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THE ASYMPTOTICS OF L-STATISTICS FOR NON I.I.D. VARIABLES WITH HEAVY TAILS

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Abstract. The purpose of this paper is to study the asymptotic behaviour of linear combinations of order statistics (*L-statistics*)

$$L_n := \sum_{i=1}^{k_n} c_{i,n} X_{i:k_n}$$

with real scores $c_{i,n}$ for variables with heavy tails. The order statistics $X_{i:k_n}$ correspond to a non i.i.d. triangular array $(X_{i,n})_{1 \leqslant i \leqslant k_n}$ of infinitesimal and rowwise independent random variables. We give sufficient conditions for the convergence of L-statistics to non-normal limit laws and it is shown that only the extremes contribute to the limit distribution, whereas the middle parts vanish. As an example we consider the case, where the extremal partial sums belong to the domain of attraction of a stable law. We also study L-statistics with scores defined by $c_{i,n} := J\big(i/(n+1)\big)$ with a regularly varying function J, a case which has often been treated in the literature.

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1. INTRODUCTION

Let $X_{1,n}, \ldots, X_{k_n,n}$ denote an infinitesimal triangular array of rowwise independent, possibly not i.i.d. real random variables (r.v.s). Denote by $X_{1:k_n} \leqslant \ldots \leqslant X_{k_n:k_n}$ the corresponding order statistics of the n-th row for positive integers k_n such that $k_n \to \infty$ as $n \to \infty$. For real scores $c_{i,n}$ we will discuss the asymptotic behaviour of *L-statistics*

$$L_n := \sum_{i=1}^{k_n} c_{i,n} X_{i:k_n}. \tag{1.1}$$

Important examples are, amongst others, empirical counterparts of general nonparametric L-functionals; see Van der Vaart [25], sec. 22.1, as well as Witting and Müller-Funk [29], ex. 7.145. Moreover, L-statistics are used for testing statistical hypotheses, see e.g. Mitra and Anis [16] or Tchirina [24].

For special situations, central limit theorems for L-statistics are well studied in the literature. In particular, the case of i.i.d. r.v.s has been treated by Shorack [20], Stigler [23], Helmers and Ruymgaart [7] and in the book of Shorack and Wellner [22]. Their work was continued by Mason and Shorack [15], Viharos [26]–[28] as well as Li et al. [12]. Under special assumptions Mason and Shorack [14] and [15], sec. 2, already obtained non-normal limit laws for L_n in the i.i.d. case. A key condition in most of these papers is the regular variation of a score function J defining the scores $c_{i,n} = J\big(i/(n+1)\big)$.

Central limit theorems for L_n under independent but not identically distributed random variables can be found in Shorack [21], Stigler [23] and Ruymgaart and van Zuijlen [19]. Here Shorack [21] (see also Shorack and Wellner [22], p. 821 ff.) used reduced empirical processes in order to handle non i.i.d. variables, whereas Stigler [23], sec. 4, used Hájek's projection lemma combined with variance and covariance arguments. These papers studied r.v.s with light tails leading to normal limit laws.

In the more general case of not necessarily identically distributed r.v.s, Janssen [9] found sufficient conditions for convergence of extreme sums of triangular arrays with distributional convergent partial sums to infinitely divisible laws. This corresponds to the case where $c_{i,n} \in \{0, 1\}$ holds in various settings. He has shown that the extreme order statistics have an important influence on the asymptotics of partial sums, whenever non-normal limits occur. Functional limit laws of this kind are published in [10].

In this paper we first prove tightness of well-centered L-statistics. We give a complete overview of the spectrum of their weak accumulation points without normal parts. Moreover, we give sufficient conditions for the scores $c_{i,n}$, so that only the extreme parts of the L-statistics contribute to the limit law. Roughly speaking, the upper and lower extremes substancially contribute to the limit law of L_n which is represented by an infinite sum. This series representation is linked to the series representation in an earlier work [9] for infinitely divisible laws without normal components. As examples, we study extremal partial sums which belong to the domain of attraction of a stable law and exemplify the limit behaviour of L-statistics with scores $c_{i,n} = J\big(i/(n+1)\big)$ given by a regularly varying function J. Section 2 states the main results and all proofs are given in Section 3.

2. MAIN RESULTS

It will turn out that the concept of shift-compactness is substantial for this paper; see, for instance, Le Cam [11]. Let us shortly recall its definition.

DEFINITION 2.1. (i) We call a sequence of real r.v.s $(Y_n)_{n\in\mathbb{N}}$ *shift-compact* iff there exists a real sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that $(Y_n - a_n)_{n\in\mathbb{N}}$ is tight, i.e. their distributions are relatively compact with respect to the topology of weak convergence.

(ii) A triangular array $(X_{i,n})_{1\leqslant i\leqslant k_n}$ of real r.v.s is called *infinitesimal* iff $\max_{1\leqslant i\leqslant k_n} P(|X_{i,n}|>\varepsilon)\to 0$ holds for each $\varepsilon>0$ as $n\to\infty$.

The following crucial lemma motivates further investigations of L-statistics. Here we use the notation $Y^+ := Y\vee 0 := \max(Y,0)$ and $Y^- := -Y\vee 0$ to denote the positive and negative parts, respectively, of the real r.v. $Y = Y^+ - Y^-$. In addition, let $X^+_{i:k_n} := (X_{i:k_n})^+$ which is order preserving. Also define $Y^{*-} := \min(Y,0) = -(Y^-)$ with $Y = Y^+ + Y^{*-}$. This procedure is also order preserving with

$$X^{*-}_{i:k_n} := (X_{i:k_n})^{*-} \quad \text{and} \quad X_{i:k_n} = X^+_{i:k_n} + X^{*-}_{i:k_n}.$$

Observe that also

$$|X_{i:k_n}| = X^+_{i:k_n} - X^{*-}_{i:k_n}.$$

LEMMA 2.1. *Suppose that $(X_{i,n})_{1\leqslant i\leqslant k_n}$ is an infinitesimal triangular array with shift-compact partial sums $\sum_{i=1}^{k_n} X_{i,n}$.*

(i) *Let $(c_{i,n})_{1\leqslant i\leqslant k_n}$ be uniformly bounded coefficients with $|c_{i,n}|\leqslant k$ for all $i\leqslant k_n$ and $n\in\mathbb{N}$. Then the sequence of L-statistics $L_n=\sum_{i=1}^{k_n} c_{i,n}X_{i:k_n}$ is shift-compact. Moreover, the same result holds for $\sum_{i=1}^{k_n} c_{i,n}\,|X_{i:k_n}|$, $\sum_{i=1}^{k_n} c_{i,n}X^+_{i:k_n}$ and $\sum_{i=1}^{k_n} c_{i,n}X^{*-}_{i:k_n}$.*

(ii) *Suppose that each $X_{i,n}$ has finite variance and that, in addition, $c_{i,n}\geqslant \delta>0$ holds for some δ and all $i\leqslant k_n$ and $n\in\mathbb{N}$. If $\limsup_{n\to\infty}\operatorname{Var}(L_n)<\infty$ holds, then, conversely, the partial sums $\sum_{i=1}^{k_n} X_{i,n}$ are shift-compact.*

All proofs are presented in Section 3.

For a real r.v. Y and truncation points $\pm\tau, \tau>0$, we denote by

$$\mathbb{E}_\tau(Y) := \mathbb{E}\big(Y\mathbf{1}_{(-\tau,\tau)}(Y)\big)$$

the expectation of the truncated variable.

In view of Lemma 2.1 (i) it is interesting to know which type of limit laws of L-statistics given by (1.1) may occur.

2.1. Cases without normal part. We will see that new classes of limit distributions may occur if the limit distribution of the partial sums $\sum_{i=1}^{k_n} X_{i,n}$ does not have a normal part.

We first recall some well-known results about infinitesimal triangular arrays $(X_{i,n})_{1\leqslant i\leqslant n}$ of rowwise independent, real-valued r.v.s; see Gnedenko and Kolmogorov [5] or Petrov [17]. Suppose that we have the convergence in distribution of partial sums to some infinitely divisible r.v. Y:

$$\sum_{i=1}^{n} X_{i,n} - a_n \xrightarrow{\mathcal{D}} Y, \tag{2.1}$$

where (and throughout the paper) $\xrightarrow{\mathcal{D}}$ means convergence in distribution. Here the centering constants a_n can be substituted by truncated expectations in the limit.

Then there exists a shift α such that the convergence $\sum_{i=1}^{k_n}[X_{i,n} - \mathbb{E}_\tau(X_{i,n})] \xrightarrow{\mathcal{D}} Y + \alpha$ holds for all continuity points $\tau \neq 0$ of the Lévy measure η of Y. Now the Lévy–Khinchin formula implies that the law $\mu := \mathcal{L}(Y + \alpha)$ has the characteristic function

$$\widehat{\mu}(t) := \exp\bigg(- \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus 0} \big[\big(\exp(iut) - 1 - iut \big) \mathbf{1}_{(-\tau,\tau)}(u) \\ + \big(\exp(iut) - 1 \big) \mathbf{1}_{(-\tau,\tau)^c}(u) \big] d\eta(u) \bigg),$$

where η denotes a Lévy measure. It is also convenient to write

$$\mu = \mu_1 * N(0,\sigma^2) * \mu_2 \tag{2.2}$$

with negative and positive Poisson parts μ_1 and μ_2, respectively; see Janssen [9], p. 1767. Here $*$ indicates convolution of distributions. Defining $\eta_1 := \eta_{|(-\infty,0)}$, we will analyse the quantile function $\psi_1 : (0,\infty) \to (-\infty, 0]$ of the Lévy measure η_1 given by

$$\psi_1(y) := \inf\{t : \eta_1(-\infty, t] \geqslant y\} \wedge 0. \tag{2.3}$$

In the same way, we will also study the right-continuous inverse function $\psi_2 : (0,\infty) \to [0,\infty)$ of $\eta_2 := \eta_{|(0,\infty)}$, given by $\psi_2(y) := \sup\{t : \eta_2[t,\infty) \geqslant y\} \vee 0$.

Assume that $(Y_n)_{n\in\mathbb{N}}$ and $(\widetilde{Y}_n)_{n\in\mathbb{N}}$ are two mutually independent sequences of i.i.d. standard exponentially distributed r.v.s with partial sums $S_n := \sum_{i=1}^n Y_i$ and $\widetilde{S}_n := \sum_{i=1}^n \widetilde{Y}_i$, respectively. Under (2.1) we have the convergence in distribution of the r lowest and upper extremes:

$$(X_{1:k_n}, \ldots, X_{r:k_n}, X_{k_n:k_n}, \ldots, X_{k_n+1-r:k_n}) \\ \xrightarrow{\mathcal{D}} \big(\psi_1(S_1), \ldots, \psi_1(S_r), \psi_2(\widetilde{S}_1), \ldots, \psi_2(\widetilde{S}_r)\big) \tag{2.4}$$

as $n \to \infty$. The proof follows from a Poisson limit law for multinomial distributions. An early proof for the convergence of the lower extremes is due to Loève [13]; see also Janssen [9]. A modern approach establishes the proof via the convergence of the point process of extremes $\sum_{i=1}^r \varepsilon_{X_{i:k_n}} + \sum_{i=1}^r \varepsilon_{X_{k_n+1-i:k_n}}$ to the Poisson point process $\sum_{i=1}^r \varepsilon_{\psi_1(S_i)} + \sum_{i=1}^r \varepsilon_{\psi_2(\widetilde{S}_i)}$; see Resnick [18], p. 222 ff.

Now we introduce the infinite sums Γ_1 and Γ_2 given by $|c_i| \leqslant k,\ |d_j| \leqslant k$ for all i, j:

$$\Gamma_1 := \sum_{i=1}^{\infty} c_i \big[\psi_1(S_i) - \mathbb{E}_\tau\big(\psi_1(S_i)\big)\big], \quad \Gamma_2 := \sum_{j=1}^{\infty} d_j \big[\psi_2(\widetilde{S}_j) - \mathbb{E}_\tau\big(\psi_2(\widetilde{S}_j)\big)\big], \tag{2.5}$$

where again $\mathbb{E}_\tau(\cdot)$ denotes truncated expectations. The existence of the independent r.v.s Γ_1 and Γ_2 is shown in the following lemma.

LEMMA 2.2. *The infinite sums Γ_1 and Γ_2 converge almost surely (a.s.).*

REMARK 2.1. *Lemma* 2.2 *extends the known series representation for infinitely divisible laws without Gaussian components* (see Janssen [9], sec. 4). *Via integration by parts it is related to an equivalent integral representation given by Poisson processes* (see [9], Remark 4.2). *In a special case a related integral representation of* Γ_1 *and* Γ_2 *as a limit of L-statistics is also known* (cf. Mason and Shorack [15], sec. 2). *We will see that the extremes given in* (2.4) *contribute by a one-to-one correspondence to the series* (2.5), *which is a nice interpretation of the limit behaviour of L-statistics.*

In contrast to earlier work we do not assume that the coefficients $c_{i,n}$ are generated by a score generating function J. Except for Example 2.2 assume throughout that $|c_{i,n}| \leqslant k$ uniformly holds in i and n.

Motivated by Lemma 2.1 (i), our aim is to study accumulation points of our L-statistics (1.1). By setting $c_{i,n} = 0$ for $i > k_n$ and $i < 0$ we obtain two sequences $x_n := (c_{i,n})_{i\in\mathbb{N}}$ and $y_n := (c_{k_n-j+1,n})_{j\in\mathbb{N}}$ in the compact Polish space $[-k,k]^{\mathbb{N}}$. Turning to subsequences we may assume without loss of generality the convergence of $c_{i,n} \to c_i$ and $c_{n-j+1,n} \to d_j$ as $n \to \infty$ for all $i, j \in \mathbb{N}$. In this subsection we will see that the well-centered L-statistics converge in distribution if the limit distribution μ in (2.2) is not trivial and has no Gaussian part. This is made precise in the following theorem.

THEOREM 2.1. *Let* $(X_{i,n})_{1\leqslant i\leqslant k_n}$ *be an infinitesimal array of rowwise independent r.v.s satisfying* (2.1) *for a real sequence* $(a_n)_{n\in\mathbb{N}}$ *and a non-trivial r.v.* Y *without Gaussian part*, *i.e.* $\sigma^2 = 0, \eta \neq 0$. *Suppose that the upper and lower scores converge to some limit scores*, *i.e. we have*

$$c_{i,n} \underset{n\to\infty}{\longrightarrow} c_i, \quad c_{n+1-j} \underset{n\to\infty}{\longrightarrow} d_j \quad \textit{for all } i, j. \tag{2.6}$$

Then the corresponding L-statistics converge in distribution:

$$\sum_{i=1}^{k_n} c_{i,n}[X_{i:k_n} - \mathbb{E}_\tau(X_{i:k_n})] \xrightarrow{\mathcal{D}} \Gamma_1 + \Gamma_2 \quad \textit{as } n \to \infty. \tag{2.7}$$

We also obtain the following two corollaries.

COROLLARY 2.1. *Under the assumptions of Theorem* 2.1 *we have the convergence in distribution of the positive and negative parts*:

$$L_n^+ := \sum_{j=1}^{k_n} c_{k_n+1-j,n}[X^+_{k_n+1-j:k_n} - \mathbb{E}_\tau(X^+_{j:k_n})] \xrightarrow{\mathcal{D}} \Gamma_2,$$

$$L_n^- := -\sum_{i=1}^{k_n} c_{i,n}[X^{*-}_{i:k_n} - \mathbb{E}_\tau(X^{*-}_{i:k_n})] \xrightarrow{\mathcal{D}} \Gamma_1$$

as $n \to \infty$. *Moreover*, L_n^+ *and* L_n^- *are asymptotically independent.*

COROLLARY 2.2. *Suppose that the assumptions of Theorem* 2.1 *are fulfiled and define middle parts*

$$M_n := \sum_{i=r_n}^{k_n-s_n} c_{i,n}[X_{i:k_n} - \mathbb{E}_\tau(X_{i:k_n})] \quad \text{for } 1 \leqslant r_n \leqslant k_n - s_n.$$

If $\min(r_n, s_n) \to \infty$, *then* $M_n \to 0$ *in probability as* $n \to \infty$.

REMARK 2.2. *Let* $Y_{i,n} = |X_{i,n}|$. *Then the shift-compactness of* $\sum_{i=1}^{k_n} X_{i,n}$ *also implies the shift-compactness of* $\sum_{i=1}^{k_n} Y_{i,n}$ *and* $\sum_{i=1}^{k_n} c_{i,n} Y_{i:k_n}$.

As an example we study the case of a non-normal stable limit distribution.

EXAMPLE 2.1. Suppose that (2.6) holds and let $Y_1, Y_2, \ldots$ denote a sequence of i.i.d. r.v.s such that

$$b_n^{-1} \sum_{i=1}^{n} Y_i - a_n \xrightarrow{\mathcal{D}} Y$$

converges to some non-degenerate r.v. Y. Then it is well-known that Y is a stable r.v. with index $0 < \alpha \leqslant 2$; see Feller [4], p. 165f. Here $\alpha = 2$ corresponds to the normal case (central limit theorem) with $\eta = 0$. For $0 < \alpha < 2$ and $x > 0$ we have $\eta_1\big((-\infty, -x]\big) = k_1 x^{-\alpha}$ and $\eta_2\big([x, \infty)\big) = k_2 x^{-\alpha}$ for constants $k_i \geqslant 0$, $i = 1, 2$, $\max(k_1, k_2) > 0$ and $\sigma^2 = 0$. It is easy to see that $\psi_1(u) = -k_1^{1/\alpha} u^{-1/\alpha}$ and $\psi_2(u) = k_2^{1/\alpha} u^{-1/\alpha}$ hold true. Thus we obtain the convergence in distribution

$$b_n^{-1} \sum_{i=1}^{n} c_{i,n} \big[Y_{i:n} - \mathbb{E}\big(Y_{i:n} \mathbf{1}_{(-b_n\tau, b_n\tau)}(Y_{i:n})\big)\big] \xrightarrow{\mathcal{D}} \Gamma_1 + \Gamma_2,$$

where

$$\Gamma_1 := -k_1^{1/\alpha} \sum_{i=1}^{\infty} c_i [S_i^{-1/\alpha} - \mathbb{E}_{\tau k_1^{-1/\alpha}}(S_i^{-1/\alpha})],$$

$$\Gamma_2 := k_2^{1/\alpha} \sum_{j=1}^{\infty} d_j [\widetilde{S}_j^{-1/\alpha} - \mathbb{E}_{\tau k_2^{-1/\alpha}}(\widetilde{S}_j^{-1/\alpha})].$$

In addition, for $0 < \alpha < 1$ it can be shown that the convergence holds without centering constants, i.e. we have

$$b_n^{-1} \sum_{i=1}^{n} c_{i,n} Y_{i:n} \xrightarrow{\mathcal{D}} -k_1^{1/\alpha} \sum_{i=1}^{\infty} c_i S_i^{-1/\alpha} + k_2^{1/\alpha} \sum_{j=1}^{\infty} d_j \widetilde{S}_j^{-1/\alpha}.$$

For $c_{i,n} = 1$ this corresponds to Lemma 4.1 (c) of Janssen [9]. Furthermore, these results extend earlier considerations about extremal partial sums for r.v.s which belong to the domain of attraction of a stable law; see Janssen [8] and the references therein. The middle part asymptotically vanishes (see Corollary 2.2).

2.2. Cases with normal part. In this subsection we treat the case where the normal part in (2.2) does not vanish, i.e. the law of the limit r.v. Y satisfies the condition $\mathcal{L}(Y+\alpha) = \mu_1 * N(0,\sigma^2) * \mu_2$ with $\sigma^2 \geqslant 0$. In this situation, the scores $c_{n,i}$ must fulfil stronger conditions than in Theorem 2.1 to obtain (2.7).

THEOREM 2.2. *Let* $(X_{i,n})_{1\leqslant i\leqslant k_n}$ *be an infinitesimal array of rowwise independent r.v.s satisfying* (2.1) *for a real sequence* $(a_n)_{n\in\mathbb{N}}$ *and a non-degenerate r.v.* Y. *In addition, assume that the scores* $c_{i,n}$ *fulfil* (2.6) *and*

$$\max_{b_n\leqslant i\leqslant k_n-b_n}(c_{i,n}) \underset{n\to\infty}{\longrightarrow} 0 \tag{2.8}$$

for all sequences $b_n \in \mathbb{N}$ *with* $b_n \leqslant k_n/2$ *and* $b_n \to \infty$. *Then* (2.7) *holds true.*

As application of the above theorem we study the limit behaviour of L-statistics with scores $c_{i,n} = f\big(i/(n+1)\big)$ given by a regularly varying function f which may be unbounded. Thus a new normalisation is necessary.

EXAMPLE 2.2. Let $f:(0,1)\to\mathbb{R}^+$ be a function regularly varying at 0 with index $-\alpha, \alpha>0$, and regularly varying at 1 with index α. So we have $f(x) = x^{-\alpha}l_1(x)$ as $x\downarrow 0$ and $f(1-x) = x^{-\alpha}l_2(x)$ as $x\downarrow 0$, where $l_1,\ l_2$ are slowly varying at zero. Suppose further that we have the convergence

$$\frac{f\big(1-1/(n+1)\big)}{f\big(1/(n+1)\big)} \underset{n\to\infty}{\longrightarrow} k$$

for some $k\geqslant 0$ and that f is bounded on every compact subinterval of $(0,1)$, i.e. for all $\varepsilon>0$ there exists $k_\varepsilon \geqslant 0$ with $\sup_{\varepsilon\leqslant x\leqslant 1-\varepsilon}|f(x)| \leqslant k_\varepsilon$.

Then the convergence in distribution

$$\frac{1}{f\big(1/(n+1)\big)}\sum_{i=1}^{n} f\left(\frac{i}{n+1}\right)[X_{i:n} - \mathbb{E}_\tau(X_{i:n})] \xrightarrow{\mathcal{D}} \Gamma_1 + \Gamma_2$$

holds as $n\to\infty$. Here the limit r.v.s on the right-hand side are as in (2.5) with scores $c_i = i^{-\alpha}$ and $d_j = kj^{-\alpha}$.

3. THE PROOFS

The following lemma is used as a main tool in Janssen [9]. For i.i.d. variables it is due to Bickel [1]. The proof follows from Hájek [6], Lemma 3.1.

LEMMA 3.1. *Suppose that each variable* $X_{i,n}$ *of the rowwise independent array* $(X_{i,n})_{1\leqslant i\leqslant k_n}$ *has finite variance. Then the corresponding order statistics are non-negatively correlated, i.e.* $\mathrm{Cov}(X_{i:k_n}, X_{j:k_n}) \geqslant 0$ *holds for all* $i, j \leqslant k_n$.

The proof of Lemma 2.1 is splitted into two parts. The extremes have to be dealt with separately by a truncation argument, whereas central parts of the sums require the following variance arguments.

LEMMA 3.2. *Let* $(X_{i,n})_{1\leqslant i\leqslant k_n}$ *be as in Lemma* 3.1.

(i) *If* $|c_{i,n}|\leqslant k$ *holds for all* $i\leqslant k_n$, *then the following inequality holds true:*

$$\operatorname{Var}\Big(\sum_{i=1}^{k_n} c_{i,n}X_{i:k_n}\Big)\leqslant k^2\operatorname{Var}\Big(\sum_{i=1}^{k_n} X_{i,n}\Big). \tag{3.1}$$

(ii) *If* $c_{i,n}\geqslant\delta>0$, *we have*

$$\operatorname{Var}\Big(\sum_{i=1}^{k_n} X_{i,n}\Big)\leqslant \frac{1}{\delta^2}\operatorname{Var}\Big(\sum_{i=1}^{k_n} c_{i,n}X_{i:k_n}\Big). \tag{3.2}$$

Proof. By Lemma 3.1 we have

$$\begin{aligned}\operatorname{Var}\Big(\sum_{i=1}^{k_n} c_{i,n}X_{i:k_n}\Big) &= \sum_{i=1}^{k_n}\sum_{j=1}^{k_n} c_{i,n}c_{j,n}\ \operatorname{Cov}(X_{i:k_n},X_{j:k_n})\\ &\leqslant k^2\sum_{i=1}^{k_n}\sum_{j=1}^{k_n}\operatorname{Cov}(X_{i:k_n},X_{j:k_n})\\ &= k^2\operatorname{Var}\Big(\sum_{i=1}^{k_n} X_{i:k_n}\Big)=k^2\operatorname{Var}\Big(\sum_{i=1}^{k_n} X_{i,n}\Big).\end{aligned}$$

This proves the first part. Under the conditions of the second part the same reasoning yields

$$\operatorname{Var}\Big(\sum_{i=1}^{k_n} c_{i,n}X_{i:k_n}\Big)\geqslant\delta^2\operatorname{Var}\Big(\sum_{i=1}^{k_n} X_{i,n}\Big)$$

and the proof is completed. ■

REMARK 3.1. *It is easy to see that the shift-compactness of two sequences* $(R_n)_{n\in\mathbb{N}}$ *and* $(S_n)_{n\in\mathbb{N}}$ *implies the shift-compactness of the sum* $(R_n+S_n)_{n\in\mathbb{N}}$.

First we prove the existence of Γ_1 and Γ_2.

Proof of Lemma 2.2. We will give the proof for ψ_1 and $0\leqslant c_i\leqslant 1$. The general result follows by linearity. Assume first that $\eta_1\big((-\infty,-\tau]\big)=0$ holds. The quantile transformation of the Lévy measure implies

$$\int_0^\infty \psi_1(u)du=\int_{-\infty}^0 x^2 d\eta(x)<\infty. \tag{3.3}$$

Now we introduce the Poisson process $N(\cdot)$ and the process $R(\cdot)$ defined by

$$N(t):=\sum_{i=1}^\infty \mathbf{1}_{[0,t]}(S_i),\qquad R(t):=\sum_{i=1}^\infty c_i\mathbf{1}_{[0,t]}(S_i),\qquad 0\leqslant t. \tag{3.4}$$

Direct arguments show that $V(t) := \mathrm{Var}\,\big(R(t)\big) \leqslant \mathrm{Var}\,\big(N(t)\big) = t$, since

$$\mathrm{Cov}\,\big(\mathbf{1}_{[0,t]}(S_i), \mathbf{1}_{[0,t]}(S_j)\big) \geqslant 0 \qquad \text{for all } i, j \in \mathbb{N}$$

follows by a slight generalisation of Lemma 3.1. Consider the process $M(t) := R(t) - \mathbb{E}\big(R(t)\big)$ which is adapted to the filtration $\mathcal{F}_t = \sigma\big(N(s) : s \leqslant t\big)$. Since $M(t)$ has independent increments, it is easy to see that it is also a martingale with respect to $\mathcal{F}_t$. Next we introduce stopping times $\tau_n := \inf\{t : N(t) \geqslant n\}$. The optional stopping theorem yields that $M_n := M(\tau_n)$ is also a martingale with respect to the same filtration. We notice that

$$n \mapsto \sum_{i=1}^{n} c_i\big[\psi_1(S_i) - \mathbb{E}\big(\psi_1(S_i)\big)\big] = \int_0^{\tau_n} \psi_1 dM =: \alpha_n$$

is a further martingale. Observe that

$$\mathrm{Var}\,\Big(\int_0^t \psi_1 dM\Big) = \int_0^t \psi_1(u)^2 dV(u) \leqslant \int_0^t \psi_1^2(u)du \leqslant \int_{-\infty}^{0} x^2 d\eta(x)$$

holds by (3.3) for each $t \geqslant 0$. Thus $(\alpha_n)_{n\in\mathbb{N}}$ is a martingale with uniformly bounded second moment. The martingale convergence theorem gives us the a.s. convergence of $(\alpha_n)_{n\in\mathbb{N}}$ and the convergence in L^2 in this special case.

Let now η_1 be an arbitrary Lévy measure on $(-\infty, 0)$. Define

$$\tilde{\eta} := \eta_{1|(-\tau,0)} + \delta\varepsilon_{-\tau} \qquad \text{for } \delta := \eta_1\big((-\infty, -\tau]\big).$$

Its quantile function is given by

$$\tilde{\psi} = \psi_1(u)\mathbf{1}_{(\delta,\infty)}(u) - \tau\mathbf{1}_{(0,\delta]}(u) \qquad \text{for } u \geqslant 0.$$

The first part of our proof implies a.s. convergence of the infinite sum

$$\begin{aligned}\sum_{i=1}^{\infty} & c_i\big[\tilde{\psi}(S_i) - E_\tau\big(\tilde{\psi}(S_i)\big)\big] \\ &= \sum_{i=1}^{\infty} c_i\big[\psi_1(S_i)\mathbf{1}_{(\delta,\infty)}(S_i) - E_\tau\big(\psi_1(S_i)\big)\big] - \tau\sum_{i=1}^{\infty} c_i\mathbf{1}_{(0,\delta]}(S_i),\end{aligned}$$

since $\psi_1(u) \leqslant -\tau$ holds for $u \leqslant \delta$. Notice that both infinite sums $\sum_{i=1}^{\infty} c_i\mathbf{1}_{(0,\delta]}(S_i)$ and $\sum_{i=1}^{\infty} c_i\psi_1(S_i)\mathbf{1}_{(0,\delta]}(S_i)$ are a.s. convergent, since $S_i \to \infty$ a.s. as $i \to \infty$. A combination of these arguments proves the result in its general form. ■

Before we prove Lemma 2.1 we state the following helpful result.

LEMMA 3.3. *Let $(X_{i,n})_{1\leqslant i\leqslant n}$ be an infinitesimal triangular array of rowwise independent, real r.v.s that fulfil* (2.1) *and suppose that* (2.6) *holds. If $-\delta < 0$ is a continuity point of the Lévy measure of Y, then the convergence*

$$\sum_{i=1}^{k_n} c_{i,n} X_{i:k_n} \mathbf{1}_{(-\infty,-\delta]}(X_{i:k_n}) \xrightarrow{\mathcal{D}} \sum_{i=1}^{\infty} c_i \psi_1(S_i) \mathbf{1}_{(-\infty,-\delta]}\big(\psi_1(S_i)\big) \tag{3.5}$$

holds as $n \to \infty$.

Proof. Observe first that, by (2.4) and the continuous mapping theorem, the convergence (3.5) holds for a finite number of weighted extremes. Moreover, for each $\varepsilon > 0$ there exists some $j \in \mathbb{N}$ with $P\big(\psi_1(S_j) \leqslant -\delta\big) \leqslant \varepsilon$. Now the portmanteau theorem together with (2.4) imply that $\limsup_{n\to\infty} P(X_{j:k_n} \leqslant -\delta) \leqslant \varepsilon$. An application of Theorem 4.2 of Billingsley [2] gives the result. ■

Proof of Lemma 2.1. Part (ii) is a direct consequence of Lemma 3.2 (ii) and the Chebyshev inequality. The first part requires different arguments.

First, we assume that $X_{i,n} \leqslant 0$ holds for $1 \leqslant i \leqslant k_n$. Let us remark that Prohorov's theorem implies that the sequence of L-statistics is shift-compact iff for each subsequence there exists a further subsequence such that the adequate shifted L-statistics converge in distribution along this subsequence.

Let now $\sum_{i=1}^{k_n} X_{i,n}$ be shift-compact. Turning to subsequences we assume without loss of generality that

$$\sum_{i=1}^{k_n} X_{i,n} - a_n \xrightarrow{\mathcal{D}} Y \tag{3.6}$$

is convergent in distribution to some infinitely divisible r.v. Y. Let η be the Lévy measure of the Lévy–Khinchin representation of the characteristic function of Y. The conditions for the convergence of (3.6) given by Gnedenko and Kolmogorov [5], p. 116 ff., state that for every continuity point ε with $\eta(\{\varepsilon\}) = 0$ the following assertions hold:

$$\sum_{i=1}^{k_n} P(X_{i,n} \leqslant \varepsilon) \underset{n\to\infty}{\longrightarrow} \eta\big((-\infty, -\varepsilon]\big), \tag{3.7}$$

$$\sup_{n\in\mathbb{N}} \operatorname{Var}\Big(\sum_{i=1}^{k_n} X_{i,n} \mathbf{1}_{(-\varepsilon,\varepsilon)}(X_{i,n})\Big) < \infty. \tag{3.8}$$

As a direct consequence of (3.7) and (3.8), Poisson's limit law for independent Bernoulli variables implies the convergence in distribution

$$N_n := \sum_{i=1}^{k_n} \mathbf{1}_{(-\infty,-\varepsilon]}(X_{i,n}) \xrightarrow{\mathcal{D}} R, \tag{3.9}$$

where R is a Poisson distributed r.v. with parameter $\lambda := \eta\big((-\infty, -\varepsilon]\big)$. Hence $\mathrm{Var}(N_n) \to \lambda$ holds as $n \to \infty$.

To apply Remark 3.1, we use an order-preserving truncation argument which allows us to treat middle and extreme parts separately. Let us define

(3.10)
$$\varphi_\varepsilon(x) := x\mathbf{1}_{(-\varepsilon,0]}(x) - \varepsilon\mathbf{1}_{(-\infty,-\varepsilon]}(x), \qquad \varrho_\varepsilon(x) := \big(x - \varphi_\varepsilon(x)\big)\mathbf{1}_{(-\infty,0)}(x).$$

Then the non-positive r.v.s $X_{i,n}$ can be splitted into two parts:

$$X_{i,n} = Z_{i,n} + \widetilde{Z}_{i,n}, \qquad Z_{i,n} := \varphi_\varepsilon(X_{i,n}),\ \widetilde{Z}_{i,n} := \varrho_\varepsilon(X_{i,n}). \tag{3.11}$$

If we turn to the order statistics of the $Z_{i,n}$ and $\widetilde{Z}_{i,n}$, we see that the equalities

$$Z_{i:k_n} = \varphi_\varepsilon(X_{i:k_n}), \qquad \widetilde{Z}_{i:k_n} = \varrho_\varepsilon(X_{i:k_n}) \qquad \text{for } 1 \leqslant i \leqslant k_n$$

imply $X_{i:k_n} = Z_{i:k_n} + \widetilde{Z}_{i:k_n}$, $1 \leqslant i \leqslant k_n$. Next we apply the inequality $\mathrm{Var}(S+T) \leqslant 3\big(\mathrm{Var}(S) + \mathrm{Var}(T)\big)$ to $Z_{i,n} = X_{i,n}\mathbf{1}_{(-\varepsilon,0]}(X_{i,n}) - \varepsilon\mathbf{1}_{(-\infty,-\varepsilon]}(X_{i,n})$. Thus (3.2), (3.8) and (3.9) imply $\sup_{n\in\mathbb{N}} \mathrm{Var}\left(\sum_{i=1}^{k_n} Z_{i,n}\right) < \infty$. By Lemma 3.2 we obtain $\sup_{n\in\mathbb{N}} \mathrm{Var}\left(\sum_{i=1}^{k_n} c_{i,n} Z_{i:k_n}\right) < \infty$. Hence $\sum_{i=1}^{k_n} c_{i,n} Z_{i:k_n}$ is shift-compact.

By Remark 3.1 it remains to prove that $\sum_{i=1}^{k_n} c_{i,n}\varrho_\varepsilon(X_{i:k_n})$ is shift-compact. Observe that we can assume without loss of generality that (2.6) holds as $(c_{i,n})_{n\in\mathbb{N}}$ is a sequence in the compact set $[-k,k]^{\mathbb{N}}$. Now, for each $t > 0$ we have

$$\sum_{i=1}^{k_n} P\big(\varrho_\varepsilon(X_{i,n}) \leqslant -t\big) \leqslant \sum_{i=1}^{k_n} P\left(X_{i,n} + \varepsilon \leqslant -t\right) \to \eta\big((-\infty, -t-\varepsilon]\big) < \infty.$$

By turning to subsequences, we infer from Theorem 1 (Section 25) of Gnedenko and Kolmogorov [5] that the sequence $\sum_{i=1}^{k_n} \varrho_\varepsilon(X_{i,n})$, $n \in \mathbb{N}$, converges to a compound Poisson limit law and is therefore shift-compact. Applying Lemma 3.3 for $\varepsilon = \delta$ yields the desired shift-compactness of $\sum_{i=1}^{k_n} \varrho_\varepsilon(X_{i:k_n})$.

For the general case consider the decomposition

$$X_{i,n} = X_{i,n}^{+} + X_{i,n}^{*-}. \tag{3.12}$$

By Lemma 3.3 and (3.8), the inequality $\mathrm{Var}(|Y|) \leqslant \mathrm{Var}(Y)$ and shift-compactness of $\sum_{i=1}^{k_n} X_{i,n}$ yield shift-compactness of $\sum_{i=1}^{k_n} |X_{i,n}|$. This implies the shift-compactness of $\sum_{i=1}^{k_n} X_{i,n}^{+}$ and $\sum_{i=1}^{k_n} X_{i,n}^{*-}$ by linearity. By Remark 3.1 the decomposition (3.12) completes the proof. ∎

P r o o f o f T h e o r e m 2.1. Suppose that, in addition to our general assumptions, the condition (2.6) holds for our scores. It is sufficient to give the proof for the negative parts L_n^- (see Corollary 2.1). The positive part L_n^+ can be treated similarly and (2.4) will imply the asymptotic independence of positive and negative parts. Since our L-statistics are shift-compact, we only have to identify the set of possible weak accumulation points of our sequence (which always exists along subsequences). Moreover, we can assume without loss of generality that $0 \leqslant c_{i,n} \leqslant 1$ holds. We will now use Theorem 4.2 of Billingsley [2] and an obvious extension; see Janssen [9], Appendix, Lemma B. In order to split L_n^- into two parts, we fix some continuity point $-\delta$ of η_1, $-\tau < -\delta < 0$. For each $k \in \mathbb{N}$ we have the convergence

$$\mathbb{E}\big(\sum_{i=1}^{k} c_{i,n} X_{i:k_n} \mathbf{1}_{(-\tau,-\delta]}(X_{i:k_n})\big) \to \mathbb{E}\Big(\sum_{i=1}^{k} c_i \psi_1(S_i) \mathbf{1}_{(-\tau,-\delta]}\big(\psi_1(S_i)\big)\Big).$$

By the monotone convergence theorem we have

$$\mathbb{E}\Big(\sum_{i=1}^{k} c_i \psi_1(S_i) \mathbf{1}_{(-\tau,-\delta]}\big(\psi_1(S_i)\big)\Big) \to \mathbb{E}\Big(\sum_{i=1}^{\infty} c_i \psi_1(S_i) \mathbf{1}_{(-\tau,-\delta]}\big(\psi_1(S_i)\big)\Big)$$

as $k \to \infty$. By the arguments given in Lemma 5.2 of Janssen [9] and label (5.15) therein, the upper tail can be made small for sufficiently large k, i.e.

$$\limsup_{n\to\infty} \big|\mathbb{E}\big(\sum_{i=k+1}^{k_n} c_{i,n} X_{i:k_n} \mathbf{1}_{(-\tau,-\delta]}(X_{i:k_n})\big)\big| \leqslant \limsup_{n\to\infty} \tau \sum_{i=k+1}^{k_n} P(X_{i:k_n} \leqslant -\delta) \underset{k\to\infty}{\longrightarrow} 0.$$

Thus the convergence

$$\mathbb{E}\big(\sum_{i=1}^{k_n} c_{i,n} X_{i:k_n} \mathbf{1}_{(-\tau,-\delta]}(X_{i:k_n})\big) \to \mathbb{E}\Big(\sum_{i=1}^{\infty} c_i \psi_1(S_i) \mathbf{1}_{(-\tau,-\delta]}\big(\psi_1(S_i)\big)\Big)$$

follows. This together with Lemma 3.3 shows the convergence in distribution

$$\sum_{i=1}^{k} c_{i,n} \big[X_{i:k_n} \mathbf{1}_{(-\infty,-\delta]}(X_{i:k_n}) - \mathbb{E}_\tau\big(X_{i:k_n} \mathbf{1}_{(-\infty,-\delta]}(X_{i:k_n})\big)\big] \xrightarrow{\mathcal{D}} L_\delta^-,$$

where

$$L_\delta^- := \sum_{i=1}^{\infty} c_i \Big[\psi_1(S_i) \mathbf{1}_{(-\infty,-\delta]}\big(\psi_1(S_i)\big) - \mathbb{E}_\tau\Big(\psi_1(S_i) \mathbf{1}_{(-\infty,-\delta]}\big(\psi_1(S_i)\big)\Big)\Big].$$

Moreover, for $\delta \downarrow 0$

$$\limsup_{n\to\infty} \operatorname{Var}\big(\sum_{i=1}^{k_n} c_{i,n} X_{i:k_n} \mathbf{1}_{(-\delta,0)}(X_{i:k_n})\big) \leqslant \limsup_{n\to\infty} k^2 \operatorname{Var}\big(\sum_{i=1}^{k_n} X_{i,n} \mathbf{1}_{(-\delta,0)}(X_{i,n})\big)$$

can be made arbitrarily small. Observe also that $L_\delta^- \xrightarrow{\mathcal{D}} \Gamma_1$ holds for $\delta \downarrow 0$, which can be shown in the same way as in the proof of Lemma 2.2. Furthermore, as in that proof, we may assume that the Lévy measure η_1 is supported on $(-\tau, 0)$. Then $L_\delta^- = M_\delta$ is an L_2-convergent martingale as $\delta \downarrow 0$. Again the same truncation method as in the proof of Lemma 2.2 can be applied. These arguments complete the proof for the convergence $L_n^- \xrightarrow{\mathcal{D}} \Gamma_1$. ■

Proof of Corollary 2.1. Confer the proof of Theorem 2.1. ■

Proof of Corollary 2.2. Define new coefficients

$$c'_{i,n} = c_{i,n}\mathbf{1}_{[r_n, k_n - s_n]}(i).$$

Then we have $c_i = d_j = 0$ for each i and j of our limit scores. Hence $\Gamma_1 + \Gamma_2 = 0$ holds by Theorem 2.1. ■

Proof of Theorem 2.2. As in the proofs of Lemma 2.1 and Theorem 2.1 we may restrict ourselves to non-positive variables $X_{i,n} \leqslant 0$. For $\tau = \varepsilon$ we may use the decomposition (3.11). It is easy to see that the variables $Z_{i,n} := \varrho_\tau(X_{i,n})$ correspond to a case with compound Poisson limit distribution for the partial sums and we can apply the above results. Thus we just have to treat the case $-\tau < X_{i,n} \leqslant 0$ as in the proof of Lemma 2.1.

Observe that by Theorem 1 (Section 25) of Gnedenko and Kolmogorov [5] we have

$$\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \operatorname{Var}\Big(\sum_{i=1}^{k_n} X_{i,n}\mathbf{1}_{(-\varepsilon,\varepsilon)}(X_{i,n})\Big) = \sigma^2.$$

Hence we may assume that $\limsup_{n\to\infty} \operatorname{Var}\big(\sum_{i=1}^{k_n} X_{i,n}\big) < \infty$. Otherwise, τ can be decreased. By using (2.4), (2.6) and standard arguments we find a sequence $r_n \leqslant k_n/2$, $r_n \to \infty$, such that

$$\sum_{i=1}^{r_n} c_{i,n}[X_{i:k_n} - \mathbb{E}(X_{i:k_n})] \xrightarrow{\mathcal{D}} \Gamma_1 \text{ and } \sum_{i=k_n+1-r_n}^{k_n} c_{i,n}[X_{i:k_n} - \mathbb{E}(X_{i:k_n})] \xrightarrow{\mathcal{D}} 0$$

in distribution as $n \to \infty$. Moreover, our assumptions and Lemma 3.1 imply

$$\operatorname{Var}\Big(\sum_{i=r_n+1}^{k_n - r_n} c_{i,n}X_{i:k_n}\Big) \leqslant \max_{r_n+1 \leqslant i \leqslant k_n - r_n} (c_{i,n})^2 \operatorname{Var}\Big(\sum_{i=1}^{k_n} X_{i,n}\Big) \to 0$$

as $n \to \infty$. Thus the middle part asymptotically vanishes, which completes the proof. ■

Proof of Example 2.2. We show that the scores

$$c_{i,n} := \frac{f\big(i/(n+1)\big)}{f\big(1/(n+1)\big)}, \quad 1 \leqslant i \leqslant k_n,$$

fulfil the conditions of Theorem 2.2. By the regular variation of f it is easy to see that (2.6) holds with given scores $c_i = i^{-\alpha}$ and $d_j = kj^{-\alpha}$. Hence we only have to prove (2.8). Suppose first that $\liminf_{n\to\infty} b_n/n = \kappa > 0$ holds. Then, for sufficiently large n, we can find an $\varepsilon > 0$ such that

$$\max_{b_n \leqslant i \leqslant n-b_n} |c_{i,n}| \leqslant \max_{\varepsilon n \leqslant i \leqslant (1-\varepsilon)n} |c_{i,n}| \leqslant k_\varepsilon\, f\left(\frac{1}{n+1}\right)^{-1} \underset{n\to\infty}{\longrightarrow} 0,$$

which implies (2.8). Thus it remains to study the case when $b_n/n \to 0$. As above, by the boundedness of f we infer that $\max_{\varepsilon n \leqslant i \leqslant (1-\varepsilon)n} |c_{i,n}| \to 0$ holds for every $0 < \varepsilon < 1$ as $n \to \infty$. Moreover, since f is regularly varying at 0, we can find a δ with $0 < \delta < \alpha$ and some $\varepsilon > 0$ such that the inequality $x^{-\alpha-\delta} \leqslant f(x) \leqslant x^{-\alpha+\delta}$ holds for all $0 < x \leqslant \varepsilon$. Hence, for sufficiently large n we have

$$|c_{i,n}| = \frac{f\big(i/(n+1)\big)}{f\big(1/(n+1)\big)} \leqslant \frac{\big(i/(n+1)\big)^{-\alpha+\delta}}{\big(1/(n+1)\big)^{-\alpha-\delta}} \leqslant i^{-\alpha+\delta}\left(\frac{1}{n+1}\right)^{2\delta} \leqslant i^{-\alpha+\delta},$$

which implies

$$\max_{b_n \leqslant i \leqslant \varepsilon n} |c_{i,n}| \leqslant b_n^{-\alpha+\delta} \to 0 \quad \text{as } n \to \infty.$$

In the same way, by the regular variation of f at 1 we obtain the convergence $\max_{(1-\varepsilon)n \leqslant i \leqslant n-b_n} |c_{i,n}| \to 0$. This completes the proof. ■

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