such that $\|x+\delta x\| \leqslant 1$. Apply $2^{\text{o}\prime}$ to $y = 0$ and a centered random variable T which takes two values: δx and $-2x/(1+\|x\|^2)$. We get

$$\begin{aligned} U^0(x,0) \leqslant{} & \frac{\delta\|x\|(1+\|x\|^2)}{2\|x\|+\delta\|x\|(1+\|x\|^2)}U^0\left(-\frac{x(1-\|x\|^2)}{1+\|x\|^2},\frac{2\|x\|}{1+\|x\|^2}\right) \\ & + \frac{2\|x\|}{2\|x\|+\delta\|x\|(1+\|x\|^2)}U^0\left(x+\delta x,\delta\|x\|\right). \end{aligned}$$

By (i) and (ii), the first term on the right equals

$$\frac{\delta\|x\|(1-\|x\|^2)}{2\|x\|+\delta\|x\|(1+\|x\|^2)}U^0(x,0).$$

The second summand can be bounded from above by

$$\frac{2\|x\|}{2\|x\|+\delta\|x\|(1+\|x\|^2)}U^0\left(x+\delta x,0\right),$$

because $M(x+\delta x,0) \subset M(x+\delta x,\delta\|x\|)$. Plugging these two facts into the inequality above yields

$$\frac{U^0(x+\delta x,0)}{U^0(x,0)} \geqslant 1+\delta\|x\|^2. \tag{3.4}$$

This gives

$$\frac{U^0\big(x(1+k\delta),0\big)}{U^0\big(x\big(1+(k-1)\delta\big),0\big)} \geqslant 1+\delta\big(1+(k-1)\delta\big)\|x\|^2,$$

provided $\|x(1+k\delta)\| \leqslant 1$. Consequently, if N is an integer such that the condition $\|x(1+N\delta)\| \leqslant 1$ holds true, then

$$\frac{U^0\big(x(1+N\delta),0\big)}{U^0(x,0)} \geqslant \prod_{k=1}^{N}\Big(1+\delta\big(1+(k-1)\delta\big)\|x\|^2\Big). \tag{3.5}$$

Now we turn to (3.3). Assume first that $s>0$. Put $x=sz$, $\delta=(t/s-1)/N$ and let $N\to\infty$ in the inequality above to obtain

$$\frac{U^0(tz,0)}{U^0(sz,0)} \geqslant \exp\bigg(\frac{1}{2}\|z\|^2(t^2-s^2)\bigg),$$

which is the claim. Next, suppose that $s=0$. For any $0<s'<t$ we have, by $2^{\circ\prime}$,

$$\begin{aligned} U^0(0,0) &\leqslant \frac{1}{2}U^0(s'z,\|s'z\|)+\frac{1}{2}U^0(-s'z,\|s'z\|) \\ &= U^0(s'z,\|s'z\|) \leqslant U^0(s'z,0), \end{aligned}$$

where in the latter passage we have used the inclusion $M(s'z,0)\subset M(s'z,\|s'z\|)$. Thus,

$$\frac{U^0(tz,0)}{U^0(0,0)} \geqslant \frac{U^0(tz,0)}{U^0(s'z,0)} \geqslant \exp\Big(\frac{1}{2}\|z\|^2\big(t^2-(s')^2\big)\Big)$$

and it remains to let $s'\to 0$. ■

REMARK 3.1. *Suppose that $\mathcal{B}=\mathbb{R}$. It is easy to see that $U^0(1,0)\leqslant 1$: consider f starting from* 1 *and satisfying $\mathbb{P}(df_1=-1)=\mathbb{P}(df_1=1)=1/2$, $df_2=df_3\equiv\ldots\equiv 0$. Thus, by $3^{\circ\prime}$ and* (3.3), *we have*

$$\beta(\mathbb{R})^{-1} \leqslant U^0(0,0) \leqslant U^0(1,0)/\sqrt{e} \leqslant 1/\sqrt{e},$$

that is, $\beta(\mathbb{R})\geqslant\sqrt{e}$. This implies the sharpness of (1.2) *in the Hilbert-space-valued setting.*

Now we will work under the assumption $\beta(\mathcal{B})=\sqrt{e}$. Then we are able to derive the explicit formula for U^0.

LEMMA 3.3. *If $\beta(\mathcal{B})=\sqrt{e}$, then*

$$U^0(x,y)=\begin{cases}\sqrt{1-y^2}\exp\big(\|x\|^2/[2(1-y^2)]-\frac{1}{2}\big) & \text{if } \|x\|^2+y^2<1,\\ \|x\| & \text{if } \|x\|^2+y^2\geqslant 1.\end{cases}$$

P r o o f. First let us focus on the set $\{(x, y) : \|x\|^2 + y^2 \geqslant 1\}$. By $1^{\mathrm{o}\prime}$ we have $U^0(x, y) \geqslant \|x\|$. To get the reverse estimate, consider a martingale f such that $f_0 \equiv x$, df_1 takes values $-x$ and x, and $df_2 = df_3 \equiv \ldots \equiv 0$. Then $y^2 - \|x\|^2 + S^2(f) = y^2 + \|x\|^2 \geqslant 1$ (so $f \in M(x, y)$) and $\|f\|_1 = \|x\|$, which implies $U^0(x, y) \leqslant \|x\|$ by the definition of U^0. Now suppose that $\|x\|^2 + y^2 < 1$. Using the second and third part of the previous lemma, we may write

$$U^0(x,y) = \sqrt{1-y^2}\,U^0\left(\frac{x}{\sqrt{1-y^2}}, 0\right) \geqslant U^0(0,0)\sqrt{1-y^2}\exp\left(\frac{\|x\|^2}{2(1-y^2)}\right),$$

so, by $3^{\mathrm{o}\prime}$,

$$U^0(x,y) \geqslant \sqrt{1-y^2}\exp\left(\frac{\|x\|^2}{2(1-y^2)} - \frac{1}{2}\right).$$

To get the reverse bound, we use the homogeneity of U^0 and (3.3) again:

$$\begin{aligned} U^0(x,y) &= \sqrt{1-y^2}\,U^0\left(\frac{x}{\sqrt{1-y^2}}, 0\right) \\ &\leqslant \sqrt{1-y^2}\,U^0\left(\frac{x}{|x|}, 0\right)\exp\left(\frac{1}{2}\left(\frac{\|x\|^2}{1-y^2} - 1\right)\right) \\ &= \sqrt{1-y^2}\exp\left(\frac{\|x\|^2}{2(1-y^2)} - \frac{1}{2}\right), \end{aligned}$$

where in the last line we have used the equality $U^0(\overline{x}, 0) = \|\overline{x}\|$ valid for $\overline{x}$ of norm one (we have just established this in the first part of the proof). For completeness, let us mention here that if $x = 0$, then $x/|x|$ should be replaced above by any vector of norm one. ∎

LEMMA 3.4. *Suppose that* $\beta(\mathcal{B}) = \sqrt{e}$ *and let us assume that* $x, y \in \mathcal{B}$ *and* $\alpha > 0$ *satisfy* $\|x\| < 1$, $\|x + \alpha x + y\|^2 + \|\alpha x + y\|^2 < 1$ *and* $\|x + \alpha x - y\|^2 + \|\alpha x - y\|^2 < 1$. *Then*

$$(3.6)\quad \begin{aligned} 2 + 2\alpha\|x\|^2 \leqslant{}& \sqrt{1-\|\alpha x + y\|^2}\exp\left(\frac{\|x+\alpha x+y\|^2}{2(1-\|\alpha x+y\|^2)} - \frac{\|x\|^2}{2}\right) \\ &+ \sqrt{1-\|\alpha x - y\|^2}\exp\left(\frac{\|x+\alpha x-y\|^2}{2(1-\|\alpha x-y\|^2)} - \frac{\|x\|^2}{2}\right). \end{aligned}$$

P r o o f. Consider a random variable T such that

$$\mathbb{P}\left(T = -\frac{2x}{1+\|x\|^2}\right) = p, \quad \mathbb{P}(T = \alpha x + y) = \mathbb{P}(T = \alpha x - y) = \frac{1-p}{2},$$

where $p \in (0,1)$ is chosen so that $\mathbb{E}T = 0$. That is,

$$p = \frac{\alpha(1+\|x\|^2)}{2+\alpha(1+\|x\|^2)}.$$

By $2^{\mathrm{o}\prime}$, we have $U^0(x,0) \leqslant \mathbb{E}U^0(x+T, \|T\|)$. Since $\|x+T\|^2 + \|T\|^2 < 1$ almost surely, the previous lemma implies that this can be rewritten in the equivalent form:

$$\exp\left(\frac{\|x\|^2}{2}\right) \leqslant p\sqrt{1-\left(\frac{2\|x\|}{1+\|x\|^2}\right)^2}\exp\left(\frac{\big\|x\big((-1+\|x\|^2)/(1+\|x\|^2)\big)\big\|^2}{2\big(1-\big(2\|x\|/(1+\|x\|^2)\big)^2\big)}\right)$$
$$+\frac{1-p}{2}\sqrt{1-\|\alpha x+y\|^2}\exp\left(\frac{\|x+\alpha x+y\|^2}{2(1-\|\alpha x+y\|^2)}\right)$$
$$+\frac{1-p}{2}\sqrt{1-\|\alpha x-y\|^2}\exp\left(\frac{\|x+\alpha x-y\|^2}{2(1-\|\alpha x-y\|^2)}\right).$$

However, the first term on the right equals

$$\frac{\alpha(1-\|x\|^2)}{2+\alpha(1+\|x\|^2)}\exp\left(\frac{\|x\|^2}{2}\right)$$

and, in addition, $(1-p)/2 = \big(2+\alpha(1+\|x\|^2)\big)^{-1}$. Consequently, it suffices to multiply both sides of the inequality above by $\big(2+\alpha(1+\|x\|^2)\big)\exp(-\|x\|^2/2)$; the claim follows. ■

Now we are ready to complete the proof of Theorem 1.1. Suppose that a, b belong to the unit ball K of $\mathcal{B}$ and take $\varepsilon \in (0, 1/2)$. Applying (3.6) to $x = \varepsilon a$, $y = \varepsilon^2 b$ and $\alpha = \varepsilon$ gives

$$2+2\varepsilon^3\|a\|^2 \leqslant \sqrt{1-\varepsilon^4\|a+b\|^2}\exp\big(m(a,b)\big) + \sqrt{1-\varepsilon^4\|a-b\|^2}\exp\big(m(a,-b)\big), \tag{3.7}$$

where

$$m(a,b) = \frac{\varepsilon^2\|a+\varepsilon(a+b)\|^2}{2(1-\varepsilon^4\|a+b\|^2)} - \frac{\varepsilon^2\|a\|^2}{2}$$
$$= \frac{\varepsilon^2}{2}\big(\|a+\varepsilon(a+b)\|^2 - \|a\|^2\big) + \frac{\varepsilon^6\|a+\varepsilon(a+b)\|^2\|a+b\|^2}{2(1-\varepsilon^4\|a+b\|^2)}.$$

It is easy to see that there exists an absolute constant M_1 such that

$$\sup_{a,b\in K}|m(a,b)| \leqslant M_1\varepsilon^3.$$

Consequently, there is a universal $M_2 > 0$ such that if ε is sufficiently small, then

$$\exp\big(m(a,b)\big) \leqslant 1 + m(a,b) + m(a,b)^2$$
$$\leqslant 1 + \frac{\varepsilon^2}{2}\big(\|a+\varepsilon(a+b)\|^2 - \|a\|^2\big) + M_2\varepsilon^6$$

for any a, $b \in K$. Since $\sqrt{1-x} \leqslant 1 - x/2$ for $x \in (0,1)$, the inequality (3.7) implies

$$\begin{aligned}
&2 + 2\varepsilon^3\|a\|^2 \\
&\quad\leqslant\ (1-\varepsilon^4\|a+b\|^2/2)\left(1 + \frac{\varepsilon^2}{2}\big(\|a+\varepsilon(a+b)\|^2 - \|a\|^2\big) + M_2\varepsilon^6\right) \\
&\qquad + (1-\varepsilon^4\|a-b\|^2/2)\left(1 + \frac{\varepsilon^2}{2}\big(\|a+\varepsilon(a-b)\|^2 - \|a\|^2\big) + M_2\varepsilon^6\right).
\end{aligned}$$

This, after some manipulations, leads to

$$\begin{aligned}
&\|a+\varepsilon(a+b)\|^2 + \|a+\varepsilon(a-b)\|^2 - 2\|a(1+\varepsilon)\|^2 \\
&\qquad\qquad \geqslant \varepsilon^2(\|a+b\|^2 + \|a-b\|^2 - 2\|a\|^2) - 2\varepsilon^4 M_3,
\end{aligned}$$

where M_3 is a positive constant not depending on ε, a and b. Equivalently,

$$\begin{aligned}
&\left\|a + \frac{\varepsilon}{1+\varepsilon}b\right\|^2 + \left\|a - \frac{\varepsilon}{1+\varepsilon}b\right\|^2 - 2\|a\|^2 - 2\left\|\frac{\varepsilon}{1+\varepsilon}b\right\|^2 \\
&\quad \geqslant \frac{\varepsilon^2}{(1+\varepsilon)^2}(\|a+b\|^2 + \|a-b\|^2 - 2\|a\|^2 - 2\|b\|^2) - 2\frac{\varepsilon^4}{(1+\varepsilon)^2}M_3.
\end{aligned}$$

Next, let $c \in \mathcal{B}$, $\gamma > 0$ and substitute $a = \gamma c$; we assume that γ is small enough to ensure that $a \in K$. If we divide both sides by γ^2 and substitute $\delta = \varepsilon(1+\varepsilon)^{-1}\gamma^{-1}$, we obtain

$$\begin{aligned}
&\|c+\delta b\|^2 + \|c-\delta b\|^2 - 2\|c\|^2 - 2\|\delta b\|^2 \\
&\qquad \geqslant \delta^2(\|\gamma c+b\|^2 + \|\gamma c-b\|^2 - 2\|\gamma c\|^2 - 2\|b\|^2) - 2\varepsilon^2\delta^2 M_3 \\
&\qquad \geqslant \delta^2(\|\gamma c+b\|^2 + \|\gamma c-b\|^2 - 2\|\gamma c\|^2 - 2\|b\|^2) - 2\delta^4 M_3.
\end{aligned}$$

Let γ and ε go to 0 so that δ remains fixed. As the result, we infer that, for any $\delta > 0$, $b \in K$ and $c \in \mathcal{B}$,

$$\|c+\delta b\|^2 + \|c-\delta b\|^2 - 2\|c\|^2 - 2\|\delta b\|^2 \geqslant -2\delta^4 M_3. \tag{3.8}$$

Now, let N be a large positive integer and consider a symmetric random walk $(g_n)_{n\geqslant 0}$ over integers, starting from 0. Let $\tau = \inf\{n : |g_n| = N\}$. The inequality (3.8), applied to $\delta = N^{-1}$, implies that for any $a \in \mathcal{B}$ and $b \in K$ the process

$$(\xi_n)_{n\geqslant 0} = \left(\left\|a + \frac{bg_{\tau\wedge n}}{N}\right\|^2 - \left\{\frac{\|b\|^2}{N^2} - \frac{M_3}{N^4}\right\}(\tau\wedge n)\right)_{n\geqslant 0}$$

is a submartingale. Since $\mathbb{E}(\tau\wedge n) = \mathbb{E}g_{\tau\wedge n}^2$, we obtain

$$\mathbb{E}\left(\left\|a + \frac{bg_{\tau\wedge n}}{N}\right\|^2 - \left\{\frac{\|b\|^2}{N^2} - \frac{M_3}{N^4}\right\}g_{\tau\wedge n}^2\right) = \mathbb{E}\xi_n \geqslant \mathbb{E}\xi_0 = \|a\|^2.$$

Letting $n \to \infty$ and using Lebesgue's dominated convergence theorem gives

$$\frac{1}{2}(\|a+b\|^2 + \|a-b\|^2) - \|b\|^2 + \frac{M_3}{N^2} \geqslant \|a\|^2.$$

It suffices to let N go to ∞ to obtain

$$\|a+b\|^2 + \|a-b\|^2 \geqslant 2\|a\|^2 + 2\|b\|^2.$$

We have assumed that b belongs to the unit ball K, but, by homogeneity, the above estimate extends to any $b \in \mathcal{B}$. Putting $a+b$ and $a-b$ in the place of a and b, respectively, we obtain the reverse estimate

$$\|a+b\|^2 + \|a-b\|^2 \leqslant 2\|a\|^2 + 2\|b\|^2.$$

This implies that the parallelogram identity is satisfied, and hence $\mathcal{B}$ is a Hilbert space.

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