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J_1 CONVERGENCE OF PARTIAL SUM PROCESSES WITH A REDUCED NUMBER OF JUMPS

BY

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Abstract. Various functional limit theorems for partial sum processes of strictly stationary sequences of regularly varying random variables in the space of càdlàg functions $D[0,1]$ with one of the Skorokhod topologies have already been obtained. The mostly used Skorokhod J_1 topology is inappropriate when clustering of large values of the partial sum processes occurs. When all extremes within each cluster of high-threshold excesses do not have the same sign, Skorokhod M_1 topology also becomes inappropriate. In this paper we alter the definition of the partial sum process in order to shrink all extremes within each cluster to a single one, which allows us to obtain the functional J_1 convergence. We also show that this result can be applied to some standard time series models, including the GARCH$(1,1)$ process and its squares, the stochastic volatility models and m-dependent sequences.

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1. INTRODUCTION

Let $(X_n)_{n\geqslant 1}$ be a strictly stationary sequence of real-valued random variables and define by $S_n = X_1 + \ldots + X_n,\ n \geqslant 1$, its accompanying sequence of partial sums. If the sequence (X_n) is i.i.d. then it is well known (see, for example, Gnedenko and Kolmogorov [11], Rvačeva [18], Feller [10]) that there exist real sequences (a_n) and (b_n) such that

$$(1.1)\qquad \frac{S_n - b_n}{a_n} \xrightarrow{d} S \quad \text{as } n \to \infty$$

for some non-degenerate α-stable random variable S with $\alpha \in (0,2)$ if and only if X_1 is regularly varying with index $\alpha \in (0,2)$, that is,

$$\mathrm{P}(|X_1| > x) = x^{-\alpha} L(x),$$

where $L(\,\cdot\,)$ is a slowly varying function at infinity and

$$\frac{\mathrm{P}(X_1 > x)}{\mathrm{P}(|X_1| > x)} \to p \quad \text{and} \quad \frac{\mathrm{P}(X_1 < -x)}{\mathrm{P}(|X_1| > x)} \to q \tag{1.2}$$

as $x \to \infty$, with $p \in [0,1]$ and $q = 1 - p$. As α is less than 2, the variance of X_1 is infinite.

The functional generalization of (1.1) has been studied extensively in probability literature. Define the partial sum processes

$$V_n(t) = \frac{1}{a_n} \sum_{k=1}^{\lfloor nt \rfloor} (X_k - b_n), \qquad t \in [0,1],$$

where the sequences (a_n) and (b_n) are chosen so that

$$n\mathrm{P}(|X_1| > a_n) \to 1 \quad \text{and} \quad b_n = \mathrm{E}(X_1 \, 1_{\{|X_1| \leqslant a_n\}}).$$

Here $\lfloor x \rfloor$ represents the integer part of the real number x. In functional limit theory one investigates the asymptotic behavior of the processes $V_n(\,\cdot\,)$ as $n \to \infty$. Since the sample paths of $V_n(\,\cdot\,)$ are elements of the space $D[0,1]$ of all right-continuous real-valued functions on $[0,1]$ with left limits, it is natural to consider the weak convergence of distributions of $V_n(\,\cdot\,)$ with the one of Skorokhod topologies on $D[0,1]$ introduced in Skorokhod [21].

Skorokhod [22] established a functional limit theorem for the processes $V_n(\,\cdot\,)$ for infinite variance i.i.d. regularly varying sequences (X_n). Under some weak dependence conditions, weak convergence of partial sum processes was obtained by Durrett and Resnick [9], Leadbetter and Rootzén [13] and Tyran-Kamińska [23]. Their functional limit theorems hold in Skorokhod J_1 topology, which is appropriate when large values of the partial sum processes do not cluster. When clustering of large values occurs then J_1 convergence fails to hold, but the functional limit theorem might still hold in the weaker Skorokhod M_1 topology. Avram and Taqqu [1] obtained a functional limit theorem with Skorokhod M_1 topology for sums of moving averages with nonnegative coefficients. Recently Basrak et al. [3] gave sufficient conditions for functional limit theorem with M_1 topology to hold for stationary, regularly varying sequences for which all extremes within each cluster of high-threshold excesses have the same sign.

In this paper we alter the definition of the partial sum process in the manner that all extremes within each cluster shrink to a single one, which allows us to recover the J_1 convergence. Note that the process $V_n(t)$ jumps at every $t = k/n$ ($k \leqslant n$) with $(X_k - b_n)/a_n$ being the size of the jump. Now we reduce the number of jumps (or, alternatively, increase the intervals between jumps) by introducing a sequence of positive integers (r_n) such that $r_n \to \infty$ and $k_n := \lfloor n/r_n \rfloor \to \infty$ as $n \to \infty$, and defining new partial sum processes

$$W_n(t) = \frac{1}{a_n} \sum_{k=1}^{\lfloor k_n t \rfloor} (S_{r_n}^k - c_n), \qquad t \in [0,1],$$

where $S_{r_n}^k = X_{(k-1)r_n+1} + \ldots + X_{kr_n}$ (k, $n \in \mathbb{N}$), and c_n are centering constants which will be specified later. The process $W_n(t)$ jumps at every $t = k/k_n$ with $(S_{r_n}^k - c_n)/a_n$ being the size of the jump. In other words, we break $X_1, X_2, \ldots$ into blocks of r_n consecutive random variables and treat the sums of random variables within each block as we treated single random variables X_i in the process $V_n(\,\cdot\,)$. One jump of the process $W_n(\,\cdot\,)$ corresponds to r_n consecutive jumps of the process $V_n(\,\cdot\,)$. In this way we have partially smoothed the trajectories of partial sum processes such that each cluster can consist of only one excess. Our method is in fact closely related to the scheme of summing triangular arrays with stationary rows (see Samur [19]).

Functional limit theorems for the processes $V_n(\,\cdot\,)$ base on the regular variation property of X_1. Therefore, to obtain the functional limit theorem for the processes $W_n(\,\cdot\,)$ we need to impose a similar condition on S_{r_n}. For this purpose we will assume S_{r_n} satisfies a certain large deviation condition (see relation (2.5) in the sequel).

The paper is organized as follows. In Section 2 we introduce some basic results on point processes and regular variation. We also describe precisely the large deviation condition that we impose on S_{r_n}. In Section 3 we state and prove the functional limit theorem for the processes $W_n(\,\cdot\,)$ in the J_1 topology. Here we also discuss several examples of stationary sequences covered by our theorem. Finally, in Section 4 (Appendix) we prove that the mixing conditions used in our main theorem are implied by some conditions which are given in terms of the standardly used α-mixing and ρ-mixing conditions.

2. PRELIMINARIES

At the beginning we introduce here some basic notions and results on point processes which will be used later on. For more background on the theory of point processes we refer to Kallenberg [12]. Let $\mathbb{E} = \overline{\mathbb{R}} \setminus \{0\}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. For $x, y \in \mathbb{E}$ define

$$\rho(x,y) = \max\left\{\left|\frac{1}{|x|} - \frac{1}{|y|}\right|, |\text{sign}(x) - \text{sign}(y)|\right\}, \tag{2.1}$$

where $\text{sign}(z) = z/|z|$. With the metric ρ, $\mathbb{E}$ becomes a locally compact, complete and separable metric space. A set $B \subseteq \mathbb{E}$ is *relatively compact* if it is bounded away from the origin, that is, if there exists $u > 0$ such that $B \subseteq \mathbb{E} \setminus [-u, u]$. Denote by $\mathcal{B}(\mathbb{E})$ the σ-algebra generated by ρ-open sets. Let $M_+(\mathbb{E})$ be the class of all Radon measures on $\mathbb{E}$, i.e., all nonnegative measures that are finite on relatively compact subsets of $\mathbb{E}$. A useful topology for $M_+(\mathbb{E})$ is the vague topology which renders $M_+(\mathbb{E})$ a complete separable metric space. If $\mu_n \in M_+(\mathbb{E})$, $n \geqslant 0$, then μ_n *converges vaguely* to μ_0 (written $\mu_n \xrightarrow{v} \mu_0$) if $\int f d\mu_n \to \int f d\mu_0$ for all $f \in C_K^+(\mathbb{E})$, where $C_K^+(\mathbb{E})$ denotes the class of all nonnegative continuous real functions on $\mathbb{E}$

with compact support. One metric that induces the vague topology is given by

(2.2)
$$d_v(\mu_1,\mu_2)=\sum_{k=1}^{\infty} 2^{-k}\Big(\Big|\int_{\mathbb{E}} f_k(x)\,\mu_1(dx)-\int_{\mathbb{E}} f_k(x)\mu_2(dx)\Big|\wedge 1\Big), \quad \mu_1,\mu_2\in M_+(\mathbb{E}),$$

for some sequence of functions $f_k \in C_K^+(\mathbb{E})$, where $a \wedge b = \min\{a,b\}$. We call d_v the *vague metric*.

A *Radon point measure* is an element of $M_+(\mathbb{E})$ of the form $m = \sum_i \delta_{x_i}$, where δ_x is the Dirac measure. Denote by $M_p(\mathbb{E})$ the class of all Radon point measures. Since $M_p(\mathbb{E})$ is a subset of $M_+(\mathbb{E})$, we endow it with the relative topology. Let $\mathcal{M}_p(\mathbb{E})$ be the Borel σ-field of subsets of $M_p(\mathbb{E})$ generated by open sets. A *point process* on $\mathbb{E}$ is a measurable map from a given probability space to the measurable space $\big(M_p(\mathbb{E}),\mathcal{M}_p(\mathbb{E})\big)$. A standard example of point process is the Poisson process. Suppose μ is a given Radon measure on $\mathbb{E}$. Then N is a Poisson process with mean (intensity) measure μ or, synonymously, a Poisson random measure ($\mathrm{PRM}(\mu)$) if for all $A \in \mathcal{B}(\mathbb{E})$

$$\mathrm{P}\big(N(A)=k\big)=\begin{cases}\exp\big(-\mu(A)\big)\big(\mu(A)\big)^k/k! & \text{for } \mu(A)<\infty,\\ 0 & \text{for } \mu(A)=\infty,\end{cases}$$

and if $A_1,\dots,A_k \in \mathcal{B}(\mathbb{E})$ are mutually disjoint, then $N(A_1),\dots,N(A_k)$ are independent random variables.

A sequence of point processes (N_n) on $\mathbb{E}$ *converges in distribution* to a point process N on $\mathbb{E}$ (written $N_n \xrightarrow{d} N$) if $\mathrm{E}f(N_n) \to \mathrm{E}f(N)$ for every bounded continuous function $f\colon M_p(\mathbb{E}) \to \mathbb{R}$. The point processes convergence is characterized by convergence of Laplace functionals. Denote by $\mathcal{B}_+$ the set of bounded measurable functions $f\colon \mathbb{E} \to [0,\infty)$. For a point process N on $\mathbb{E}$ the *Laplace functional* of N is the nonnegative function on $\mathcal{B}_+$ given by

$$\Psi_N(f)=\mathrm{E}e^{-N(f)}, \quad f \in \mathcal{B}_+,$$

where $N(f)=\int_{\mathbb{E}} f(x)N(dx)$. Then it follows that given point processes N_n, $n \geqslant 0$,

$$N_n \xrightarrow{d} N_0 \quad \text{iff} \quad \Psi_{N_n}(f) \to \Psi_{N_0}(f) \text{ for all } f \in C_K^+(\mathbb{E}) \tag{2.3}$$

(see Kallenberg [12], Theorem 4.2).

Let (X_n) be a strictly stationary sequence of regularly varying random variables with index $\alpha \in (0,2)$, and let (a_n) be a sequence of positive real numbers such that $n\mathrm{P}(|X_1|>a_n) \to 1$ as $n \to \infty$. Regular variation can be expressed in terms of vague convergence of measures on $\mathbb{E}$:

$$n\mathrm{P}(a_n^{-1}X_1 \in \cdot\,) \xrightarrow{v} \mu(\,\cdot\,) \quad \text{as } n \to \infty,$$

the Radon measure μ on $\mathbb{E}$ being given by

$$\mu(dx) = \big(p\alpha x^{-\alpha-1}1_{(0,\infty)}(x) + q\alpha(-x)^{-\alpha-1}1_{(-\infty,0)}(x)\big)dx,$$

where p and q are as in (1.2).

Using standard regular variation arguments it can be shown that for every $\lambda > 0$ it follows that $a_{\lfloor \lambda n \rfloor}/a_n \to \lambda^{1/\alpha}$ as $n \to \infty$. Therefore, a_n can be represented as $a_n = n^{1/\alpha}L'(n)$, where $L'(\,\cdot\,)$ is a slowly varying function at infinity.

Throughout the whole paper we will assume the sequence (X_n) satisfies the following large deviation type relations:

$$\text{(2.4)} \qquad \begin{aligned} k_n \mathrm{P}(S_{r_n} > x a_n) &\to\ c_+ x^{-\alpha}, \\ k_n \mathrm{P}(S_{r_n} < -x a_n) &\to\ c_- x^{-\alpha}, \end{aligned} \qquad x > 0,$$

as $n \to \infty$, where $c_+, c_- \geqslant 0$ are some constants, (r_n) is a sequence of positive integers such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$, and $k_n = \lfloor n/r_n \rfloor$. Some sufficient conditions for the relations in (2.4) to hold are given in Bartkiewicz et al. [2] and Davis and Hsing [7]. It is easy to see (for example, by Lemma 6.1 in Resnick [17]) that (2.4) is equivalent to

$$\text{(2.5)} \qquad k_n \mathrm{P}(a_n^{-1} S_{r_n} \in \cdot\,) \xrightarrow{v} \nu(\,\cdot\,) \quad \text{as } n \to \infty,$$

where ν is the measure,

$$\text{(2.6)} \qquad \nu(dx) = \big(c_+\alpha x^{-\alpha-1}1_{(0,\infty)}(x) + c_-\alpha(-x)^{-\alpha-1}1_{(-\infty,0)}(x)\big)dx.$$

LEMMA 2.1. *Let $\alpha \in (0,1)$ and assume the relation* (2.5) *holds. Then, for any $u > 0$,*

$$\lim_{n\to\infty} k_n \mathrm{E}\bigg(\frac{|S_{r_n}|}{a_n} 1_{\{|S_{r_n}|/a_n \leqslant u\}}\bigg) = \int\limits_{|x|\leqslant u} |x|\nu(dx).$$

P r o o f. Fix $u > 0$. Define

$$\nu_n(\,\cdot\,) = k_n \mathrm{P}(a_n^{-1} S_{r_n} \in \cdot\,), \quad n \in \mathbb{N},$$

and

$$f_\delta(x) = |x| 1_{\overline{B}(\delta,\,u)}(x), \quad x \in \mathbb{E},\ \delta \in (0,u),$$

where $B(\delta,\,u) = \{x \in \mathbb{E} : \delta < |x| < u\}$ (and $\overline{B}(\delta,\,u) = \{x \in \mathbb{E} : \delta \leqslant |x| \leqslant u\}$). By the relation (2.5) we have $\nu_n \xrightarrow{v} \nu$ as $n \to \infty$, and this yields

$$\text{(2.7)} \qquad \int\limits_{\mathbb{E}} f_\delta(x)\nu_n(dx) \to \int\limits_{\mathbb{E}} f_\delta(x)\nu(dx)$$

as $n \to \infty$ (see Kallenberg [12], 15.7.3). Define

$$f(x) = |x| 1_{\overline{B}(u)}(x), \qquad x \in \mathbb{E},$$

where $B(r) = \{x \in \mathbb{E} : |x| < r\}$. For any $\delta \in (0, u)$ it follows that

$$(2.8) \quad \Big| \int_{\mathbb{E}} f(x)\nu_n(dx) - \int_{\mathbb{E}} f(x)\nu(dx) \Big|$$
$$\leqslant \Big| \int_{B(\delta)} f(x)\nu_n(dx) - \int_{B(\delta)} f(x)\nu(dx) \Big| + \Big| \int_{B(\delta)^c} f(x)\nu_n(dx) - \int_{B(\delta)^c} f(x)\nu(dx) \Big|$$
$$\leqslant \Big| \int_{B(\delta)} f(x)\nu_n(dx) \Big| + \Big| \int_{B(\delta)} f(x)\nu(dx) \Big|$$
$$+ \Big| \int_{\overline{B}(\delta, u)} f(x)\nu_n(dx) - \int_{\overline{B}(\delta, u)} f(x)\nu(dx) \Big|.$$

For the first term on the right-hand side of (2.8) we have

$$\Big| \int_{B(\delta)} f(x)\, \nu_n(dx) \Big| = \int_{\mathbb{E}} |x| 1_{B(\delta)}(x)\, \nu_n(dx) = k_n \mathrm{E}\bigg[\frac{|S_{r_n}|}{a_n} 1_{\{|S_{r_n}| < \delta a_n\}} \bigg]$$
$$= k_n \mathrm{E}\bigg[\frac{|S_{r_n}|}{a_n} 1_{\{|S_{r_n}| < \delta a_n\}} 1_{\{\cap_{j=1}^{r_n} \{|X_j| \leqslant \delta a_n\}\}} \bigg]$$
$$+ k_n \mathrm{E}\bigg[\frac{|S_{r_n}|}{a_n} 1_{\{|S_{r_n}| < \delta a_n\}} 1_{\{\cup_{j=1}^{r_n} \{|X_j| > \delta a_n\}\}} \bigg].$$

This term is bounded above by

$$k_n \mathrm{E}\bigg[\frac{|S_{r_n}|}{a_n} 1_{\{\cap_{j=1}^{r_n} \{|X_j| \leqslant \delta a_n\}\}} \bigg] + k_n \delta \mathrm{P}\Big(\bigcup_{j=1}^{r_n} \{|X_j| > \delta a_n\} \Big)$$
$$\leqslant k_n \sum_{j=1}^{r_n} \mathrm{E}\bigg[\frac{|X_j|}{a_n} 1_{\{|X_j| \leqslant \delta a_n\}} \bigg] + k_n \delta \sum_{j=1}^{r_n} \mathrm{P}(|X_j| > \delta a_n)$$
$$= k_n r_n \mathrm{E}\bigg[\frac{|X_1|}{a_n} 1_{\{|X_1| \leqslant \delta a_n\}} \bigg] + k_n r_n \delta \mathrm{P}(|X_1| > \delta a_n)$$
$$= \delta \cdot \frac{k_n r_n}{n} \cdot n\mathrm{P}(|X_1| > \delta a_n) \cdot \bigg[\frac{\mathrm{E}[|X_1| 1_{\{|X_1| \leqslant \delta a_n\}}]}{\delta a_n \mathrm{P}(|X_1| > \delta a_n)} + 1 \bigg].$$

From the definition of the sequences (r_n) and (k_n) we infer that $k_n r_n / n \to 1$ as $n \to \infty$. Since X_1 is a regularly varying random variable with index α, it follows immediately that $n\mathrm{P}(|X_1| > \delta a_n) \to \delta^{-\alpha}$ as $n \to \infty$. By Karamata's theorem we have

$$\lim_{n \to \infty} \frac{\mathrm{E}[|X_1| 1_{\{|X_1| \leqslant \delta a_n\}}]}{\delta a_n \mathrm{P}(|X_1| > \delta a_n)} = \frac{\alpha}{1 - \alpha}.$$

Hence we obtain

$$\limsup_{n\to\infty} \Big| \int_{B(\delta)} f(x)\nu_n(dx) \Big| \leqslant \delta^{1-\alpha}\bigg(\frac{\alpha}{1-\alpha}+1\bigg),$$

and therefore, since $\alpha \in (0,1)$,

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \Big| \int_{B(\delta)} f(x)\nu_n(dx) \Big| = 0. \tag{2.9}$$

By the representation of the measure ν in (2.6) we get

$$\int_{|x|<\delta} |x|\nu(dx) = (c_- + c_+)\frac{\alpha}{1-\alpha}\,\delta^{1-\alpha}.$$

Hence for the second term on the right-hand side of (2.8) we have

$$\Big| \int_{B(\delta)} f(x)\,\nu(dx) \Big| = \int_{|x|<\delta} |x|\nu(dx) \to 0 \quad \text{as } \delta \to 0. \tag{2.10}$$

By (2.7), for the third term on the right-hand side of (2.8) we get

$$\begin{aligned} \Big| \int_{\overline{B}(\delta,u)} f(x)\nu_n(dx) - \int_{\overline{B}(\delta,u)} f(x)\nu(dx) \Big| \\ = \Big| \int_{\mathbb{E}} f_\delta(x)\nu_n(dx) - \int_{\mathbb{E}} f_\delta(x)\nu(dx) \Big| \to 0 \end{aligned} \tag{2.11}$$

as $n \to \infty$. Now, from (2.8) using (2.9)–(2.11), we obtain

$$\lim_{\delta\to 0}\limsup_{n\to\infty} \Big| \int_{\mathbb{E}} f(x)\nu_n(dx) - \int_{\mathbb{E}} f(x)\nu(dx) \Big| = 0.$$

Hence we get immediately $\int_{\mathbb{E}} f(x)\nu_n(dx) \to \int_{\mathbb{E}} f(x)\nu(dx)$ as $n \to \infty$, i.e.,

$$k_n \mathrm{E}\bigg(\frac{|S_{r_n}|}{a_n} 1_{\{|S_{r_n}|/a_n \leqslant u\}}\bigg) \to \int_{|x|\leqslant u} |x|\,\nu(dx) \quad \text{as } n \to \infty. \ \blacksquare$$

3. MAIN THEOREM

Let (X_n) be a strictly stationary sequence of regularly varying random variables with index $\alpha \in (0,2)$. Assume (2.4) holds. The theorem below gives conditions under which a stochastic sum process constructed from the sequence $(S_{r_n}^k)$ satisfies a nonstandard functional limit theorem in the space $D[0,1]$ of real-valued càdlàg functions equipped with the Skorokhod J_1 topology, with a non-Gaussian α-stable Lévy process as a limit. Recall that the distribution of a Lévy process

$W(\,\cdot\,)$ is characterized by its characteristic triplet, i.e., the characteristic triplet of the infinitely divisible distribution of $W(1)$. The characteristic function of $W(1)$ and the characteristic triplet (a, μ, b) are related in the following way:

$$\mathrm{E}[e^{izW(1)}] = \exp\Big(-\frac{1}{2}az^2 + ibz + \int_{\mathbb{R}} \big(e^{izx} - 1 - izx1_{[-1,1]}(x)\big)\mu(dx)\Big)$$

for $z \in \mathbb{R}$; here $a \geqslant 0$, $b \in \mathbb{R}$ are constants, and μ is a measure on $\mathbb{R}$ satisfying

$$\mu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (|x|^2 \wedge 1)\mu(dx) < \infty,$$

that is, μ is a Lévy measure. For a textbook treatment of Lévy processes we refer to Bertoin [4] and Sato [20].

The metric d_{J_1} that generates the J_1 topology on $D[0,1]$ is defined in the following way. Let Δ be the set of strictly increasing continuous functions $\lambda\colon [0,1] \to [0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$, and let $e \in \Delta$ be the identity map on $[0,1]$, i.e., $e(t) = t$ for all $t \in [0,1]$. For $x, y \in D[0,1]$ define

$$d_{J_1}(x,y) = \inf\{\|x \circ \lambda - y\|_{[0,1]} \vee \|\lambda - e\|_{[0,1]} : \lambda \in \Delta\},$$

where $\|x\|_{[0,1]} = \sup\{|x(t)| : t \in [0,1]\}$ and $a \vee b = \max\{a, b\}$. Then d_{J_1} is a metric on $D[0,1]$ and is called the *Skorokhod J_1 metric*.

The mixing condition appropriate for the result in this section is similar to the condition $\mathcal{A}(a_n)$ of Davis and Hsing [7], and hence we denote it by $\mathcal{A}^*(a_n)$ and say that a strictly stationary sequence of random variables (X_n) satisfies the *mixing condition* $\mathcal{A}^*(a_n)$ if there exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$, and such that for every $f \in C_K^+(\mathbb{E})$ (putting $k_n = \lfloor n/r_n \rfloor$), as $n \to \infty$,

$$\mathrm{E}\exp\big(-\sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^k)\big) - \Big(\mathrm{E}\exp\big(-f(a_n^{-1}S_{r_n})\big)\Big)^{k_n} \to 0. \tag{3.1}$$

In case $\alpha \in [1,2)$, we will need to assume that the contribution of the smaller increments of the partial sum process is close to its expectation. We use the following "vanishing small values" condition which will be denoted by $\mathcal{VSV}(a_n)$: there exists a sequence of positive integers (r_n) with $r_n \to \infty$ and $k_n = \lfloor n/r_n \rfloor \to \infty$ as $n \to \infty$, such that for all $\delta > 0$

$$\lim_{u\downarrow 0}\limsup_{n\to\infty} \mathrm{P}\bigg[\max_{1\leqslant j\leqslant k_n}\bigg|\sum_{k=1}^{j}\bigg(\frac{S_{r_n}^k}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}} - \mathrm{E}\bigg(\frac{S_{r_n}^k}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}}\bigg)\bigg)\bigg| > \delta\bigg] = 0. \tag{3.2}$$

In the Appendix we discuss some sufficient conditions for the mixing condition $\mathcal{A}^*(a_n)$ and condition $\mathcal{VSV}(a_n)$ to hold.

THEOREM 3.1. *Let* (X_n) *be a strictly stationary sequence of regularly varying random variables with index* $\alpha \in (0,2)$, *and let* (a_n) *be a sequence of positive real numbers such that* $n\mathrm{P}(|X_1| > a_n) \to 1$ *as* $n \to \infty$. *Suppose there exists a sequence of positive integers* (r_n) *such that, as* $n \to \infty$, $r_n \to \infty$, $k_n = \lfloor n/r_n \rfloor \to \infty$ *and*

$$k_n \mathrm{P}\bigg(\frac{S_{r_n}}{a_n} \in \cdot\bigg) \xrightarrow{v} \nu(\,\cdot\,). \tag{3.3}$$

Suppose that the mixing condition $\mathcal{A}^*(a_n)$ *holds, and if* $\alpha \in [1,2)$, *then suppose that the condition* $\mathcal{VSV}(a_n)$ *also holds, both of them with the same sequence* (r_n) *as in* (3.3). *Then for a stochastic process defined by*

$$W_n(t) = \sum_{k=1}^{\lfloor k_n t \rfloor} \frac{S_{r_n}^k}{a_n} - \lfloor k_n t \rfloor \mathrm{E}\bigg(\frac{S_{r_n}}{a_n} 1_{\{|S_{r_n}|/a_n \leqslant 1\}}\bigg), \qquad t \in [0,1],$$

it follows that

$$W_n \xrightarrow{d} W_0 \quad \text{as } n \to \infty$$

in $D[0,1]$ *endowed with the* J_1 *topology, where* $W_0(\,\cdot\,)$ *is an* α*-stable Lévy process with characteristic triplet* $(0,\nu,0)$.

P r o o f. Let, for any $n \in \mathbb{N}$, $(Z_{n,k})_k$ be a sequence of i.i.d. random variables such that $Z_{n,1} \overset{d}{=} S_{r_n}$. By the relation (3.3) we have

$$k_n \mathrm{P}\bigg(\frac{Z_{n,1}}{a_n} \in \cdot\bigg) \xrightarrow{v} \nu(\,\cdot\,) \quad \text{as } n \to \infty.$$

Theorem 5.3 in Resnick [17] then implies, as $n \to \infty$,

$$\widetilde{\xi}_n := \sum_{k=1}^{k_n} \delta_{a_n^{-1} Z_{n,k}} \xrightarrow{d} \mathrm{PRM}(\nu) \tag{3.4}$$

on $\mathbb{E}$. Define the point process $\xi_n = \sum_{k=1}^{k_n} \delta_{a_n^{-1} S_{r_n}^k}$. For any $f \in C_K^+(\mathbb{E})$ we have

$$\begin{aligned}\Psi_{\xi_n}(f) - \Psi_{\widetilde{\xi}_n}(f) &= \mathrm{E}\exp\Big(-\sum_{k=1}^{k_n} f(a_n^{-1} S_{r_n}^k)\Big) - \Big(\mathrm{E}\exp\big(-f(a_n^{-1} Z_{n,1})\big)\Big)^{k_n} \\ &= \mathrm{E}\exp\Big(-\sum_{k=1}^{k_n} f(a_n^{-1} S_{r_n}^k)\Big) - \Big(\mathrm{E}\exp\big(-f(a_n^{-1} S_{r_n})\big)\Big)^{k_n}.\end{aligned}$$

Hence, the mixing condition $\mathcal{A}^*(a_n)$ implies $\Psi_{\xi_n}(f) - \Psi_{\widetilde{\xi}_n}(f) \to 0$ as $n \to \infty$. Then by the relations (2.3) and (3.4) we obtain, as $n \to \infty$,

$$\sum_{k=1}^{k_n} \delta_{a_n^{-1} S_{r_n}^k} \xrightarrow{d} \mathrm{PRM}(\nu). \tag{3.5}$$

Suppose $U_1, \ldots, U_{k_n}$ are i.i.d. random variables uniformly distributed on $[0,1]$ with order statistics $U_{1:k_n} \leqslant U_{2:k_n} \leqslant \ldots \leqslant U_{k_n:k_n}$ which are independent of $(S_{r_n}^k)$. From (3.5), using Lemma 4.3 of Resnick [16], we obtain, as $n \to \infty$,

$$\sum_{k=1}^{k_n} \delta_{(U_k,\, a_n^{-1} S_{r_n}^k)} \xrightarrow{d} \mathrm{PRM}(\mathbb{LEB} \times \nu).$$

By the independence of (U_k) and $(S_{r_n}^k)$, we have

$$\sum_{k=1}^{k_n} \delta_{(U_{k:k_n},\, a_n^{-1} S_{r_n}^k)} \stackrel{d}{=} \sum_{k=1}^{k_n} \delta_{(U_k,\, a_n^{-1} S_{r_n}^k)}$$

as random elements of $M_+([0,1] \times \mathbb{E})$. Therefore,

$$\sum_{k=1}^{k_n} \delta_{(U_{k:k_n},\, a_n^{-1} S_{r_n}^k)} \xrightarrow{d} \mathrm{PRM}(\mathbb{LEB} \times \nu). \tag{3.6}$$

Using the arguments from Step 3 in the proof of Theorem 6.3 in Resnick [17] we get

$$d_v\Big(\sum_{k=1}^{k_n} \delta_{(k/k_n,\, a_n^{-1} S_{r_n}^k)}, \sum_{k=1}^{k_n} \delta_{(U_{k:k_n},\, a_n^{-1} S_{r_n}^k)} \Big) \xrightarrow{P} 0 \qquad \text{as } n \to \infty, \tag{3.7}$$

where d_v is the vague metric on $M_+([0,1] \times \mathbb{E})$ (cf. (2.2)). From (3.6) and (3.7), using Slutsky's theorem (see, for instance, Theorem 3.4 in Resnick [17]), we obtain, as $n \to \infty$,

$$\sum_{k=1}^{k_n} \delta_{(k/k_n,\, a_n^{-1} S_{r_n}^k)} \xrightarrow{d} \mathrm{PRM}(\mathbb{LEB} \times \nu) = \sum_k \delta_{(t_k, j_k)}$$

on $[0,1] \times \mathbb{E}$. Hence, using the same arguments as in the proof of Theorem 7.1 in Resnick [17], we obtain, as $n \to \infty$,

$$W_n^{(u)} \xrightarrow{d} W_0^{(u)} \tag{3.8}$$

in $D[0,1]$ with the J_1 topology, where

$$W_n^{(u)}(\,\cdot\,) := \sum_{k=1}^{\lfloor k_n \cdot \rfloor} \frac{S_{r_n}^k}{a_n} 1_{\{|S_{r_n}^k|/a_n > u\}} - \lfloor k_n \cdot \rfloor \mathrm{E}\bigg(\frac{S_{r_n}}{a_n} 1_{\{u < |S_{r_n}|/a_n \leqslant 1\}} \bigg),$$

and

$$W_0^{(u)}(\,\cdot\,) := \sum_{t_k \leqslant \cdot} j_k 1_{\{|j_k| > u\}} - (\,\cdot\,) \int_{u < |x| \leqslant 1} x \nu(dx).$$

By the Lévy–Itô representation of a Lévy process (see Section 5.5.3 in Resnick [17] or Theorem 19.2 in Sato [20]), there exists a Lévy process $W_0(\,\cdot\,)$ with characteristic triplet $(0,\nu,0)$ such that

$$\sup_{t\in[0,1]} |W_0^{(u)}(t) - W_0(t)| \to 0$$

almost surely as $u \downarrow 0$. Since uniform convergence implies Skorokhod J_1 convergence, we get $d_{J_1}(W_0^{(u)}, W_0) \to 0$ almost surely as $u \downarrow 0$. Therefore, since almost sure convergence implies convergence in distribution,

$$W_0^{(u)} \xrightarrow{d} W_0 \quad \text{as } u \to 0 \tag{3.9}$$

in $D[0,1]$ with the J_1 topology.

If we show that

$$\lim_{u\downarrow 0} \limsup_{n\to\infty} \mathrm{P}[d_{J_1}(W_n^{(u)}, W_n) > \delta] = 0$$

for any $\delta > 0$, then from (3.8), (3.9) and Theorem 3.5 in Resnick [17] we will have, as $n \to \infty$,

$$W_n \xrightarrow{d} W_0$$

in $D[0,1]$ with the J_1 topology. Since the J_1 metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$, it suffices to show that

$$\lim_{u\downarrow 0} \limsup_{n\to\infty} \mathrm{P}\Big(\sup_{t\in[0,1]} |W_n^{(u)}(t) - W_n(t)| > \delta \Big) = 0.$$

We have

$$\begin{aligned} &\mathrm{P}\Big(\sup_{t\in[0,1]} |W_n^{(u)}(t) - W_n(t)| > \delta \Big) \\ &\quad = \mathrm{P}\bigg[\max_{1\leqslant j\leqslant k_n} \bigg| \sum_{k=1}^{j} \bigg(\frac{S_{r_n}^k}{a_n} 1_{\{|S_{r_n}^k|/a_n \leqslant u\}} - \mathrm{E}\bigg(\frac{S_{r_n}^k}{a_n} 1_{\{|S_{r_n}^k|/a_n \leqslant u\}} \bigg) \bigg) \bigg| > \delta \bigg]. \end{aligned} \tag{3.10}$$

For $\alpha \in [1,2)$ this relation is simply the condition $\mathcal{VSV}(a_n)$. Therefore, it remains to show (3.10) for the case when $\alpha \in (0,1)$. Hence assume $\alpha \in (0,1)$. For arbitrary (and fixed) $\delta > 0$ define

$$I(u,n) = \mathrm{P}\bigg[\max_{1\leqslant j\leqslant k_n} \bigg| \sum_{k=1}^{j} \bigg(\frac{S_{r_n}^k}{a_n} 1_{\{|S_{r_n}^k|/a_n \leqslant u\}} - \mathrm{E}\bigg(\frac{S_{r_n}^k}{a_n} 1_{\{|S_{r_n}^k|/a_n \leqslant u\}} \bigg) \bigg) \bigg| > \delta \bigg].$$

Using stationarity and Chebyshev's inequality we get the bound

$$
\begin{aligned}
I(u,n) &\leqslant \mathrm{P}\biggl[\max_{1\leqslant j\leqslant k_n}\sum_{k=1}^{j}\biggl|\frac{S_{r_n}^k}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}}-\mathrm{E}\biggl(\frac{S_{r_n}^k}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}}\biggr)\biggr|>\delta\biggr]\\
&=\mathrm{P}\biggl[\sum_{k=1}^{k_n}\biggl|\frac{S_{r_n}^k}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}}-\mathrm{E}\biggl(\frac{S_{r_n}^k}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}}\biggr)\biggr|>\delta\biggr]\\
&\leqslant \delta^{-1}\mathrm{E}\biggl[\sum_{k=1}^{k_n}\biggl|\frac{S_{r_n}^k}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}}-\mathrm{E}\biggl(\frac{S_{r_n}^k}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}}\biggr)\biggr|\biggr]\\
&\leqslant 2\delta^{-1}\sum_{k=1}^{k_n}\mathrm{E}\biggl(\frac{|S_{r_n}^k|}{a_n}1_{\{|S_{r_n}^k|/a_n\leqslant u\}}\biggr)\\
&=2\delta^{-1}k_n\mathrm{E}\biggl(\frac{|S_{r_n}|}{a_n}1_{\{|S_{r_n}|/a_n\leqslant u\}}\biggr).
\end{aligned}
$$

Using Lemma 2.1 we obtain

$$
\begin{aligned}
\lim_{n\to\infty}k_n\mathrm{E}\biggl(\frac{|S_{r_n}|}{a_n}1_{\{|S_{r_n}|/a_n\leqslant u\}}\biggr)&=\int_{|x|\leqslant u}|x|\nu(dx)\\
&=(c_-+c_+)\frac{\alpha}{1-\alpha}u^{1-\alpha}\to 0 \quad \text{as } u\to 0.
\end{aligned}
$$

Hence

$$\lim_{u\downarrow 0}\limsup_{n\to\infty}I(u,n)=0,$$

which completes the proof, with the note that the α-stability of the process $W_0(\,\cdot\,)$ follows from Theorem 14.3 in Sato [20] and the representation of the measure ν in (2.6). ∎

REMARK 3.1. *Theorem* 3.1 *covers a wide range of stationary sequences. In Bartkiewicz et al.* [2], *some sufficient conditions for the relation* (3.3) *to hold are given* (see their Theorem 1 and Section 3.2.2 in [2]) *as well as several examples of standard time series models that satisfy these conditions, including m-dependent sequences, GARCH$(1,1)$ process and its squares, solutions to stochastic recurrence equations and stochastic volatility models. These conditions are the following*:

(C1) *The process (X_n) is regularly varying with index $\alpha\in(0,2)$, i.e., for every $d\geqslant 1$, the d-dimensional random vector $\mathbf{X}=(X_1,\dots,X_d)$ is multivariate regularly varying with index α. This means that for some (and then for every) norm $\|\cdot\|$ on $\mathbb{R}^d$ there exists a random vector $\mathbf{\Theta}$ on the unit sphere $\mathbb{S}^{d-1}=\{\mathbf{x}\in\mathbb{R}^d:\|\mathbf{x}\|=1\}$ such that, for every $u>0$ and as $x\to\infty$,*

$$\frac{\mathrm{P}(\|\mathbf{X}\|>ux,\mathbf{X}/\|\mathbf{X}\|\in\cdot\,)}{\mathrm{P}(\|\mathbf{X}\|>x)}\xrightarrow{w}u^{-\alpha}\mathrm{P}(\mathbf{\Theta}\in\cdot\,),$$

where the arrow "$\xrightarrow{w}$" denotes weak convergence of finite measures.

(C2) *There exists a sequence of positive integers* (r_n) *such that* $r_n \to \infty$, $k_n = \lfloor n/r_n \rfloor \to \infty$ *and, for every* $x \in \mathbb{R}$,

$$\left|\varphi_n(x) - \big(\varphi_{nr_n}(x)\big)^{k_n}\right| \to 0$$

as $n \to \infty$, *where* $\varphi_{nj}(x) = \mathrm{E}e^{ixa_n^{-1}S_j}$, $j = 1, 2, \ldots$, *and* $\varphi_n(x) = \varphi_{nn}(x)$.

(C3) *For every* $x \in \mathbb{R}$,

$$\lim_{d\to\infty} \limsup_{n\to\infty} \frac{n}{r_n} \sum_{j=d+1}^{r_n} \mathrm{E}\left|\overline{xa_n^{-1}(S_j - S_d)} \cdot \overline{xa_n^{-1}X_1}\right| = 0,$$

where the sequence $(r_n)_n$ *is the same as in* (C2) *and for an arbitrary random variable* Z *we put* $\overline{Z} = (Z \wedge 2) \vee (-2)$.

(C4) *The limits*

$$\lim_{n\to\infty} n\mathrm{P}(S_d > a_n) = b_+(d) \quad \textit{and} \quad \lim_{n\to\infty} n\mathrm{P}(S_d \leqslant -a_n) = b_-(d), \qquad d \geqslant 1,$$

$$\lim_{d\to\infty} \big(b_+(d) - b_+(d-1)\big) = c_+ \quad \textit{and} \quad \lim_{d\to\infty} \big(b_-(d) - b_-(d-1)\big) = c_-$$

exist.

(C5) *For* $\alpha > 1$ *assume* $\mathrm{E}X_1 = 0$ *and, for* $\alpha = 1$,

$$\lim_{d\to\infty} \limsup_{n\to\infty} n \left|\mathrm{E}\big(\sin(a_n^{-1}S_d)\big)\right| = 0.$$

With appropriate (*and standard*) *assumptions*, *which are precisely described in* [2], *the above-mentioned time series models are strongly mixing with geometric rate*, *which suffices the mixing condition* $\mathcal{A}^*(a_n)$ *to hold* (see Proposition 4.1 below). *Therefore*, *for* $\alpha \in (0, 1)$, *all conditions of Theorem* 3.1 *are satisfied and the conclusion of the theorem follows. Naturally*, *for* $\alpha \in [1, 2)$ *one has also to verify the condition* $\mathcal{VSV}(a_n)$.

4. APPENDIX

In this section we give some sufficient conditions for the mixing condition $\mathcal{A}^*(a_n)$ and condition $\mathcal{VSV}(a_n)$ to hold. These conditions are principally based on the well-known strong or α-mixing and ρ-mixing conditions. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. For any σ-field $\mathcal{A} \subset \mathcal{F}$, let $L_2(\mathcal{A})$ denote the space of square-integrable, $\mathcal{A}$-measurable, real-valued random variables. For any two σ-fields $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ define

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B)| : A \in \mathcal{A}, B \in \mathcal{B}\}$$

and

$$\rho(\mathcal{A}, \mathcal{B}) = \sup\left\{\frac{|\mathrm{E}(XY) - \mathrm{E}X\mathrm{E}Y|}{\sqrt{\mathrm{E}X^2\mathrm{E}Y^2}} : X \in L_2(\mathcal{A}), Y \in L_2(\mathcal{B})\right\}.$$

Let now $(X_n)_{n\in\mathbb{Z}}$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathrm{P})$, and write $\mathcal{F}_k^l = \sigma(\{X_i : k \leqslant i \leqslant l\})$ for $-\infty \leqslant k \leqslant l \leqslant \infty$. Then we say the sequence $(X_n)_n$ is α*-mixing* (or *strongly mixing*) if

$$\alpha(n) = \sup_{j\in\mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^j, \mathcal{F}_{j+n}^\infty) \to 0,$$

and $(X_n)_n$ is ρ*-mixing* if

$$\rho(n) = \sup_{j\in\mathbb{Z}} \rho(\mathcal{F}_{-\infty}^j, \mathcal{F}_{j+n}^\infty) \to 0$$

as $n \to \infty$. Note that when the sequence (X_n) is strictly stationary, one has simply $\alpha(n) = \alpha(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^\infty)$, and similarly for $\rho(n)$.

PROPOSITION 4.1. *Suppose (X_n) is a strictly stationary sequence of regularly varying random variables with index $\alpha \in (0,2)$, and (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Assume the relation* (3.3) *holds for some sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$, and $k_n = \lfloor n/r_n \rfloor = o(n^t)$ for some $0 < t < 1$. If the sequence (X_n) is strongly mixing with exponential rate, i.e., $\alpha_n \leqslant C\rho^n$ for some $\rho \in (0,1)$ and $C > 0$, where (α_n) is the sequence of α-mixing coefficients of (X_n), then the mixing condition $\mathcal{A}^*(a_n)$ holds.*

Proof. Let (l_n) be an arbitrary (and fixed) sequence of positive real numbers such that $l_n \sim n^q$, i.e.,

$$l_n/n^q \to 1 \text{ as } n \to \infty, \qquad \text{where } q = \min\{1/\alpha, (1-t)/(1+\alpha)\}/2.$$

Let n be large enough such that $l_n < r_n$ (note that for large n it follows that $l_n < n^{1-t} < r_n$). We break $X_1, X_2, \ldots$ into blocks of r_n consecutive random variables. The last l_n variables in each block will be dropped. Then we shall show that doing so, the new blocks will be almost independent (as $n \to \infty$) and this will imply the relation (3.1) for the new blocks. The error which occurs by cutting off the ends of the original blocks will be small, and this will imply the condition (3.1) also for the original blocks.

Take an arbitrary $f \in C_K^+(\mathbb{E})$. Since its support is bounded away from zero, there exists some $r > 0$ such that $f(x) = 0$ for $|x| \leqslant r$, and since f is bounded, there exists some $M > 0$ such that $|f(x)| < M$ for all $x \in \mathbb{E}$. For all $k, n \in \mathbb{N}$ define

$$S_{r_n,l_n}^k = X_{kr_n - l_n + 1} + \ldots + X_{kr_n}.$$

S_{r_n,l_n}^k is the sum of the last l_n random variables in the k-th block. By stationarity we have

$$S_{r_n}^k - S_{r_n,l_n}^k \stackrel{d}{=} S_{r_n}^1 - S_{r_n,l_n}^1 = S_{r_n - l_n}.$$

This relation and the inequality

$$|\mathrm{E}gh - \mathrm{E}g\mathrm{E}h| \leqslant 4C_1C_2\alpha_m$$

for an $\mathcal{F}_{-\infty}^{j}$-measurable function g and an $\mathcal{F}_{j+m}^{\infty}$-measurable function h such that $|g| \leqslant C_1$ and $|h| \leqslant C_2$ (see Lemma 1.2.1 in Lin and Lu [14]), applied k_n times, give

$$\Big|\mathrm{E}\exp\big(-\sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^k - a_n^{-1}S_{r_n,l_n}^k)\big) - \Big(\mathrm{E}\exp\big(-f(a_n^{-1}S_{r_n-l_n})\big)\Big)^{k_n}\Big| \leqslant 4k_n\alpha_{l_n+1}. \tag{4.1}$$

Then

$$\Big|\mathrm{E}\exp\big(-\sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^k)\big) - \Big(\mathrm{E}\exp\big(-f(a_n^{-1}S_{r_n})\big)\Big)^{k_n}\Big| \tag{4.2}$$

$$\leqslant \big|\mathrm{E}\exp\big(-\sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^k)\big) - \mathrm{E}\exp\big(-\sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^k - a_n^{-1}S_{r_n,l_n}^k)\big)\big|$$

$$+ \Big|\mathrm{E}\exp\big(-\sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^k - a_n^{-1}S_{r_n,l_n}^k)\big) - \Big(\mathrm{E}\exp\big(-f(a_n^{-1}S_{r_n-l_n})\big)\Big)^{k_n}\Big|$$

$$+ \Big|\Big(\mathrm{E}\exp\big(-f(a_n^{-1}S_{r_n-l_n})\big)\Big)^{k_n} - \Big(\mathrm{E}\exp\big(-f(a_n^{-1}S_{r_n})\big)\Big)^{k_n}\Big|$$

$$=: I_1(n) + I_2(n) + I_3(n).$$

By Lemma 4.3 in Durrett [8] and stationarity we have

$$\begin{aligned} I_1(n) &\leqslant \mathrm{E}\big(\sum_{k=1}^{k_n} |e^{-f(a_n^{-1}S_{r_n}^k)} - e^{-f(a_n^{-1}S_{r_n}^k - a_n^{-1}S_{r_n,l_n}^k)}|\big) \\ &= k_n\mathrm{E}|e^{-f(a_n^{-1}S_{r_n})} - e^{-f(a_n^{-1}S_{r_n-l_n})}| \\ &= k_n\mathrm{E}|e^{-f(a_n^{-1}S_{r_n})}(1 - e^{f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})})| \\ &\leqslant k_n\mathrm{E}|1 - e^{f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})}|. \end{aligned}$$

It can be shown that for any $t > 0$ there exists a constant $C = C(t) > 0$ such that $|1 - e^{-x}| \leqslant C|x|$ for all $|x| < t$. Since for all $x, y \in \mathbb{E}$ we have $|f(x) - f(y)| < 2M$, there exists a positive constant C such that

$$I_1(n) \leqslant Ck_n\mathrm{E}|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})|. \tag{4.3}$$

Further,

$$
\begin{aligned}
(4.4)\quad & \mathrm{E}|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})| \\
&= \mathrm{E}[|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})| 1_{\{a_n^{-1}|S_{r_n-l_n}|>r/2\}} 1_{\{a_n^{-1}|S_{r_n}|>r/4\}}] \\
&\quad + \mathrm{E}[f(a_n^{-1}S_{r_n-l_n}) 1_{\{a_n^{-1}|S_{r_n-l_n}|>r/2\}} 1_{\{a_n^{-1}|S_{r_n}|\leqslant r/4\}}] \\
&\quad + \mathrm{E}[f(a_n^{-1}S_{r_n}) 1_{\{a_n^{-1}|S_{r_n-l_n}|\leqslant r/2\}} 1_{\{a_n^{-1}|S_{r_n}|>r\}}] \\
&\leqslant \mathrm{E}[|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})| 1_{\{a_n^{-1}|S_{r_n-l_n}|>r/2\}} 1_{\{a_n^{-1}|S_{r_n}|>r/4\}}] \\
&\quad + M\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n} > \frac{r}{4}\bigg) + M\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n} > \frac{r}{2}\bigg).
\end{aligned}
$$

Since the set $S = \{x \in \mathbb{E} : |x| > r/4\}$ is relatively compact and any continuous function on a compact set is uniformly continuous, it follows that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in S$ such that $\rho(x, y) \leqslant \delta$, where ρ is the metric on $\mathbb{E}$ defined in (2.1). If $|x| > r/2$, $|y| > r/4$ and $\mathrm{sign}(x) = \mathrm{sign}(y)$, then $x, y \in S$ and

$$
\rho(x, y) = \frac{\big||x| - |y|\big|}{|xy|} \leqslant \frac{8}{r^2}|x - y|. \tag{4.5}
$$

Define

$$
g_n(x, y) = |f(a_n^{-1}x) - f(a_n^{-1}y)|.
$$

Let $\epsilon > 0$ be arbitrary. Then

$$
\begin{aligned}
&\mathrm{E}[|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})| 1_{\{a_n^{-1}|S_{r_n-l_n}|>r/2\}} 1_{\{a_n^{-1}|S_{r_n}|>r/4\}}] \\
&= \mathrm{E}[g_n(S_{r_n}, S_{r_n-l_n}) 1_{\{a_n^{-1}|S_{r_n-l_n}|>r/2,\, a_n^{-1}|S_{r_n}|>r/4\}} 1_{\{\mathrm{sign}(S_{r_n-l_n}) \neq \mathrm{sign}(S_{r_n})\}}] \\
&\quad + \mathrm{E}[g_n(S_{r_n}, S_{r_n-l_n}) 1_{\{a_n^{-1}S_{r_n-l_n}>r/2,\, a_n^{-1}S_{r_n}>r/4\}} 1_{\{a_n^{-1}|S_{r_n}-S_{r_n-l_n}|\leqslant \delta r^2/8\}}] \\
&\quad + \mathrm{E}[g_n(S_{r_n}, S_{r_n-l_n}) 1_{\{a_n^{-1}S_{r_n-l_n}<-r/2,\, a_n^{-1}S_{r_n}<-r/4\}} 1_{\{a_n^{-1}|S_{r_n}-S_{r_n-l_n}|\leqslant \delta r^2/8\}}] \\
&\quad + \mathrm{E}[g_n(S_{r_n}, S_{r_n-l_n}) 1_{\{a_n^{-1}S_{r_n-l_n}>r/2,\, a_n^{-1}S_{r_n}>r/4\}} 1_{\{a_n^{-1}|S_{r_n}-S_{r_n-l_n}|>\delta r^2/8\}}] \\
&\quad + \mathrm{E}[g_n(S_{r_n}, S_{r_n-l_n}) 1_{\{a_n^{-1}S_{r_n-l_n}<-r/2,\, a_n^{-1}S_{r_n}<-r/4\}} 1_{\{a_n^{-1}|S_{r_n}-S_{r_n-l_n}|>\delta r^2/8\}}].
\end{aligned}
$$

By stationarity and the relation (4.5) this is bounded above by

$$2M\mathrm{P}\bigg(\frac{|S_{r_n}-S_{r_n-l_n}|}{a_n}>\frac{3r}{4}\bigg)$$

$$+\,\mathrm{E}[g_n(S_{r_n},S_{r_n-l_n})1_{\{a_n^{-1}S_{r_n-l_n}>r/2\}}1_{\{a_n^{-1}S_{r_n}>r/4\}}1_{\{\rho(a_n^{-1}S_{r_n},\,a_n^{-1}S_{r_n-l_n})\leqslant\delta\}}]$$

$$+\,\mathrm{E}[g_n(S_{r_n},S_{r_n-l_n})1_{\{a_n^{-1}S_{r_n-l_n}<-r/2\}}1_{\{a_n^{-1}S_{r_n}<-r/4\}}1_{\{\rho(a_n^{-1}S_{r_n},\,a_n^{-1}S_{r_n-l_n})\leqslant\delta\}}]$$

$$+\,4M\mathrm{P}\bigg(\frac{|S_{r_n}-S_{r_n-l_n}|}{a_n}>\frac{\delta r^2}{8}\bigg)$$

$$\leqslant 2M\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n}>\frac{3r}{4}\bigg)+\epsilon\mathrm{P}\bigg(\frac{|S_{r_n}|}{a_n}>\frac{r}{4}\bigg)+4M\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n}>\frac{\delta r^2}{8}\bigg).$$

Therefore, from (4.3) and (4.4) we obtain

$$I_1(n)\leqslant 8MCk_n\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n}>\gamma\bigg)+\epsilon Ck_n\mathrm{P}\bigg(\frac{|S_{r_n}|}{a_n}>\frac{r}{4}\bigg), \tag{4.6}$$

where $\gamma=\min\{r/4,\delta r^2/8\}>0$.

Recall that, since X_1 is regularly varying with index $\alpha\in(0,2)$, it follows that $\mathrm{P}(|X_1|>x)=x^{-\alpha}L(x)$ for any $x>0$, where $L(\,\cdot\,)$ is a slowly varying function. It follows also that $a_n=n^{1/\alpha}L'(n)$, where $L'(\,\cdot\,)$ is a slowly varying function. Hence, taking an arbitrary $0<s<\min\{\alpha,\,\alpha(1-t-q-\alpha q)/(1-\alpha q)\}$, we have

$$k_n\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n}>\gamma\bigg)\leqslant k_nl_n\mathrm{P}(|X_1|>\gamma a_n/l_n)=k_nl_n\bigg(\frac{\gamma a_n}{l_n}\bigg)^{-\alpha}L\bigg(\frac{\gamma a_n}{l_n}\bigg)$$
$$=k_nl_n\bigg(\frac{\gamma a_n}{l_n}\bigg)^{s-\alpha}\cdot c_n,$$

where

$$c_n=\bigg(\frac{\gamma a_n}{l_n}\bigg)^{-s}L\bigg(\frac{\gamma a_n}{l_n}\bigg).$$

Since $a_n/l_n\to\infty$ as $n\to\infty$, by Proposition 1.3.6 in Bingham et al. [5] we infer that $c_n\to 0$ as $n\to\infty$. Further,

$$k_nl_n\bigg(\frac{\gamma a_n}{l_n}\bigg)^{s-\alpha}=\frac{k_n(l_n)^{1+\alpha-s}}{\gamma^{\alpha-s}a_n^{\alpha-s}}=\bigg(\frac{l_n}{n^q}\bigg)^{1+\alpha-s}\cdot\frac{k_n}{n^t}\cdot\frac{n^t(n^q)^{1+\alpha-s}}{\gamma^{\alpha-s}n^{(\alpha-s)/\alpha}\big(L'(n)\big)^{\alpha-s}}$$
$$\leqslant\bigg(\frac{l_n}{n^q}\bigg)^{1+\alpha-s}\cdot\frac{k_n}{n^t}\cdot\frac{1}{\gamma^{\alpha-s}n^p\big(L'(n)\big)^{\alpha-s}},$$

where $p = (\alpha - s)/\alpha - t - (1 + \alpha - s)q$. It can easily be checked that $p > 0$. This and the fact that $l_n \sim n^q$ and $k_n = o(n^t)$, by Proposition 1.3.6 in Bingham et al. [5], imply that $k_n l_n (\gamma a_n / l_n)^{s-\alpha} \to 0$ as $n \to \infty$. Hence

$$k_n \mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n} > \gamma \bigg) \to 0 \quad \text{as } n \to \infty. \tag{4.7}$$

From the relation (3.3) we infer that, as $n \to \infty$,

$$k_n \mathrm{P}\bigg(\frac{|S_{r_n}|}{a_n} > \frac{r}{4} \bigg) \to \nu(\{x \in \mathbb{E} : |x| > r/4\}) =: A < \infty. \tag{4.8}$$

Thus, by the relations (4.6)–(4.8), we obtain

$$\limsup_{n\to\infty} I_1(n) \leqslant AC\epsilon,$$

and since $\epsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} I_1(n) = 0. \tag{4.9}$$

From the assumption that (X_n) is strongly mixing with exponential rate it follows that $k_n \alpha_{l_n+1} \to 0$ as $n \to \infty$, and hence from (4.1) we obtain

$$\lim_{n\to\infty} I_2(n) = 0. \tag{4.10}$$

Using again Lemma 4.3 in Durrett [8] we have

$$I_3(n) \leqslant k_n \mathrm{E}|e^{-f(a_n^{-1} S_{r_n})} - e^{-f(a_n^{-1} S_{r_n - l_n})}|.$$

Repeating the same procedure as for $I_1(n)$ we get

$$\lim_{n\to\infty} I_3(n) = 0. \tag{4.11}$$

Taking into account the relations (4.9)–(4.11), from (4.2) we infer that, as $n \to \infty$,

$$\mathrm{E} \exp\Big(- \sum_{k=1}^{k_n} f(a_n^{-1} S_{r_n}^k) \Big) - \Big(\mathrm{E} \exp\big(- f(a_n^{-1} S_{r_n}) \big) \Big)^{k_n} \to 0,$$

and this completes the proof. ∎

PROPOSITION 4.2. *Suppose (X_n) is a strictly stationary sequence of regularly varying random variables with index of regular variation $\alpha \in (1, 2)$, and (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Let (r_n) be a sequence of positive integers such that $r_n \to \infty$ as $n \to \infty$. If $r_n = o(n^s)$ for some $0 < s < 2/\alpha - 1$, and the sequence (ρ_n) of ρ-mixing coefficients of (X_n) decreases to zero as $n \to \infty$ and*

$$\sum_{j \geqslant 0} \rho_{\lfloor 2^{j/3} \rfloor} < \infty, \tag{4.12}$$

then the condition $\mathcal{VSV}(a_n)$ holds.

P r o o f. Let $n \in \mathbb{N}$ and $u > 0$ be arbitrary. Define

$$Z_k = Z_k(u,n) = \frac{S^k_{r_n}}{a_n} 1_{\{|S^k_{r_n}|/a_n \leqslant u\}} - \mathrm{E}\Big(\frac{S^k_{r_n}}{a_n} 1_{\{|S^k_{r_n}|/a_n \leqslant u\}} \Big), \qquad k \in \mathbb{N}.$$

Take an arbitrary $\delta > 0$ and, as in the proof of Theorem 3.1, define

$$I(u,n) = \mathrm{P}\big[\max_{1 \leqslant j \leqslant k_n} \big| \sum_{k=1}^{j} Z_k \big| > \delta \big].$$

Corollary 2.1 in Peligrad [15] then implies

$$I(u,n) \leqslant \delta^{-2} C \exp\big(8 \sum_{j=0}^{\lfloor \log_2 k_n \rfloor} \widetilde{\rho}_{\lfloor 2^{j/3} \rfloor}\big)\, k_n \mathrm{E}(Z_1^2),$$

where $(\widetilde{\rho}_k)$ is a sequence of ρ-mixing coefficients of (Z_k), and C is some positive constant (here we put $\log_2 0 := 0$). Now standard calculations show that, for any $k \in \mathbb{N}$,

$$\widetilde{\rho}_k \leqslant \rho_{(k-1)r_n+1},$$

and since the sequence (ρ_k) is non-increasing, we have $\widetilde{\rho}_k \leqslant \rho_k$. Hence, using the assumption (4.12), we obtain

$$I(u,n) \leqslant CL\delta^{-2}\, k_n \mathrm{E}(Z_1^2) \tag{4.13}$$

for some positive constant L. Further we have

$$\begin{aligned} \mathrm{E}(Z_1^2) &\leqslant \mathrm{E}\Big(\frac{|S_{r_n}|^2}{a_n^2} 1_{\{|S_{r_n}|/a_n \leqslant u\}} \Big) \\ &= \mathrm{E}\Big(\frac{|S_{r_n}|^2}{a_n^2} 1_{\{|S_{r_n}|/a_n \leqslant u\}} 1_{\{\cap_{i=1}^{r_n} \{|X_i| \leqslant u a_n\}\}} \Big) \\ &\quad + \mathrm{E}\Big(\frac{|S_{r_n}|^2}{a_n^2} 1_{\{|S_{r_n}|/a_n \leqslant u\}} 1_{\{\cup_{i=1}^{r_n} \{|X_i| > u a_n\}\}} \Big) \\ &\leqslant \mathrm{E}\Big(\Big| \sum_{i=1}^{r_n} \frac{X_i}{a_n} 1_{\{|X_i|/a_n \leqslant u\}} \Big|^2 \Big) + u^2 \mathrm{P}\big(\bigcup_{i=1}^{r_n} \{|X_i| > u a_n\} \big). \end{aligned} \tag{4.14}$$

Note that

$$\mathrm{E}\Big(\Big| \sum_{i=1}^{r_n} \frac{X_i}{a_n} 1_{\{|X_i|/a_n \leqslant u\}} \Big|^2 \Big) \tag{4.15}$$

$$= \mathrm{E}\Big(\Big| \sum_{i=1}^{r_n} \frac{X_i 1_{\{|X_i| \leqslant u a_n\}} - \mathrm{E}(X_i 1_{\{|X_i| \leqslant u a_n\}})}{a_n} + \sum_{i=1}^{r_n} \mathrm{E}\Big(\frac{X_i}{a_n} 1_{\{|X_i|/a_n \leqslant u\}} \Big) \Big|^2 \Big)$$

$$= \mathrm{E}(I_1^2) + 2\mathrm{E}(I_1) I_2 + I_2^2,$$

where

$$I_1 = \sum_{i=1}^{r_n} \frac{X_i 1_{\{|X_i| \leqslant ua_n\}} - \mathrm{E}(X_i 1_{\{|X_i| \leqslant ua_n\}})}{a_n} \text{ and } I_2 = \sum_{i=1}^{r_n} \mathrm{E}\bigg(\frac{X_i}{a_n} 1_{\{|X_i|/a_n \leqslant u\}} \bigg).$$

Since I_1 is a sum of centered random variables, by Theorem 2.1 in Peligrad [15], we have

$$\mathrm{E}(I_1^2) \leqslant C \exp\Big(8 \sum_{j=0}^{\lfloor \log_2 r_n \rfloor} \rho_{\lfloor 2^{j/3} \rfloor}(n,u)\Big)\, r_n \mathrm{E}\bigg(\frac{X_1^2}{a_n^2} 1_{\{|X_1|/a_n \leqslant u\}} \bigg) \tag{4.16}$$

for all $n \in \mathbb{N}$, where $\big(\rho_j(n,u)\big)_j$ is the sequence of ρ-mixing coefficients of

$$\bigg(\frac{X_j}{a_n} 1_{\{|X_j|/a_n \leqslant u\}} - \mathrm{E}\bigg(\frac{X_j}{a_n} 1_{\{|X_j|/a_n \leqslant u\}} \bigg) \bigg)_j .$$

Since the function $f = f_{n,u} \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{a_n} 1_{\{|x|/a_n \leqslant u\}} - \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\{|X_1|/a_n \leqslant u\}} \bigg)$$

is measurable, it follows that

$$\sigma\bigg(\frac{X_j}{a_n} 1_{\{|X_j|/a_n \leqslant u\}} - \mathrm{E}\bigg(\frac{X_j}{a_n} 1_{\{|X_j|/a_n \leqslant u\}} \bigg) \bigg) \subseteq \sigma(X_j)$$

(see Theorem 4 in Chow and Teicher [6]). Consequently, we obtain immediately $\rho_j(n,u) \leqslant \rho_j$ for all $j, n \in \mathbb{N}$ and $u > 0$. Thus from (4.16), by a new application of the assumption (4.12), we get

$$\mathrm{E}(I_1^2) \leqslant CL\, r_n \mathrm{E}\bigg(\frac{X_1^2}{a_n^2} 1_{\{|X_1|/a_n \leqslant u\}} \bigg). \tag{4.17}$$

Note also that $\mathrm{E}(I_1) = 0$. Since $\alpha \in (1,2)$, it follows that $\mathrm{E}|X_i| < \infty$. Hence

$$I_2^2 \leqslant M \frac{r_n^2}{a_n^2} \tag{4.18}$$

for some positive constant M. Now the relations (4.14), (4.15), (4.17) and (4.18) imply

$$\begin{aligned} & k_n \mathrm{E}(Z_1^2) \\ & \leqslant CL\, k_n r_n \mathrm{E}\bigg(\frac{X_1^2}{a_n^2} 1_{\{|X_1|/a_n \leqslant u\}} \bigg) + M \frac{k_n r_n^2}{a_n^2} + u^2 k_n r_n P(|X_1| > ua_n) \\ & = u^2 \cdot \frac{k_n r_n}{n} \cdot n\mathrm{P}(|X_1| > ua_n) \cdot \bigg[CL \frac{\mathrm{E}[X_1^2 1_{\{|X_1| \leqslant ua_n\}}]}{u^2 a_n^2 P(|X_1| > ua_n)} + 1 \bigg] + M \frac{k_n r_n^2}{a_n^2}. \end{aligned}$$

Therefore, using the regular variation property of X_1, Karamata's theorem and the fact that $k_n r_n / n \to 1$ and

$$\frac{k_n r_n^2}{a_n^2} = \frac{k_n r_n}{n} \cdot \frac{r_n}{n^s} \cdot \frac{1}{n^{2/\alpha - 1 - s}\big(L'(n)\big)^2} \to 0$$

as $n \to \infty$ (here we used again the representation $a_n = n^{1/\alpha} L'(n)$ with $L'(\cdot)$ being a slowly varying function at infinity), we obtain

$$\limsup_{n\to\infty} k_n \mathrm{E}(Z_1^2) \leqslant u^{2-\alpha}\bigg(\frac{CL\alpha}{2-\alpha} + 1\bigg).$$

Letting $u \downarrow 0$, we see that $\lim_{u\downarrow 0} \limsup_{n\to\infty} k_n \mathrm{E}(Z_1^2) = 0$. Therefore, by (4.13), we get

$$\lim_{u\downarrow 0} \limsup_{n\to\infty} I(u,n) = 0,$$

and the relation (3.2) holds. ■

REMARK 4.1. *A careful analysis of the proof of Proposition* 4.2 *shows that the additional condition on the sequence* (r_n) (*namely*, $r_n = o(n^s)$ *for some* $0 < s < 2/\alpha - 1$) *can be dropped if we assume that the random variables* X_i *are symmetric. Indeed*, *then we can directly apply Theorem* 2.1 *of Peligrad* [15] *to the term* $\mathrm{E}\big(\big|\sum_{i=1}^{r_n} a_n^{-1} X_i 1_{\{|X_i| \leqslant u a_n\}}\big|^2\big)$, *and hence we do not need to introduce* I_2 (*to which the additional condition on* r_n *is related*). *This also holds for* $\alpha = 1$.

REFERENCES

[1] F. Avram and M. Taqqu, *Weak convergence of sums of moving averages in the* α*-stable domain of attraction*, Ann. Probab. 20 (1992), pp. 483–503.

[2] K. Bartkiewicz, A. Jakubowski, T. Mikosch, and O. Wintenberger, *Stable limits for sums of dependent infinite variance random variables*, Probab. Theory Related Fields 150 (2011), pp. 337–372.

[3] B. Basrak, D. Krizmanić, and J. Segers, *A functional limit theorem for partial sums of dependent random variables with infinite variance*, Ann. Probab. 40 (2012), pp. 2008–2033.

[4] J. Bertoin, *Lévy Processes*, Cambridge Tracts in Math., Vol. 121, Cambridge University Press, Cambridge 1996.

[5] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge, 1989.

[6] Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, third edition, Springer, New York 1997.

[7] R. A. Davis and T. Hsing, *Point process and partial sum convergence for weakly dependent random variables with infinite variance*, Ann. Probab. 23 (1995), pp. 879–917.

[8] R. Durrett, *Probability: Theory and Examples*, second edition, Duxbury Press, Wadsworth Publishing Company, 1996.

[9] R. Durrett and S. I. Resnick, *Functional limit theorems for dependent variables*, Ann. Probab. 6 (1978), pp. 829–846.

[10] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. 2, Wiley, New York 1971.

[11] B. V. Gnedenko and A. N. Kolmogorov, *Limit Theorems for Sums of Independent Random Variables*, Addison-Wesley, Cambridge 1954.

[12] O. Kallenberg, *Random Measures*, third edition, Akademie-Verlag, Berlin 1983.

[13] M. R. Leadbetter and H. Rootzén, *Extremal theory for stochastic processes*, Ann. Probab. 16 (1988), pp. 431–478.

[14] Z. Y. Lin and C. R. Lu, *Limit Theory for Mixing Dependent Random Variables*, Math. Appl., Springer, New York 1997.

[15] M. Peligrad, *Convergence of stopped sums of weakly dependent random variables*, Electron. J. Probab. 4 (1999), pp. 1–13.

[16] S. I. Resnick, *Point processes, regular variation and weak convergence*, Adv. in Appl. Probab. 18 (1986), pp. 66–138.

[17] S. I. Resnick, *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*, Springer Ser. Oper. Res. Financ. Eng., New York 2007.

[18] E. L. Rvačeva, *On domains of attraction of multi-dimensional distributions*, in: *Selected Translations in Mathematical Statistics and Probability*, Vol. 2, American Mathematical Society, Providence, R.I., 1962, pp. 183–205.

[19] J. D. Samur, *Convergence of sums of mixing triangular arrays of random vectors with stationary rows*, Ann. Probab. 12 (1984), pp. 390–426.

[20] K. Sato, *Lévy Processes and Infinitely Divisible Distributions,* Cambridge Stud. Adv. Math., Vol. 68, Cambridge University Press, Cambridge 1999.

[21] A. V. Skorokhod, *Limit theorems for stochastic processes*, Theory Probab. Appl. 1 (1956), pp. 261–290.

[22] A. V. Skorokhod, *Limit theorems for stochastic processes with independent increments*, Theory Probab. Appl. 2 (1957), pp. 145–177.

[23] M. Tyran-Kamińska, *Convergence to Lévy stable processes under some weak dependence conditions*, Stochastic Process. Appl. 120 (2010), pp. 1629–1650.

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