PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 32, Fasc. 1 (2012), pp. 167–186

SELFSIMILAR PROCESSES WITH STATIONARY INCREMENTS IN THE SECOND WIENER CHAOS

BY

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Abstract. We study selfsimilar processes with stationary increments in the second Wiener chaos. We show that, in contrast with the first Wiener chaos which contains only one such process (the fractional Brownian motion), there is an infinity of selfsimilar processes with stationary increments living in the Wiener chaos of order two. We prove some limit theorems which provide a mechanism to construct such processes.

2000 AMS Mathematics Subject Classification: Primary: 60F05; Secondary: 60H05, 91G70.

Key words and phrases: Selfsimilar processes, stationary increments, second Wiener chaos, limit theorems, multiple stochastic integrals, weak convergence.

1. INTRODUCTION

The selfsimilar processes with stationary increments have been widely studied. Let $H > 0$. A stochastic process $Y = (Y_t)_{t\geqslant 0}$ is H-*selfsimilar* if for any $c > 0$ the processes $(Y_{ct})_{t\geqslant 0}$ and $(c^H Y_t)_{t\geqslant 0}$ have the same finite-dimensional distributions. Here H is called the *selfsimilarity parameter* of Y. The process $(Y_t)_{t\geqslant 0}$ has *stationary increments* if $(Y_t)_{t\geqslant 0}$ and $(Y_{t+h} - Y_h)_{t\geqslant 0}$ have the same finite-dimensional distributions for every $h > 0$. Let $H \in (0, 1]$. All H-selfsimilar processes with stationary increments and with finite variances have the same covariance function given by

$$R(t,s) = \frac{C}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \quad \text{for all } s,t \geqslant 0,$$

where C is the second moment of the process at time one. We refer to the monographs [4] and [10] for a complete exposition on selfsimilar processes. Since the

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Gaussian processes are characterized by their covariance, there is only one Gaussian selfsimilar process with stationary increments (and with unit variance at time one). This is the fractional Brownian motion. The Gaussian processes live in the first Wiener chaos, that is, they can basically be expressed as single integrals, with a deterministic integrand, with respect to the Wiener process.

The purpose of this paper is to discuss selfsimilar processes in the second Wiener chaos. The elements of the second Wiener chaos are double iterated stochastic integrals with respect to the Wiener process. The law of such processes is not given anymore by their covariance function, therefore the fact that two selfsimilar processes with stationary increments in the second Wiener chaos have the same covariance does not imply the equivalence of finite-dimensional distribution of these processes. It is then expected to have more than one selfsimilar process in the second Wiener chaos. We will actually show that there exists an infinity of such processes.

This paper is organized as follows. In Section 2 we study the so-called non-symmetric Rosenblatt process, which depends on two parameters, and by suitable choosing these parameters, we obtain an infinity of selfsimilar processes with stationary increments in the second Wiener chaos. The analysis of the laws of these processes is based on the cumulants and this is done in Section 3. Sections 4 and 5 contain the proofs of some non-central limit theorems in which selfsimilar processes with stationary increments appear as limits. Our results extend those from [2], [3] or [11]. We finish our paper with some thoughts about how many selfsimilar processes with stationary increments are in the second Wiener chaos and how they can be obtained as limits in non-central-type limit theorems.

2. A CLASS OF SELFSIMILAR PROCESSES WITH STATIONARY INCREMENTS IN THE SECOND WIENER CHAOS

The purpose of this section is to discuss a particular class of selfsimilar processes with stationary increments living in the second Wiener chaos. This class contains an infinite number of elements and all of them have different finite-dimensional distributions. We introduce our set as follows. Let $H_1, H_2 \in (0,1)$ be such that

$$H_1 + H_2 > 1. \tag{2.1}$$

Consider the stochastic process $Y^{H_1,H_2} = (Y_t^{H_1,H_2})_{t\geqslant 0}$ given by, for every $t \geqslant 0$,

$$Y_t^{H_1,H_2} = c(H_1,H_2) \int_{\mathbb{R}^2} \Big(\int_0^t (u-y_1)_+^{H_1/2-1} (u-y_2)_+^{H_2/2-1} du \Big) dB_{y_1} dB_{y_2}, \tag{2.2}$$

where the integral above is a multiple Wiener–Itô stochastic integral of order two. We refer to [8] for the definition and the basic properties of multiple Wiener–Itô integrals.

Let us denote by f_t the kernel of $Y_t^{H_1,H_2}$, that is,

$$f_t(y_1,y_2) = c(H_1,H_2)\int_0^t (u-y_1)_+^{H_1/2-1}(u-y_2)_+^{H_2/2-1}du \tag{2.3}$$

for every $y_1, y_2 \in \mathbb{R}$. The kernel f_t is in general not symmetric with respect to the variables y_1, y_2 (except the case $H_1 = H_2$). We denote by $\tilde{f}_t$ its symmetrization:

$$\tilde{f}_t(y_1,y_2) = \frac{1}{2}\big(f_t(y_1,y_2) + f_t(y_2,y_1)\big).$$

In this way, using the usual notation for multiple integrals, we can write $Y_t^{H_1,H_2} = I_2(f_t)$ for every $t \geqslant 0$. The condition (2.1) assures that the kernel f_t belongs to $L^2\big([0,\infty)^2\big)$ for every t (this can be seen in the sequel of this section), and thus the double integral in (2.2) is well defined.

The constant $c(H_1,H_2)$ will be chosen such that $\mathbf{E}[Y_1^2] = 1$. This constant plays actually an important role in our paper. It will be explicitly calculated later.

PROPOSITION 2.1. *Assume* (2.1) *is satisfied. Then the process* $(Y_t^{H_1,H_2})_{t\geqslant 0}$ *is* $\frac{1}{2}(H_1+H_2)$ *selfsimilar and has stationary increments.*

P r o o f. Let $c > 0$. We have

$$\begin{aligned} Y_{ct}^{H_1,H_2} &= c(H_1,H_2)\int_{\mathbb{R}^2}\Big(\int_0^{ct}(u-y_1)_+^{H_1/2-1}(u-y_2)_+^{H_2/2-1}du\Big)dB_{y_1}dB_{y_2} \\ &= c(H_1,H_2)c\int_{\mathbb{R}^2}\Big(\int_0^{t}(cu-y_1)_+^{H_1/2-1}(cu-y_2)_+^{H_2/2-1}du\Big)dB_{y_1}dB_{y_2} \\ &= c(H_1,H_2)c\int_{\mathbb{R}^2}\Big(\int_0^{t}(cu-cy_1)_+^{H_1/2-1}(cu-cy_2)_+^{H_2/2-1}du\Big)dB_{cy_1}dB_{cy_2} \\ &\overset{\mathrm{d}}{=} c^{(H_1+H_2)/2}Y_t, \end{aligned}$$

where we have used the $\frac{1}{2}$-selfsimilarity of the Wiener process B. Here $\overset{\mathrm{d}}{=}$ means equivalence of all finite-dimensional distributions. It is obvious that the process $(Y_t^{H_1,H_2})$ has stationary increments since for every $h > 0$ and $t \geqslant 0$ we have $(Y_{t+h}^{H_1,H_2} - Y_h^{H_1,H_2}) \overset{\mathrm{d}}{=} (Y_t^{H_1,H_2})$. ∎

REMARK 2.1. *The particular case $H_1 = H_2 = H$ corresponds to the Rosenblatt process as defined in* [3] *and* [11]. *In our paper we will call this process the* symmetric Rosenblatt process. *The process Y^{H_1,H_2} with $H_1 \neq H_2$ will be called a* non-symmetric Rosenblatt process. *Also note that the selfsimilar parameter of Y^{H_1,H_2} is always contained in the interval* $\big(\frac{1}{2},1\big)$.

We will need the following lemma throughout the paper.

LEMMA 2.1. *Let $v < u$ and $H_1, H_2 \in (0,1)$. Then*

$$\int_{-\infty}^{v} (u-y_1)^{H_1/2-1}(v-y_1)^{H_2/2-1}dy_1 = \beta\left(1-\frac{H_1+H_2}{2}, \frac{H_2}{2}\right)(u-v)^{(H_1+H_2)/2-1},$$

where $\beta(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$ denotes the beta function with parameters $a,b>0$. Therefore, for every $u,v>0$

$$\int_{-\infty}^{u\wedge v} (u-y_1)^{H_1/2-1}(v-y_1)^{H_2/2-1}dy_1 = \beta\left(1-\frac{H_1+H_2}{2}, \frac{H_2}{2}\right)(u-v)_+^{(H_1+H_2)/2-1} + \beta\left(1-\frac{H_1+H_2}{2}, \frac{H_1}{2}\right)(u-v)_-^{(H_1+H_2)/2-1}.$$

Proof. This follows by making the change of variables $z=(v-y_1)/(u-y_1)$ with $dy_1=(v-u)(1-z)^{-2}dz$ and from the fact that $(-x)_+ = x_-$. ■

REMARK 2.2. *Using the well-known properties of the beta and gamma functions* (*recall that* $\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}dx$ *for* $a>0$), *i.e.*,

$$\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \textit{and} \quad \Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin(\pi a)},$$

we can give a variant of the above lemma:

$$\begin{aligned}\int_{-\infty}^{u\wedge v} &(u-y_1)^{H_1/2-1}(v-y_1)^{H_2/2-1}dy_1 \\ &= \Gamma\left(1-\frac{H_1+H_2}{2}\right) \\ &\times \left[\Gamma\left(\frac{H_1}{2}\right)\Gamma\left(1-\frac{H_2}{2}\right)^{-1} + \Gamma\left(\frac{H_2}{2}\right)\Gamma\left(1-\frac{H_1}{2}\right)^{-1}\right]|u-v|^{H-1} \\ &= \Gamma\left(1-\frac{H_1+H_2}{2}\right)\Gamma\left(\frac{H_1}{2}\right)\Gamma\left(\frac{H_1}{2}\right)|u-v|^{H-1}a(H_1,H_2),\end{aligned}$$

where $a(H_1,H_2)=\sin(\pi H_2/2)$ if $v<u$ and $a(H_1,H_2)=\sin(\pi H_1/2)$ if $u<v$. Or, otherwise,

$$\int_{-\infty}^{u\wedge v} (u-y_1)^{H_1/2-1}(v-y_1)^{H_2/2-1}dy_1 = \Gamma\left(1-\frac{H_1+H_2}{2}\right)\Gamma\left(\frac{H_1}{2}\right)\Gamma\left(\frac{H_1}{2}\right) \times |u-v|^{H-1}\big(\sin(\pi H_1/2)+\sin(\pi H_2/2)\big).$$

We will now compute the renormalizing constant appearing in (2.2).

LEMMA 2.2. *Assume that* $H_1, H_2 \in (0,1)$ *and that* (2.1) *is satisfied. The normalizing constant* $c(H_1,H_2)$ *appearing in the definition of* Y^{H_1,H_2} *in* (2.2) *is given by the formula*

$$c(H_1,H_2)^{-2} = \frac{1}{H(2H-1)}$$
$$\times \left[\beta\left(1-H_1,\frac{H_1}{2}\right)\beta\left(1-H_2,\frac{H_2}{2}\right)+\beta\left(1-H,\frac{H_1}{2}\right)\beta\left(1-H,\frac{H_2}{2}\right)\right],$$

where $2H = H_1 + H_2$.

P r o o f. Since $Y_t^{H_1,H_2} = I_2(f_t) = I_2(\tilde{f}_t)$ for every $t \geqslant 0$ with f_t given by (2.3), we have from the isometry property of multiple stochastic integrals (see [8])

$$\mathbf{E}[(Y_t^{H_1,H_2})^2] = 2\|\tilde{f}_t\|^2_{L^2(\mathbb{R}^2)} = 2\int_{\mathbb{R}^2} \tilde{f}^2(y_1,y_2)dy_1dy_2$$
$$= \frac{1}{2}c(H_1,H_2)^2$$
$$\times \int_{\mathbb{R}^2} \Big(\int_0^t (u-y_1)_+^{H_1/2-1}(u-y_2)_+^{H_2/2-1}du + \int_0^t (u-y_2)_+^{H_1/2-1}(u-y_1)_+^{H_2/2-1}du\Big)$$
$$\times \Big(\int_0^t (v-y_1)_+^{H_1/2-1}(v-y_2)_+^{H_2/2-1}dv$$
$$+\int_0^t (v-y_2)_+^{H_1/2-1}(v-y_1)_+^{H_2/2-1}dv\Big)dy_1dy_2$$
$$= \frac{1}{2}c(H_1,H_2)^2$$
$$\times \int_{\mathbb{R}^2} \Big[\Big(\int_0^t (u-y_1)_+^{H_1/2-1}(u-y_2)_+^{H_2/2-1}du \int_0^t (v-y_1)_+^{H_1/2-1}(v-y_2)_+^{H_2/2-1}dv\Big)$$
$$+\Big(\int_0^t (u-y_1)_+^{H_1/2-1}(u-y_2)_+^{H_2/2-1}du \int_0^t (v-y_2)_+^{H_1/2-1}(v-y_1)_+^{H_2/2-1}dv\Big)$$
$$+\Big(\int_0^t (u-y_2)_+^{H_1/2-1}(u-y_1)_+^{H_2/2-1}du \int_0^t (v-y_1)_+^{H_1/2-1}(v-y_2)_+^{H_2/2-1}dv\Big)$$
$$+\Big(\int_0^t (u-y_2)_+^{H_1/2-1}(u-y_1)_+^{H_2/2-1}du$$
$$\times \int_0^t (v-y_2)_+^{H_1/2-1}(v-y_1)_+^{H_2/2-1}dv\Big)\Big]dy_1dy_2$$

and, by interchanging the order of integration and noticing that the first and third summands, and the second and fourth, coincide, we obtain

$$\begin{aligned}
&c(H_1,H_2)^{-2}\mathbf{E}[(Y_t^{H_1,H_2})^2] \\
&\quad = \int_0^t\int_0^t \Big[\Big(\int_{-\infty}^{u\wedge v} dy_1 (u-y_1)^{H_1/2-1}(v-y_1)^{H_1/2-1}\Big) \\
&\qquad \times \Big(\int_{-\infty}^{u\wedge v} dy_2 (u-y_2)^{H_2/2-1}(v-y_2)^{H_2/2-1}\Big) \\
&\qquad + \Big(\int_{-\infty}^{u\wedge v} dy_1 (u-y_1)^{H_1/2-1}(v-y_1)^{H_2/2-1}\Big) \\
&\qquad \times \Big(\int_{-\infty}^{u\wedge v} dy_2 (u-y_2)^{H_2/2-1}(v-y_2)^{H_1/2-1}\Big)\Big] dudv.
\end{aligned}$$

Observe that the function inside the integral $dudv$ is symmetric. Therefore,

$$\begin{aligned}
&c(H_1,H_2)^{-2}\mathbf{E}[(Y_t^{H_1,H_2})^2] \\
&\quad = 2\int_0^t du \int_0^u \Big[\Big(\int_0^v dy_1 (u-y_1)^{H_1/2-1}(v-y_1)^{H_1/2-1}\Big) \\
&\qquad \times \Big(\int_{-\infty}^{v} dy_2 (u-y_2)^{H_2/2-1}(v-y_2)^{H_2/2-1}\Big) \\
&\qquad + \Big(\int_{-\infty}^{v} dy_1 (u-y_1)^{H_1/2-1}(v-y_1)^{H_2/2-1}\Big) \\
&\qquad \times \Big(\int_{-\infty}^{v} dy_2 (u-y_2)^{H_2/2-1}(v-y_2)^{H_1/2-1}\Big)\Big] dv.
\end{aligned}$$

We obtain, using Lemma 2.1,

$$\begin{aligned}
&c(H_1,H_2)^{-2}\mathbf{E}[(Y_t^{H_1,H_2})^2] \\
&\quad = 2\Big[\beta\left(1-H_1,\frac{H_1}{2}\right)\beta\left(1-H_2,\frac{H_2}{2}\right) \\
&\qquad + \beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)\beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)\Big] \\
&\qquad \times \int_0^t du \int_0^u (u-v)^{H_1+H_2-2} dv \\
&\quad = \Big[\beta\left(1-H_1,\frac{H_1}{2}\right)\beta\left(1-H_2,\frac{H_2}{2}\right) \\
&\qquad + \beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)\beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)\Big] \\
&\qquad \times \frac{1}{(H_1+H_2)(H_1+H_2-1)} t^{2H}.
\end{aligned}$$

If $H_1 + H_2 = 2H$, then

$$c(H_1, H_2)^{-2}\mathbf{E}[(Y_t^{H_1,H_2})^2] = \left[\beta\left(1 - H_1, \frac{H_1}{2}\right)\beta\left(1 - H_2, \frac{H_2}{2}\right) + \beta\left(1 - H, \frac{H_1}{2}\right)\beta\left(1 - H, \frac{H_2}{2}\right)\right]\frac{1}{H(2H-1)}t^{2H},$$

which implies

$$c(H_1, H_2)^{-2} = \frac{1}{H(2H-1)}\left[\beta\left(1 - H_1, \frac{H_1}{2}\right)\beta\left(1 - H_2, \frac{H_2}{2}\right) + \beta\left(1 - H, \frac{H_1}{2}\right)\beta\left(1 - H, \frac{H_2}{2}\right)\right]. \quad (2.4)$$ ∎

REMARK 2.3. *In the particular case* $H_1 = H_2 = H$ *we have*

$$c(H, H) := c(H) = \left[\frac{2}{H(2H-1)}\beta\left(1 - H, \frac{H}{2}\right)^2\right]^{-1/2}$$

and it coincides with the constant used, *e.g.*, *in* [12].

3. CUMULANTS OF THE NON-SYMMETRIC ROSENBLATT PROCESS

We will prove in this section that the processes Y^{H_1,H_2} given by (2.2) have different laws upon the values of the selfsimilar parameters H_1 and H_2. We will use the concept of *cumulant.* The cumulants of a random variable X having all moments appear as the coefficients in the Maclaurin series of $g(t) = \log \mathbf{E}e^{tX}$, $t \in \mathbb{R}$. The first cumulant c_1 is the expectation of X while the second one is the variance of X. Generally, the nth cumulant is given by $g^{(n)}(0)$. The key fact is that for random variables in the second Wiener chaos the cumulants characterize the law.

Let us consider a multiple integral $I_2(f)$ with $f \in L^2(\mathbb{R}^2)$ symmetric. Then the mth cumulant of the random variable $I_2(f)$ is given by (see [7] or [5])

$$c_m\big(I_2(f)\big) = 2^{m-1}(m-1)! \int_{\mathbb{R}^m} f(y_1, y_2)f(y_2, y_3)\ldots f(y_{m-1}, y_m)f(y_m, y_1)dy_1\ldots dy_m \quad (3.1)$$

For the following result we refer to [5]:

REMARK 3.1. *It is known that the law of a multiple integral of order two is completely determined by its cumulants in the sense that*, *if two multiple integrals of order two have the same cumulants*, *then their distributions are the same.*

We can state the main result of this section.

PROPOSITION 3.1. *Let us consider the process* $(Y_t^{H_1,H_2})_{t\geqslant 0}$ *given by* (2.2). *There exist couples* $(H_1,H_2),(H_1',H_2')\in(0,1)^2$ *with* $H_1+H_2=H_1'+H_2'=2H>1$ *such that* $(H_1,H_2)\neq(H_1',H_2')$ *and, for any* $t>0$, *the laws of the random variables* $Y_t^{H_1,H_2}$ *and* $Y_t^{H_1',H_2'}$ *are different.*

Proof. It suffices to show that for fixed t the two random variables $Y_t^{H_1,H_2}$ and $Y_t^{H_1',H_2'}$ have at least one different cumulant. The first two cumulants (that is, the expectation and the variance) of these two random variables are the same since Y^{H_1,H_2} is an H-selfsimilar process with stationary increments. Let us compute the third cumulant.

Let us consider the case $m=3$. Then, using (3.1), the expression of $\tilde{f}_t$, and changing the order of integration, we get

$$\begin{aligned}&c_3\big(I_2(\tilde{f}_t)\big)\\&=: c(H_1,H_2)^3\int_0^t\int_0^t\int_0^t[g_{H_1,H_2}(u_1,u_2,u_3)+g_{H_1,H_2}(u_3,u_2,u_1)\\&+f_{H_1,H_2}(u_1,u_2,u_3)+f_{H_1,H_2}(u_1,u_3,u_2)+f_{H_1,H_2}(u_2,u_1,u_3)\\&+f_{H_1,H_2}(u_2,u_3,u_1)+f_{H_1,H_2}(u_3,u_1,u_2)+f_{H_1,H_2}(u_3,u_2,u_1)]du_1du_2du_3,\end{aligned}$$

where we have put

$$\begin{aligned}&g_{H_1,H_2}(u_1,u_2,u_3)=\Big(\int_{\mathbb{R}}(u_1-y)_+^{H_1/2-1}(u_3-y)_+^{H_2/2-1}dy\Big)\\&\times\Big(\int_{\mathbb{R}}(u_1-y)_+^{H_2/2-1}(u_2-y)_+^{H_1/2-1}dy\Big)\Big(\int_{\mathbb{R}}(u_2-y)_+^{H_2/2-1}(u_3-y)_+^{H_1/2-1}dy\Big)\end{aligned}$$

and

$$\begin{aligned}&f_{H_1,H_2}(u_1,u_2,u_3)=\Big(\int_{\mathbb{R}}(u_1-y)_+^{H_1/2-1}(u_3-y)_+^{H_1/2-1}dy\Big)\\&\times\Big(\int_{\mathbb{R}}(u_1-y)_+^{H_2/2-1}(u_2-y)_+^{H_1/2-1}dy\Big)\Big(\int_{\mathbb{R}}(u_2-y)_+^{H_2/2-1}(u_3-y)_+^{H_2/2-1}dy\Big).\end{aligned}$$

Therefore, the function under the integral $du_1du_2du_3$ is symmetric with respect to the variables u_1,u_2,u_3. The integral $\int_0^t\int_0^t\int_0^t du_1du_2du_3$ is then equal to

$$3!\int_{\substack{u_3<u_2<u_1,\\ u_1,u_2,u_3\in[0,t]}} du_1du_3du_3.$$

Also, from Lemma 2 we obtain for $u_3<u_3<u_1$

$$g_{H_1,H_2}(u_1,u_2,u_3)=\beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)(u_1-u_3)^{(H_1+H_2)/2-1}$$

$$\times \beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)(u_1-u_2)^{(H_1+H_2)/2-1}$$
$$\times \beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)(u_2-u_3)^{(H_1+H_2)/2-1}$$

and

$$\begin{aligned} f_{H_1,H_2}(u_1,u_2,u_3) = {}& \beta\left(1-H_1,\frac{H_1}{2}\right)(u_1-u_3)^{(H_1+H_2)/2-1} \\ & \times \beta\left(1-H_2,\frac{H_1}{2}\right)(u_1-u_2)^{(H_1+H_2)/2-1} \\ & \times \beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)(u_2-u_3)^{(H_1+H_2)/2-1}. \end{aligned}$$

Thus we have

$$\begin{aligned} & c_3\big(I_2(\tilde{f}_t)\big) \\ & = 3!c(H_1,H_2)^3\Bigg[\beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)\beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right) \\ & \times \left(\beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)+\beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)\right) \\ & + 2\beta\left(1-H_1,\frac{H_1}{2}\right)\beta\left(1-H_2,\frac{H_2}{2}\right) \\ & \times \left(\beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)+\beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)\right)\Bigg] \\ & \times \int\limits_{\substack{u_3<u_2<u_1,\\ u_1,u_2,u_3\in[0,t]}} (u_1-u_3)^{(H_1+H_2)/2-1}(u_1-u_2)^{(H_1+H_2)/2-1} \\ & \times (u_2-u_3)^{(H_1+H_2)/2-1}du_1du_2du_3 \\ & = 3!c(H_1,H_2)^3\Bigg[\beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)+\beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)\Bigg] \\ & \times \Bigg[2\beta\left(1-H_1,\frac{H_1}{2}\right)\beta\left(1-H_2,\frac{H_2}{2}\right) \\ & + \beta\left(1-\frac{H_1+H_2}{2},\frac{H_1}{2}\right)\beta\left(1-\frac{H_1+H_2}{2},\frac{H_2}{2}\right)\Bigg] \\ & \times \int\limits_{\substack{u_3<u_2<u_1,\\ u_1,u_2,u_3\in[0,t]}} (u_1-u_3)^{(H_1+H_2)/2-1}(u_1-u_2)^{(H_1+H_2)/2-1} \\ & \times (u_2-u_3)^{(H_1+H_2)/2-1}du_1du_2du_3. \end{aligned}$$

Further, using gamma integrals we get

$$
\begin{aligned}
&c_3\big(I_2(\tilde{f}_t)\big)\\
&= 3!c(H_1,H_2)^3\left[\beta\left(1-H,\frac{H_1}{2}\right)+\beta\left(1-H,\frac{H_2}{2}\right)\right]\\
&\quad\times\left[2\beta\left(1-H_1,\frac{H_1}{2}\right)\beta\left(1-H_2,\frac{H_2}{2}\right)+\beta\left(1-H,\frac{H_1}{2}\right)\beta\left(1-H,\frac{H_2}{2}\right)\right]\\
&\quad\times\int\limits_{\substack{u_3<u_2<u_1,\\ u_1,u_2,u_3\in[0,t]}}(u_1-u_3)^{H-1}(u_1-u_2)^{H-1}(u_2-u_3)^{H-1}du_1du_2du_3.\\
&= 3!c(H_1,H_2)^3\frac{\Gamma(1-H)\Gamma(H_1/2)\Gamma(H_2/2)}{\big(\Gamma(1-H_1/2)\Gamma(1-H_2/2)\big)^2}\\
&\quad\times\left[\Gamma\Big(\frac{H_1}{2}\Big)\Gamma\Big(1-\frac{H_1}{2}\Big)+\Gamma\Big(\frac{H_2}{2}\Big)\Gamma\Big(1-\frac{H_2}{2}\Big)\right]\\
&\quad\times\big(2\Gamma(1-H_1)\Gamma(1-H_2)+\Gamma(1-H)^2\big)\\
&\quad\times\int\limits_{\substack{u_3<u_2<u_1,\\ u_1,u_2,u_3\in[0,t]}}(u_1-u_3)^{H-1}(u_1-u_2)^{H-1}(u_2-u_3)^{H-1}du_1du_2du_3.
\end{aligned}
$$

It is obvious, given the expression of the normalizing constant $c(H_1,H_2)$, that there exist $(H_1,H_2)\neq(H_1',H_2')$ with $c_3\big(I_2(f_{H_1,H_2})\big)\neq c_3\big(I_2(f_{H_1',H_2'})\big)$. For example, this happens when $H_1=H_2=0.4$ and $H_1'=0.3, H_2'=0.5$ because then the expression of the gamma function can be computed numerically. ■

EXAMPLE 3.1. There are other classes of selfsimilar process with stationary increments. For this example we refer to [9] and [6]. Consider α,β such that $\frac{1}{2}<\alpha<\frac{3}{4}$ and $0<2-2\alpha-\beta<1$. Define for every $t\geqslant 0$

$$
X_t=\int\limits_{\mathbb{R}^2}\Big(\int\limits_0^\infty(u-y_1)_+^{-\alpha}(u-y_2)_+^{-\alpha}(|u|^{-\beta}-|u-t|^{-\beta})du\Big)dB_{y_1}dB_{y_2}.
$$

The process $X=(X_t)_{t\geqslant 0}$ defined above is H-selfsimilar with stationary increments where $H=2-\beta-2\alpha$. The proof is immediate and follows the lines of Proposition 2.1. It can also be proved that for suitable choices of α,β the law of the process X defined above is different from the law of the process Y given by (2.2). We will come back to this process X defined above in the last section.

4. LIMIT THEOREM FOR NON-SYMMETRIC ROSENBLATT PROCESS

Let B^{H_1},B^{H_2} be two fractional Brownian motions with Hurst parameters H_1,H_2, respectively. We will assume that the selfsimilar parameters H_1 and H_2 are both bigger than $\frac{1}{2}$. We will also assume that the two fractional Brownian motions can be expressed as Wiener integrals with respect to the same Wiener process B. This implies that B^{H_1} and B^{H_2} are not independent. We have

$$(4.1)\qquad \begin{aligned} B_t^{H_1} &= c(H_1) \int_{\mathbb{R}} dB_y \int_0^t (u-y)_+^{H_1-3/2} du, \\ B_t^{H_2} &= c(H_2) \int_{\mathbb{R}} dB_y \int_0^t (u-y)_+^{H_2-3/2} du, \end{aligned}$$

where the constants $c(H_1), c(H_2)$ are such that $\mathbf{E}[(B_1^{H_1})^2] = \mathbf{E}[(B_1^{H_2})^2] = 1$. Actually, applying Lemma 2.1 with H_1, H_2 replaced by $2H_1 - 1, 2H_2 - 1$, respectively, we get

$$(4.2)\qquad c(H_1)^2 = \frac{H_1(2H_1-1)}{\beta\big(2-2H_1, H_1-\frac{1}{2}\big)}$$

and an analogous expression for $c(H_2)$.

Define, for every $N \geqslant 2, t \geqslant 0$, the sequence

$$(4.3)\qquad V_N(t) = \sum_{i=0}^{[Nt]-1} \left[\frac{(B_{(i+1)/N}^{H_1} - B_{i/N}^{H_1})(B_{(i+1)/N}^{H_2} - B_{i/N}^{H_2})}{\mathbf{E}[(B_{(i+1)/N}^{H_1} - B_{i/N}^{H_1})(B_{(i+1)/N}^{H_2} - B_{i/N}^{H_2})]} - 1 \right].$$

It is well known that in the case $H_1 = H_2 = H \in \big(\frac{3}{4}, 1\big)$ the (renormalized) sequence $\big(V_N(t)\big)_{t \geqslant 0}$ converges as $N \to \infty$, in the sense of finite-dimensional distribution, to a symmetric Rosenblatt process with selfsimilar parameter $2H - 1$. Our aim is to extend this result to the situation when $H_1 \neq H_2$. We will actually prove that, after suitable normalization, the sequence (4.3) converges in the sense of finite-dimensional distributions to the process Y^{H_1,H_2} in (2.2).

First, we need to understand the correlations structure of the fractional Brownian motions B^{H_1} and B^{H_2}.

LEMMA 4.1. *Let $t > s$. Then*

$$\mathbf{E}[(B_t^{H_1} - B_s^{H_1})(B_t^{H_2} - B_s^{H_2})] = b(H_1, H_2)|t-s|^{2H},$$

where

$$b(H_1, H_2) = \frac{c(H_1)c(H_2)}{2H(2H-1)} \left[\beta\left(2-2H, H_1 - \frac{1}{2}\right) + \beta\left(2-2H, H_2-\frac{1}{2}\right) \right]$$

with $c(H_1), c(H_2)$ given by (4.2) *and $2H = H_1 + H_2$.*

P r o o f. Since

$$B_t^{H_1} - B_s^{H_1} = c(H_1) \int_{\mathbb{R}} dB_y \int_s^t (u-y)_+^{H_1-3/2} du,$$

we obtain, using the isometry of Wiener integrals and Lemma 2.1,

$$\begin{aligned}
&\mathbf{E}[(B_t^{H_1} - B_s^{H_1})(B_t^{H_2} - B_s^{H_2})] \\
&= c(H_1)c(H_2) \int_s^t \int_s^t du dv \int_{-\infty}^{u\wedge v} (u-y)_+^{H_1-3/2}(v-y)_+^{H_2-3/2} dy \\
&= c(H_1)c(H_2) \int_s^t du \int_s^u \beta(2-2H, 2H_1-1)(u-v)^{2H-2} dv \\
&\quad + c(H_1)c(H_2) \int_s^t dv \int_s^v \beta(2-2H, 2H_2-1)(v-u)^{2H-2} du \\
&= \frac{c(H_1)c(H_2)}{2H(2H-1)} \left[\beta\left(2-2H, H_1 - \frac{1}{2}\right) + \beta\left(2-2H, H_2-\frac{1}{2}\right)\right](t-s)^{2H},
\end{aligned}$$

which completes the proof. ■

REMARK 4.1. *The above constant* $b(H_1, H_2)$ *is equal to one if* $H_1 = H_2 = H$.

The following result constitutes an extension to the non-symmetric case of the non-central limit theorem proved in [2], [3], [11].

THEOREM 4.1. *Let* V_N *be given by* (4.3) *and assume* $H_1 + H_2 = 2H > \frac{3}{2}$. *Then, as* $N \to \infty$, $c(H_1, H_2)\big(c(H_1)c(H_2)\big)^{-1} b(H_1, H_2) N^{1-2H} V_N(1)$ *converges in* $L^2(\Omega)$ *to the random variable* $Y_1^{2H_1-1, 2H_2-1}$ *given by* (2.2).

Proof. Using the product formula for multiple integrals (see [8], Proposition 1.1.2), we can express V_N as

$$\begin{aligned}
V_N(1) = N^{2H} b(H_1, H_2)^{-1} c(H_1)c(H_2) &\sum_{i=0}^{N-1} \int_{\mathbb{R}} \int_{\mathbb{R}} dB_{y_1} dB_{y_2} \\
&\times \int_{i/N}^{(i+1)/N} \int_{i/N}^{(i+1)/N} (u-y_1)_+^{H_1-3/2}(v-y_2)_+^{H_2-3/2} du dv.
\end{aligned}$$

It suffices to show that the sequence

$$N \sum_{i=0}^{N-1} \int_{\mathbb{R}} \int_{\mathbb{R}} dB_{y_1} dB_{y_2} \int_{i/N}^{(i+1)/N} \int_{i/N}^{(i+1)/N} (u-y_1)_+^{H_1-3/2}(v-y_2)_+^{H_2-3/2} du dv$$

converges in $L^2(\Omega)$, as $N \to \infty$, to

$$\int_{\mathbb{R}} \int_{\mathbb{R}} dB_{y_1} dB_{y_2} \int_0^1 (u-y_1)_+^{H_1-3/2}(u-y_2)_+^{H_2-3/2} du$$

or, equivalently, by the isometry formula for multiple integrals, that the sequence

$$a_N(y_1,y_2) = N\sum_{i=0}^{N-1}\int_{i/N}^{(i+1)/N}\int_{i/N}^{(i+1)/N}(u-y_1)_+^{H_1-3/2}(v-y_2)_+^{H_2-3/2}dudv$$

converges in $L^2(\mathbb{R}^2)$, as $N\to\infty$, to the function

$$a(y_1,y_2) = \int_0^1 (u-y_1)_+^{H_1-3/2}(u-y_2)_+^{H_2-3/2}du$$

which represents the kernel of the non-symmetric Rosenblatt process. Let us estimate the $L^2(\mathbb{R}^2)$-norm of the difference $a_N - a$. We have

$$\|a_N - a\|^2_{L^2(\mathbb{R}^2)} = \|a_N\|^2_{L^2(\mathbb{R}^2)} - 2\langle a_N, a\rangle_{L^2(\mathbb{R}^2)} + \|a\|^2_{L^2(\mathbb{R}^2)}.$$

We compute separately the three quantities above. First,

$$\|a_N\|^2_{L^2(\mathbb{R}^2)} = N^2\sum_{i,j=0}^{N-1}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\int_{i/N}^{(i+1)/N}\int_{i/N}^{(i+1)/N}(u-y_1)_+^{H_1-3/2}(v-y_2)_+^{H_2-3/2}dudv$$

$$\times\int_{j/N}^{(j+1)/N}\int_{j/N}^{(j+1)/N}(u'-y_1)_+^{H_1-3/2}(v'-y_2)_+^{H_2-3/2}du'dv'\Big)dy_1dy_2$$

$$= \beta\left(2-2H_1, H_1-\frac{1}{2}\right)\beta\left(2-2H_2, H_2-\frac{1}{2}\right)$$

$$\times N^2\sum_{i,j=0}^{N-1}\int_{i/N}^{(i+1)/N}\int_{i/N}^{(i+1)/N}dudv\int_{j/N}^{(j+1)/N}\int_{j/N}^{(j+1)/N}|u-u'|^{2H_1-2}|v-v'|^{2H_2-2}du'dv',$$

where we have used the Fubini theorem and Lemma 2.1. In the same way,

$$\langle a_N, a\rangle_{L^2(\mathbb{R}^2)} = \beta\left(2-2H_1, H_1-\frac{1}{2}\right)\beta\left(2-2H_2, H_2-\frac{1}{2}\right)$$

$$\times N\sum_{i=0}^{N-1}\int_{i/N}^{(i+1)/N}\int_{i/N}^{(i+1)/N}dudv\int_0^1|u-u'|^{2H_1-2}|v-u'|^{2H_2-2}du'$$

and

$$\|a\|^2_{L^2(\mathbb{R}^2)} = \beta\left(2-2H_1, H_1-\frac{1}{2}\right)\beta\left(2-2H_2, H_2-\frac{1}{2}\right)\int_0^1\int_0^1|u-v|^{4H-4}dudv.$$

To summarize, we get

$$\|a_N - a\|^2_{L^2(\mathbb{R}^2)} = \beta\left(2-2H_1, H_1 - \frac{1}{2}\right)\beta\left(2-2H_2, H_2-\frac{1}{2}\right)$$
$$\times \sum_{i,j=0}^{N-1} \big[N^2 \sum_{i,j=0}^{N-1} \int_{\frac{i}{N}}^{\frac{i+1}{N}} \int_{\frac{i}{N}}^{\frac{i+1}{N}} dudv \int_{j/N}^{(j+1)/N} \int_{j/N}^{(j+1)/N} |u-u'|^{2H_1-2}|v-v'|^{2H_2-2} du'dv'$$
$$- 2N \sum_{i=0}^{N-1} \int_{i/N}^{(i+1)/N} \int_{i/N}^{(i+1)/N} dudv \int_{j/N}^{(j+1)/N} |u-u'|^{2H_1-2}|v-u'|^{2H_2-2} du'$$
$$+ \int_{i/N}^{(i+1)/N} du \int_{j/N}^{(j+1)/N} |u-v|^{4H-4} dv\big],$$

and making the change of variables $\tilde{u} = (u - i/N)\,N$ (and similarly for the other variables u', v, v'), we obtain

$$\|a_N - a\|^2_{L^2(\mathbb{R}^2)} = \beta\left(2-2H_1, H_1 - \frac{1}{2}\right)\beta\left(2-2H_2, H_2-\frac{1}{2}\right) N^{2-4H}$$
$$\times \sum_{i,j=0}^{N-1} \big[\int_0^1\int_0^1 |u-u'+i-j|^{2H_1-2} dudu' \int_0^1\int_0^1 |v-v'+i-j|^{2H_2-2} dvdv'$$
$$- 2\int_0^1\int_0^1\int_0^1 |u-u'+i-j|^{2H_1-2}|v-u'+i-j|^{2H_2-2} dudvdu'$$
$$+ \int_0^1\int_0^1 |u-v+i-j|^{4H-4} dudv\big]$$
$$\leqslant \beta\left(2-2H_1, H_1 - \frac{1}{2}\right)\beta\left(2-2H_2, H_2-\frac{1}{2}\right) N^{3-4H}$$
$$\times \sum_{k\in\mathbb{Z}} \big[\int_0^1\int_0^1 |u-u'+k|^{2H_1-2} dudu' \int_0^1\int_0^1 |v-v'+k|^{2H_2-2} dvdv'$$
$$- 2\int_0^1\int_0^1\int_0^1 |u-u'+k|^{2H_1-2}|v-u'+k|^{2H_2-2} dudvdu'$$
$$+ \int_0^1\int_0^1 |u-v+k|^{4H-4} dudv\big].$$

As in [1], the proof of Proposition 3.1, we can prove that the sum over $k \in \mathbb{Z}$ is finite. Indeed, this sum can be written as

$$\sum_{k\in\mathbb{Z}} k^{4H-4} F\left(\frac{1}{k}\right),$$

where

$$F(x) = \big[\int_0^1\int_0^1 |(u-u)x+1|^{2H_1-2} dudu' \int_0^1\int_0^1 |(v-v)x+1|^{2H_2-2} dvdv'$$

$$- 2\int_0^1\int_0^1\int_0^1 |(u-u')x+1|^{2H_1-2}|(v-u')x+1|^{2H_2-2}dudvdu'$$
$$+\int_0^1\int_0^1 |(u-v)x+1|^{4H-4}dudv\big].$$

The conclusion follows since for $\frac{3}{4} < H < 1$ we can see that $F(x)$ behaves as x for x close to zero. ■

REMARK 4.2. *The condition $H_1 + H_2 > \frac{3}{2}$ is natural since it extends the classical condition $H > \frac{3}{4}$ necessary to obtain non-Gaussian limit of V_N in the symmetric case.*

Following exactly the lines of the above proof, we obtain immediately the next corollary.

COROLLARY 4.1. *Let V_N be as in* (4.3) *and assume $2H = H_1 + H_2 > \frac{3}{2}$. The sequence of stochastic processes*

$$\Big(c(H_1,H_2)\big(c(H_1)c(H_2)\big)^{-1}b(H_1,H_2)N^{1-2H}V_N(t)\Big)_{t\geqslant 0}$$

converges in the sense of finite-dimensional distributions as $N\to\infty$ to the stochastic process $(Y_t^{2H_1-1,2H_2-1})_{t\geqslant 0}$.

COROLLARY 4.2. *Consider B^{H_1}, B^{H_2} as before and assume $2H = H_1 + H_2 > \frac{3}{2}$. Set, for every $t \geqslant 0$,*

$$S_N(t) = \sum_{i=0}^{[Nt]-1}\{(B_{i+1}^{H_1}-B_i^{H_1})(B_{i+1}^{H_2}-B_i^{H_2}) - \mathbf{E}[(B_{i+1}^{H_1}-B_i^{H_1})(B_{i+1}^{H_2}-B_i^{H_2})]\}$$
$$= \sum_{i=0}^{[Nt]-1}(B_{i+1}^{H_1}-B_i^{H_1})(B_{i+1}^{H_2}-B_i^{H_2}) - b(H_1,H_2).$$

Then the sequence of stochastic processes

$$\Big(c(H_1,H_2)\big(c(H_1)c(H_2)\big)^{-1}N^{1-2H}S_N(t)\Big)_{t\geqslant 0}$$

converges as $N\to\infty$ to $(Y_t^{2H_1-1,2H_2-1})_{t\geqslant 0}$ in the sense of finite-dimensional distributions.

Proof. Since for every $t \geqslant 0$ we have $S_N(t) = I_2(g_t)$ with

$$g_t(y_1,y_2) = \int_i^{i+1}\int_i^{i+1}(u-y_1)_+^{H_1-3/2}(u-y_2)_+^{H_2-3/2}dudv, \qquad y_1,y_2\in\mathbb{R},$$

by making the change of variable as in the proof of Proposition 2.1, it can be seen that V_N has the same law as $b(H_1,H_2)^{-1}S_N$. ■

5. GENERALIZATION AND THOUGHTS: HOW MANY SELFSIMILAR PROCESSES WITH STATIONARY INCREMENTS ARE IN THE SECOND WIENER CHAOS?

It is well known that in the case $H_1 = H_2 = H$ the result in Corollary 4.2 is still true if $H_2(B^H_{i+1} - B^H_i)$ (H_2 is the Hermite polynomial of degree two; see below for the definition) is replaced by $h(B^H_{i+1} - B^H_i)$, where h is a function with Hermite rank equal to two. We propose here a more general version of Corollary 4.2 in the non-symmetric case. Let us define, for every $t \geqslant 0$,

$$W_N(t) = N^{1-2H} \sum_{i=0}^{[Nt]-1} [(B^{H_1}_{i+1} - B^{H_1}_i) g(B^{H_2}_{i+1} - B^{H_2}_i) - c_0], \tag{5.1}$$

where $c_0 = \mathbf{E}[(B^{H_1}_{i+1} - B^{H_1}_i) g(B^{H_2}_{i+1} - B^{H_2}_i)]$, and g is a deterministic function with Hermite rank equal to one, which has a finite expansion into Hermite polynomials of the form

$$g(x) = \sum_{q=1}^{M} c_q H_q(x) \tag{5.2}$$

with $M \geqslant 1$, $c_1 \neq 0$, and H_n denoting the nth Hermite polynomial

$$H_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n}\left(\exp\left(-\frac{x^2}{2}\right)\right), \qquad x \in \mathbb{R}.$$

THEOREM 5.1. *Consider two fractional Brownian motions B^{H_1} and B^{H_2} given by* (4.1) *with $H_1 + H_2 = 2H > \frac{3}{2}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a deterministic function given by* (5.2) *such that for every $q \geqslant 2$*

$$(2H_2 - 2)(q - 1) < -1. \tag{5.3}$$

Then the sequence of stochastic processes $\big(W_N(t)\big)_{t \geqslant 0}$ converges in the sense of finite-dimensional distributions as $N \to \infty$ to the process

$$c_1 c(H_1, H_2)^{-1} c(H_1) c(H_2) b(H_1, H_2)^{-1} Y^{2H_1-1, 2H_2-1}$$

in (2.2).

REMARK 5.1. *The assumption* (5.3) *excludes the existence of terms with $q = 2$ in the expansion of g.*

Proof. Again we assume $t = 1$. We have, since $H_q\big(I_1(\varphi)\big) = (q!)^{-1} I_q(\varphi^{\otimes q})$ (see, e.g., [8]),

$$W_N(1) = N^{1-2H} \sum_{i=0}^{N-1} \left[(B^{H_1}_{i+1} - B^{H_1}_i) \sum_{q=1}^{M} c_q \frac{1}{q!} I_q(f_{q,i,H_2}) - c_0\right],$$

where

$$f_{q,i,H_2}(y_1,\dots,y_q) = \int_{[i,i+1]^q} (u_1-y_1)_+^{H_2-3/2}\dots(u_q-y_q)_+^{H_2-3/2}du_1\dots du_q.$$

Thus

$$W_N(1) = N^{1-2H}\sum_{i=0}^{N-1} I_1(f_{1,i,H_1})\sum_{q=1}^{M}\frac{c_q}{q!}I_q(f_{q,i,H_2})$$
$$= N^{1-2H}c_1\sum_{i=0}^{N-1} I_1(f_{1,i,H_1})I_1(f_{1,i,H_2}) + N^{1-2H}\sum_{q=2}^{M}\sum_{i=0}^{N-1} I_1(f_{1,i,H_1})\frac{c_q}{q!}I_q(f_{q,i,H_2}).$$

From Theorem 4.1 it follows that the first summand above converges in $L^2(\Omega)$ to the desired limit. Let us show that the remaining term

$$R_N := N^{1-2H}\sum_{i=0}^{N-1} I_1(f_{1,i,H_1})I_q(f_{q,i,H_2})$$

converges to zero in $L^2(\Omega)$ for every $q \geqslant 2$. By the product formula for multiple integrals (see [8], Proposition 1.1.2), we can write

$$R_N = N^{1-2H}\sum_{i=0}^{N-1} I_{q+1}\left(f_{1,i,H_1}\otimes f_{q,i,H_2}\right) + qN^{1-2H}\sum_{i=0}^{N-1} I_{q-1}\left(f_{1,i,H_1}\otimes_1 f_{q,i,H_2}\right)$$
$$= N^{1-2H}\sum_{i=0}^{N-1}\left[I_{q+1}\left(f_{1,i,H_1}\otimes f_{q,i,H_2}\right) + c(H_1,H_2,q)I_{q-1}\left(f_{q-1,i,H_2}\right)\right]$$
$$=: R_{N,1} + R_{N,2}$$

(here $c(H_1,H_2,q)$ denotes a generic constant depending on H_1, H_2, q that may change from line to line), where we used the fact that, by Lemma 2.1,

$$(f_{1,i,H_1}\otimes_1 f_{q,i,H_2})(y_1,\dots,y_{q-1}) = \Big(\int_{\mathbb{R}} dx f_{1,i,H_1}(x)f_{1,i,H_2}(x)\Big)I_{q-1}(f_{q-1,i,H_2})$$
$$= c(H_1,H_2,q)\int_i^{i+1}\int_i^{i+1}|u-v|^{H_1+H_2-2}dudv = c(H_1,H_2,q).$$

We first treat the term $R_{N,1}$. More precisely, we show that this term converges to zero in $L^2(\Omega)$ as $N\to\infty$. Since for any square-integrable function we have $\|\tilde{f}\| \leqslant \|f\|$ (below $a_N \sim b_N$ means that the sequences a_N and b_N have the same limit as $N\to\infty$), it follows that

$$\mathbf{E}[|R_{N,1}|^2] \leqslant c(H_1,H_2,q)N^{2-4H}\sum_{i,j=0}^{N-1}\Big(\int_i^{i+1}\int_j^{j+1}|u-v|^{2H_1-2}dudv\Big)$$

$$\times \Big(\int_i^{i+1} \int_j^{j+1} |u-v|^{2H_2-2} du dv \Big)^q$$

$$\sim c(H_1,H_2,q) N^{2-4H} \sum_{i,j=0; i\neq j}^{N-1} |i-j|^{2H_1-2+(2H_2-2)q}$$

$$= c(H_1,H_2,q) N^{2-4H} \sum_{k=1}^{N-1} (N-k) k^{2H_1-2+(2H_2-2)q}$$

$$= c(H_1,H_2,q) N^{3-4H} \sum_{k=1}^{N-1} k^{2H_1-2+(2H_2-2)q}$$

$$+ c(H_1,H_2,q) N^{2-4H} \sum_{k=1}^{N-1} k^{2H_1-1+(2H_2-2)q}.$$

The sequence $N^{3-4H} \sum_{k=1}^{N-1} k^{2H_1-2+(2H_2-2)q}$ converges to zero as $N \to \infty$. Indeed, when the series $\sum_{k=1}^{\infty} k^{2H_1-1+(2H_2-2)q}$ is convergent then the sequence $N^{3-4H} \sum_{k=1}^{N-1} k^{2H_1-2+(2H_2-2)q}$ converges to zero since $H > \frac{3}{4}$. When the same series is divergent, it behaves as $N^{2H_1-2+(2H_2-2)q} + 1$ and the summand goes to zero because $3-4H+2H_1-2+(2H_2-2)q+1 = (2H_2-2)(q-1) < 0$. The second summand can be treated similarly.

Let us prove now that the term $R_{N,2}$ converges to zero in $L^2(\Omega)$ as $N \to \infty$. We have

$$\mathbf{E}\, |R_{N,2}|^2 = c(H_1,H_2,q) N^{2-4H} \sum_{i,j=0}^{N-1} \Big(\int_i^{i+1} \int_j^{j+1} |u-v|^{2H_2-2} du dv \Big)^{q-1}$$

$$\sim c(H_1,H_2,q) N^{2-4H} \sum_{i,j=0; i\neq j}^{N-1} |i-j|^{(2H_2-2)(q-1)}$$

$$= c(H_1,H_2,q) N^{3-4H} \sum_{k=1}^{N-1} k^{(2H_2-2)(q-1)}$$

$$+ c(H_1,H_2,q) N^{2-4H} \sum_{k=1}^{N-1} k^{(2H_2-2)(q-1)+1},$$

where we made again the change of summation $i-j=k$ and we noticed that the diagonal term, which behaves as N^{2-4H}, converges to zero. The fact that

$$(2H_2-2)(q-1) < -1$$

implies that the series $\sum_{k=0}^{N-1} k^{(2H_2-2)(q-1)}$ is convergent, and since $H > \frac{3}{4}$, the sequence $N^{3-4H} \sum_{k=0}^{N-1} k^{(2H_2-2)(q-1)}$ goes to zero as $N \to \infty$. The second series is bounded by the first one (since $k \leqslant N$), and thus it converges to zero. ■

In principle, Theorem 5.1 can be extended to functions g having an infinite series expansion into Hermite polynomials. But in this case, W_N is given by an infinite sum of multiple integrals and it is much more difficult to control the L^2-norm of the rest.

REMARK 5.2. (a) *Let* $H_1 + H_2 = 2H > \frac{3}{2}$ *and* $H_1, H_2 > \frac{1}{2}$. *Corollary* 4.1 *shows that* $(\xi_i^{H_1})_{i\in\mathbb{N}}$ *and* $(\xi_i^{H_2})_{i\in\mathbb{N}}$ *are two stationary Gaussian sequences with zero mean and unit variance, and with correlation function* $r_1(n) \sim c(H_1)n^{2H_1-2}$, $r_2(n) \sim c(H_1)n^{2H_2-2}$ *such that*

$$\mathbf{E}[\xi_i^{H_1}\xi_j^{H_2}] \sim c(H_1, H_2)|i-j|^{H_1+H_2}.$$

Then

$$N^{1-2H}\sum_{k=1}^{[Nt]}\{f(\xi_k^{H_1}, \xi_k^{H_2}) - \mathbf{E}[f(\xi_k^{H_1}, \xi_k^{H_2})]\}$$

with function f *given by* $f(x, y) = xy = H_1(x)H_1(y)$ *converges in the sense of finite-dimensional distribution to, modulo a constant, a non-symmetric Rosenblatt process with Hurst parameters* $2H_1 - 1$ *and* $2H_2 - 1$. *Theorem* 5.1 *shows that the result can be extended to function* f *of the form*

$$f(x, y) = H_1(x)\sum_{q=1}^{M} c_q H_q(y)$$

with suitable assumptions on q, H_1 *and* H_2.

(b) *Let us discuss the selfsimilar process with stationary increments from Example* 3.1. *This process, denoted by* $X = (X_t)_{t\geqslant 0}$, *can be also obtained as a limit in a non-central limit theorem in the following way* (see [9], pp. 127–131). *Define a stationary Gaussian sequence* $(\xi_k)_{k\in\mathbb{Z}}$ *with zero mean and unit variance and with covariance* $r_k = \mathbf{E}[\xi_0\xi_k] \sim k^{-2\alpha}$. *Set* $X_k = \xi_k^2 - 1$ *and*

$$U_m = \sum_{k\in\mathbb{Z}} a_k X_{m-k}$$

with $a_k = 0$ *if* $k = 0$, $a_k = k^{-\beta-1}$ *if* $k > 0$ *and* $a_k = -|k|^{-\beta-1}$ *if* $k < 0, \beta > 0$. *Assume that* α, β *satisfy the assumptions from Example* 3.1. *Then* $N^{-\alpha}\sum_{m=1}^{N} U_m$ *converges to the process* X *from Example* 3.1. For the proof of this fact, we refer to [9].

(c) *Taking into account the points* (a) *and* (b) *above, we can find a mechanism to construct more selfsimilar processes with stationary increments in the second Wiener chaos. For example, consider the sequence* V_N *given by* (4.3) *and from it construct a linear process as* U_m *above with suitable weight* a_k. *It is expected to find a new selfsimilar process with stationary increments as a limit.*

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Received on 5.4.2011;
revised version on 15.5.2012