PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 36, Fasc. 1 (2016), pp. 47–57

ON THE ORDER OF APPROXIMATION IN THE RANDOM CENTRAL LIMIT THEOREM FOR m-DEPENDENT RANDOM VARIABLES

BY

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Abstract. We consider a random number N_n of m-dependent random variables X_k with a common distribution and the partial sums $S_{N_n} = \sum_{j=1}^{N_n} X_j$, where the random variable N_n is independent of the sequence of random variables $\{X_k, k \geqslant 1\}$ for every $n \geqslant 1$. Under certain conditions on the random variables X_k and N_n, we obtain the limit distribution of the sequence S_{N_n} and the corresponding rate of convergence after suitable normalization.

2010 AMS Mathematics Subject Classification: Primary: 60F05; Secondary: 60G99.

Key words and phrases: Random central limit theorem, m-dependent random variables, Berry–Esseen type bound, approximation.

1. INTRODUCTION

Limit theorems for random sums have been studied for about 70 years now. In their book *Random Summation*, Gnedenko and Korolev [5] discussed most of the limit theoretic results concerning random sums of independent random variables (r.v.s) such as the random central limit theorem and their importance in various disciplines such as financial mathematics and insurance. The order of approximation is a topic of interest in statistics, and initial work in this direction was done by Tomko [16], Sreehari [14], and Landers and Rogge [7], [8] among others. However, the problem and its variants appear to be of interest even now (see Barbour and Xia [1] and Sunklodas [15] and the references therein).

Investigation of the random central limit theorem for various types of dependent r.v.s has been going on simultaneously and early results can be found in Billingsley [2], Prakasa Rao [9] and Sreehari [13], and the problem is still getting the attention of research workers (see, for example, Shang [12] and Işlak [6]). The order of approximation in the random central limit theorem for certain types of dependent r.v.s has also received some attention (see Prakasa Rao [10], [11]). The aim of this paper is to investigate the order of approximation in the random central limit theorem for a sequence of stationary m-dependent r.v.s.

Let the sequence $\{X_n\}$ be a stationary sequence of m-dependent r.v.s with $E(X_1) = \mu$, $V(X_1) = E(X_1 - \mu)^2 = \sigma^2 < \infty, \mathrm{Cov}(X_1, X_{1+j}) = a_j$, and let $\sigma^2 + 2\sum_{j=1}^{m} a_j > 0$. Then, it is known (see Diananda [4]) that

$$\frac{S_n - E(S_n)}{\sqrt{V(S_n)}} \xrightarrow{D} Z_1 \tag{1.1}$$

as $n \to \infty$, where Z_1 is the standard normal r.v. Let the sequence $\{N_n\}$ be a sequence of non-negative integer-valued r.v.s such that the r.v. N_n is independent of the sequence $\{X_k\}$ for every $n \geqslant 1$ and such that the r.v. N_n, properly normalized, converges in distribution to an r.v. Z_2 defined in Section 2. We prove that

$$\frac{S_{N_n} - E(S_{N_n})}{\sqrt{V(S_{N_n})}} \xrightarrow{D} Z^* \tag{1.2}$$

as $n \to \infty$, where Z^* is a mixture of Z_1 and Z_2, and also obtain the rate of convergence of this limit. It will be noted that if Z_2 is also a standard normal r.v., then Z^* is also standard normal and marginally different from the limit r.v. given in Işlak [6].

In Section 2, we give details of the assumptions made and prove some lemmas. The main result is given in Section 3.

2. ASSUMPTIONS AND LEMMAS

For the sequence of r.v.s $\{X_k\}$, we assume that $\beta^2 = \sigma^2 + 2\sum_{j=1}^{m} a_j > 0$. It is easy to check that (see Işlak [6])

$$V(S_n) = n\sigma^2 + 2n\sum_{j=1}^{m} a_j I(n \geqslant j+1) - 2\sum_{j=1}^{m} j a_j I(n \geqslant j+1), \tag{2.1}$$

where $I(A)$ denotes the indicator function of the set A. Observe that, for $n > m$,

$$V(S_n) = n\sigma^2 + 2n\sum_{j=1}^{m} a_j - 2\sum_{j=1}^{m} j a_j = n\beta^2(n),$$

say, and that $\beta^2(n) \to \beta^2$ as $n \to \infty$.

We now recall a result on the rate of convergence in the limit theorem given in (1.1). Let $\Phi(x)$ denote the standard normal distribution function.

THEOREM 2.1 (Chen and Shao [3]). *If* $E|X_1|^{2+\delta} < \infty$ *for some* $0 < \delta \leqslant 1$, *then*

$$\sup_x \left|P\big(S_n - ES_n \leqslant x\sqrt{V(S_n)}\big) - \Phi(x)\right| \leqslant \frac{75(10m+1)^{1+\delta} n E|X_1|^{2+\delta}}{\big[n\sigma^2 + 2n\sum_{j=1}^{m} a_j - 2\sum_{j=1}^{m} j a_j\big]^{1+\delta/2}}.$$

We assume that $EN_n/n \to \nu > 0$ as $n \to \infty$ and $V(N_n)/n \to \tau^2 < \infty$ as $n \to \infty$ and that, for large n,

(2.2) $$\sup_x \left| P\big(N_n - EN_n \leqslant x\sqrt{V(N_n)}\big) \; - \; G(x)\right| \leqslant \epsilon_n,$$

where $G(\cdot)$ is a continuous distribution function (d.f.) satisfying the condition that there exists a constant $C > 0$ such that

$$\sup_x |G(x+y) - G(x)| < Cy, \quad y > 0,$$

and $\epsilon_n \to 0$ as $n \to \infty$. In view of (2.2) and the assumptions regarding $E(N_n)$ and $V(N_n)$, it follows that

(2.3) $$\frac{N_n - EN_n}{V(N_n)} \xrightarrow{P} 0$$

as $n \to \infty$. Furthermore, we have the following result concerning $V(S_{N_n})$.

LEMMA 2.1. *Let* $p_{n,k} = P(N_n = k)$ *for* $k = 0, 1, \ldots$ *Under the conditions stated above*,

$$V(S_{N_n}) = E(N_n)\big(\sigma^2 + 2\sum_{j=1}^{m} a_j\big) - 2\sum_{j=1}^{m} j a_j + \mu^2 V(N_n) + \alpha_n(m),$$

where

$$\alpha_n(m) = \sum_{k=0}^{m} 2k p_{n,k} \sum_{j=1}^{m} a_j \left\{ I(k \geqslant j+1) - 1 \right\}$$
$$- \sum_{k=0}^{m} 2p_{n,k} \sum_{j=1}^{m} j a_j \{ I(k \geqslant j+1) - 1 \}.$$

Proof. Note that

$$V(S_{N_n}) = E\big(V(S_{N_n}|N_n)\big) + V\big(E(S_{N_n}|N_n)\big)$$
$$= \sum_{k=0}^{\infty} p_{n,k} \big[k\big\{\sigma^2 + 2\sum_{j=1}^{m} a_j I(k \geqslant j+1)\big\} - 2\sum_{j=1}^{m} j a_j I(k \geqslant j+1)\big]$$
$$+ V(\mu N_n).$$

Note that, for $k > m \geqslant j$, $I(k \geqslant j+1) = 1$, and we have

$$
\begin{aligned}
V(S_{N_n}) &= \sigma^2 EN_n + \mu^2 V(N_n) + 2\sum_{k=0}^{m} kp_{n,k} \sum_{j=1}^{m} a_j I(k \geqslant j+1) \\
&\quad + 2\sum_{k=m+1}^{\infty} kp_{n,k} \sum_{j=1}^{m} a_j - 2\sum_{k=0}^{m} p_{n,k} \sum_{j=1}^{m} ja_j I(k \geqslant j+1) \\
&\quad - 2\sum_{k=m+1}^{\infty} p_{n,k} \sum_{j=1}^{m} ja_j \\
&= EN_n\big[\sigma^2 + 2\sum_{j=1}^{m} a_j\big] - 2\sum_{j=1}^{m} ja_j + \mu^2 V(N_n) + \alpha_n(m),
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_n(m) &= \sum_{k=0}^{m} 2kp_{n,k} \sum_{j=1}^{m} a_j \{I(k \geqslant j+1) - 1\} \\
&\quad - \sum_{k=0}^{m} 2p_{n,k} \sum_{j=1}^{m} ja_j \{I(k \geqslant j+1) - 1\}. \quad ■
\end{aligned}
$$

REMARK 2.1. Observe that the sequence $|\alpha_n(m)|$ is bounded in n, and hence $\alpha_n(m)/n \to 0$ as $n \to \infty$.

We now prove two lemmas which are of independent interest.

LEMMA 2.2. *Let $U = V + tW$, $t \in R$, and G be a d.f. Then, for all $z \in R$ and $\delta > 0$,*

$$
\begin{aligned}
&|P(U \leqslant z) - G(z)| \\
&\quad < \sup_x |P(V \leqslant x) - G(x)| + \sup_x |G(x) - G(x+\delta t)| + P(|W| > \delta).
\end{aligned}
$$

P r o o f. Let $t > 0$. Then, for any $\delta > 0$,

$$
\begin{aligned}
P(U \leqslant z) &\leqslant P(U \leqslant z, |W| \leqslant \delta) + P(|W| > \delta) \\
&\leqslant P(V \leqslant z + t\delta) + P(|W| > \delta).
\end{aligned}
$$

Then, for all $z \in R$,

$$
\begin{aligned}
(2.4)\quad & P(U \leqslant z) - G(z) \\
&\leqslant |P(V \leqslant z+t\delta) - G(z+t\delta)| + |G(z+t\delta) - G(z)| + P(|W| > \delta) \\
&\leqslant \sup_x |P(V \leqslant x) - G(x)| + |G(z+t\delta) - G(z)| + P(|W| > \delta).
\end{aligned}
$$

Moreover,

$$
P(U \leqslant z) \geqslant P(U \leqslant z, |W| \leqslant \delta) \geqslant P(V \leqslant z - t\delta) - P(|W| > \delta).
$$

Then, for all $z \in R$,

$$(2.5)\quad G(z) - P(U \leqslant z) \leqslant \sup_x |P(V \leqslant x) - G(x)| + |G(z - t\delta) - G(z)| + P(|W| > \delta).$$

From the inequalities (2.4) and (2.5) we get the required result for $t > 0$ and, on similar lines, the inequalities can be checked for $t \leqslant 0$, completing the proof of the lemma. ■

LEMMA 2.3. *Let U_n and U be r.v.s with the d.f. $H(x)$ of U satisfying the condition that there exists a constant $\alpha > 0$ such that*

$$\sup_x |H(x+\theta) - H(x)| \leqslant \alpha\theta, \qquad \theta > 0,$$

and V be an r.v. independent of r.v.s U_n and U with $E|V| < \infty$. Let $g : R \to R$. Then, for any constant c and $\delta > 0$, and for all $z \in R$,

$$\big|P\big(U_n + Vg(U_n) \leqslant z\big) - P(U + cV \leqslant z)\big| \leqslant \alpha\delta E|V| + \sup_x |P(U_n \leqslant x) - P(U \leqslant x)| + P\big(|g(U_n) - c| > \delta\big).$$

P r o o f. Denote by H the d.f. of V. Then

$$(2.6)\quad P\big(U_n + Vg(U_n) \leqslant z\big) - P(U + cV \leqslant z) = \int \big[P\big(U_n + vg(U_n) \leqslant z\big) - P(U + cv \leqslant z)\big] dH(v).$$

Suppose $v > 0$. Then, for $\delta > 0$,

$$\begin{aligned} P\big(U_n + vg(U_n) \leqslant z\big) &\leqslant P\big(U_n + vg(U_n) \leqslant z, |g(U_n) - c| \leqslant \delta\big) \\ &\quad + P\big(|g(U_n) - c| > \delta\big) \\ &\leqslant P\big(U_n \leqslant z - v(c-\delta)\big) + P\big(|g(U_n) - c| > \delta\big). \end{aligned}$$

Hence

$$\begin{aligned} &P\big(U_n + vg(U_n) \leqslant z\big) - P(U + cv \leqslant z) \\ &\quad \leqslant |P(U_n + vc \leqslant z + v\delta) - P(U + vc \leqslant z + v\delta)| \\ &\qquad + |P(U + vc \leqslant z + v\delta) - P(U + vc \leqslant z)| + P\big(|g(U_n) - c| > \delta\big). \end{aligned}$$

Hence, for $v > 0$, there exists a constant $\alpha > 0$ such that

$$\begin{aligned} &P\big(U_n + vg(U_n) \leqslant z\big) - P(U + cv \leqslant z) \\ &\leqslant \sup_x |P(U_n + cv \leqslant x) - P(U + cv \leqslant x)| + \alpha v\delta + P\big(|g(U_n) - c| > \delta\big). \end{aligned}$$

Similarly we get

$$\begin{aligned}&P\big(U_n+vg(U_n)\leqslant z\big)-P(U+cv\leqslant z)\\ \geqslant&-\sup_x|P(U_n+cv\leqslant x)-P(U+cv\leqslant x)|-\alpha v\delta-P\big(|g(U_n)-c|>\delta\big),\end{aligned}$$

so that, for all $v>0$,

$$\begin{aligned}&\big|P\big(U_n+vg(U_n)\leqslant z\big)-P(U+cv\leqslant z)\big|\\ \leqslant&\sup_x|P(U_n+cv\leqslant x)-P(U+cv\leqslant x)|+\alpha v\delta+P\big(|g(U_n)-c|>\delta\big).\end{aligned}$$

Similar arguments will prove that the above inequalities hold with $-v\delta$ in place of $v\delta$ for $v\leqslant 0$. Then, from (2.6) it follows that

$$\begin{aligned}&\big|P\big(U_n+Vg(U_n)\leqslant z\big)-P(U+cV\leqslant z)\big|\\ &\leqslant\sup_x|P(U_n\leqslant x)-P(U\leqslant x)|+\alpha\delta E|V|+P\big(|g(U_n)-c|>\delta\big).\quad\blacksquare\end{aligned}$$

3. MAIN RESULT

Before we state and prove the main result, we need to introduce some notation. For any two random variables U and V, let

$$d_K(U,V)=\sup_x|P(U\leqslant x)-P(V\leqslant x)|$$

denote the Kolmogorov distance between the d.f.s of U and V. Define

$$T_n=\frac{S_{N_n}-ES_{N_n}}{\sqrt{V(S_{N_n})}}=\frac{S_{N_n}-\mu N_n}{\sqrt{V(S_{N_n})}}+\frac{(N_n-EN_n)\mu}{\sqrt{V(S_{N_n})}}$$

and

$$T_n(Z_1)=\sqrt{\frac{N_n}{V(S_{N_n})}}\beta(N_n)Z_1+\frac{(N_n-EN_n)\mu}{\sqrt{V(S_{N_n})}},$$

where Z_1 is an $N(0,1)$ r.v. independent of N_n. Furthermore, define

$$T_n'(Z_1)=\sqrt{\frac{N_n}{V(S_{N_n})}}\beta Z_1+\frac{(N_n-EN_n)\mu}{\sqrt{V(S_{N_n})}}$$

and

$$T(Z_1,Z_2)=\frac{\mu\tau}{\sqrt{\nu\beta^2+\mu^2\tau^2}}\left[\frac{\beta\sqrt{\nu}}{\mu\tau}Z_1+Z_2\right],$$

where Z_2 follows the d.f. G given at (2.2) and is independent of Z_1. The r.v. $T(Z_1,Z_2)$ is the limit r.v. Z^* in (1.2).

In the following discussion, C with or without subscript will denote a positive constant.

THEOREM 3.1. *Let $\{X_n\}$ be a stationary sequence of m-dependent r.v.s with $EX_1 = \mu,\ V(X_1) = \sigma^2,\ \mathrm{Cov}(X_1, X_{1+j}) = a_j,\ E|X_1|^{2+\delta} < \infty$ for some $\delta > 0$. Let $\{N_n\}$ be a sequence of non-negative integer-valued r.v.s such that N_n is independent of $\{X_k\}$ for every $n \geqslant 1$ and satisfying* (2.2). *Let $0 < \theta < 1$ and $\delta_n = n^{-\theta}$ be a sequence of positive numbers. Then there exists a constant $C > 0$ such that, for n large,*

$$\begin{aligned} d_K\big(T_n, T(Z_1, Z_2)\big) &= \sup_x \left| P\left(\frac{S_{N_n} - E(S_{N_n})}{\sqrt{V(S_{N_n})}} \leqslant x\right) - P\big(T(Z_1, Z_2) \leqslant x\big)\right| \\ &\leqslant d_K\left(\frac{N_n - EN_n}{\sqrt{V(N_n)}}, Z_2\right) + Cn^{-\min(\theta,\ \delta/2)} \\ &\quad + P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}} - \frac{\sqrt{\nu}}{\sqrt{\nu\beta^2 + \mu^2\tau^2}}\right| > \delta_n\right). \end{aligned}$$

P r o o f. We first obtain upper bounds for the distances $d_K\big(T_n, T_n(Z_1)\big)$ and $d_K\big(T_n'(Z_1), T(Z_1, Z_2)\big)$. We then use the second estimate to obtain an upper bound for the distance $d_K\big(T_n(Z_1), T(Z_1, Z_2)\big)$. Note that

$$T_n = \frac{S_{N_n} - \mu N_n}{\sqrt{V(S_{N_n})}} + \frac{(N_n - EN_n)\mu}{\sqrt{V(N_n)}}\sqrt{\frac{V(N_n)}{V(S_{N_n})}}.$$

Let $B_n = \{|N_n - n\nu| \leqslant n\nu/2\}$ and B_n' denote its complement. Then

$$\begin{aligned} &d_K\big(T_n, T_n(Z_1)\big) \\ &\leqslant P(B_n') + \sum_{k=n\nu/2}^{3n\nu/2} p_{n,k} \sup_x \big|P(T_n \leqslant x | N_n = k) - P\big(T_n(Z_1) \leqslant x | N_n = k\big)\big| \\ &= P(B_n') + \\ &\quad + \sum_{k=n\nu/2}^{3n\nu/2} p_{n,k} \sup_x \left| P\left(\frac{S_k - k\mu}{\beta(k)\sqrt{k}} \leqslant \frac{1}{\beta(k)}\sqrt{\frac{V(S_{N_n})}{k}}\left\{x - \frac{(k - EN_n)\mu}{\sqrt{V(S_{N_n})}}\right\}\right)\right| \\ &\quad - P\left(Z_1 \leqslant \frac{1}{\beta(k)}\sqrt{\frac{V(S_{N_n})}{k}}\left\{x - \frac{(k - EN_n)\mu}{\sqrt{V(S_{N_n})}}\right\}\right) \\ &\leqslant P(B_n') + \sum_{k=n\nu/2}^{3n\nu/2} p_{n,k} \sup_u \left| P\left(\frac{S_k - k\mu}{\beta(k)\sqrt{k}} \leqslant u\right) - P(Z_1 \leqslant u)\right|. \end{aligned}$$

Then, by Chebyshev's inequality and the bound given in Theorem 2.1, it follows that, for n sufficiently large,

$$d_K\big(T_n, T_n(Z_1)\big) \leqslant \frac{4V(N_n)}{(EN_n)^2} + \sum_{k \geqslant n\nu/2} p_{n,k} \frac{C_1 k E|X_1|^{2+\delta}}{\big(\sqrt{k}\beta(k)\big)^{2+\delta}} < \frac{C_2}{n^{\delta/2}}. \tag{3.1}$$

Next we estimate $d_K\big(T_n'(Z_1), T(Z_1, Z_2)\big)$. It can be checked that

$$\frac{N_n}{V(S_{N_n})} \xrightarrow{P} \frac{\nu}{\nu\beta^2 + \mu^2\tau^2} \tag{3.2}$$

as $n \to \infty$. Furthermore, since $V(N_n)/V(S_{N_n}) \to \tau^2/(\nu\beta^2 + \mu^2\tau^2)$ as $n \to \infty$, we obtain

$$\frac{N_n - EN_n}{\sqrt{V(N_n)}} \frac{\mu\sqrt{V(N_n)}}{\sqrt{V(S_{N_n})}} \xrightarrow{D} \frac{\mu\tau}{\sqrt{\nu\beta^2 + \mu^2\tau^2}} Z_2 \tag{3.3}$$

as $n \to \infty$. We use Lemma 2.3 with $U_n = (N_n - EN_n)\mu/\sqrt{V(S_{N_n})}, V = Z_1$, and $g(U_n) = \beta\sqrt{N_n/V(S_{N_n})}$ to get

$$\begin{aligned} &d_K\big(T_n'(Z_1), T(Z_1, Z_2)\big) \\ &\quad\leqslant P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\frac{\nu}{\nu\beta^2 + \mu^2\tau^2}}\right| > \delta_n\right) + \alpha\delta_n\, E|Z_1| \\ &\qquad + \sup_x \left|P\left(\frac{N_n - EN_n}{\sqrt{V(N_n)}} \leqslant x\right) - P(Z_2 \leqslant x)\right|. \end{aligned} \tag{3.4}$$

Finally, we estimate $d_K\big(T_n(Z_1), T(Z_1, Z_2)\big)$. Observe that

$$T_n(Z_1) - T_n'(Z_1) = Z_1\sqrt{\frac{N_n}{V(S_{N_n})}}\big(\beta(N_n) - \beta\big)$$

and

$$\sqrt{N_n}\big(\beta(N_n) - \beta\big) = -2\frac{\sum_{j=1}^{m} ja_j}{\sqrt{n}} \frac{\sqrt{n}}{\sqrt{N_n}[\beta(N_n) + \beta]} \xrightarrow{P} 0 \tag{3.5}$$

because $N_n/n \xrightarrow{P} \nu$ and $\beta(N_n) \xrightarrow{P} \beta$ as $n \to \infty$. We have

$$
\begin{aligned}
(3.6)\quad & P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}}\big(\beta(N_n)-\beta\big)\right| > \delta_n\right) \\
&\leqslant P(B_n') + P\left(B_n; \frac{C_3}{\sqrt{N_n}[\beta(N_n)+\beta]} > \delta_n\sqrt{V(S_{N_n})}\right) \\
&= P(B_n') + P\left(B_n; N_n\big(\beta(N_n)+\beta\big)^2 < \frac{C_4}{\delta_n^2 V(S_{N_n})}\right) \\
&\leqslant P(B_n') + P\left(\frac{n\nu}{2} \leqslant N_n \leqslant \frac{3n\nu}{2}; N_n < C_5 n^{2\theta-1}\right) \\
&= P(B_n') < \frac{C_6}{n}
\end{aligned}
$$

because the second probability bound above is zero for $0 < \theta < 1$. Consider

$$
d_K\big(T_n(Z_1), T(Z_1,Z_2)\big) = \int_{-\infty}^{\infty} \sup_x \big|P\big(T_n(z) \leqslant x\big) - P\big(T(z,Z_2) \leqslant x\big)\big|\, d\Phi(z) =
$$

$$
\int_{-\infty}^{\infty} \sup_x \left|P\left(T_n'(z) + z\sqrt{\frac{N_n}{V(S_{N_n})}}\big(\beta(N_n)-\beta\big) \leqslant x\right) - P\big(T(z,Z_2) \leqslant x\big)\right| d\Phi(z).
$$

Using Lemma 2.2 with $V = T_n'(z), t = z, W = \sqrt{N_n/V(S_{N_n})}\big(\beta(N_n)-\beta\big)$, and (3.6), we obtain

$$
\begin{aligned}
& d_K\big(T_n(Z_1), T(Z_1,Z_2)\big) \\
&\quad \leqslant P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}}\big(\beta(N_n)-\beta\big)\right| > \delta_n\right) \\
&\quad\quad + \int_{-\infty}^{\infty} \big[\sup_x \big|P\big(T_n'(z) \leqslant x\big) - P\big(T(z,Z_2) \leqslant x\big)\big|\big] d\Phi(z) \\
&\quad\quad + \int_{-\infty}^{\infty} \sup_x \big|P\big(T(z,Z_2) \leqslant x\big) - P\big(T(z,Z_2) \leqslant x + \delta_n z\big)\big| d\Phi(z) \\
&\quad \leqslant \sup_x \big|P\big(T_n'(Z_1) \leqslant x\big) - P\big(T(Z_1,Z_2) \leqslant x\big)\big| \\
&\quad\quad + \frac{\sqrt{\nu\beta^2 + \mu^2\tau^2}}{\mu\tau}\delta_n E|Z_1| + \frac{C_7}{n}.
\end{aligned}
$$

Using (3.4), we have

$$\begin{aligned}(3.7)\quad & d_K\big(T_n(Z_1), T(Z_1, Z_2)\big)\\ &\leqslant \frac{C_7}{n} + \alpha\delta_n E|Z_1| + \frac{\sqrt{\nu\beta^2+\mu^2\tau^2}}{\mu\tau}\delta_n E|Z_1|\\ &+ \sup_x \left| P\left(\frac{N_n - EN_n}{\sqrt{V(N_n)}} \leqslant x\right) - P(Z_2 \leqslant x)\right|\\ &+ P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\frac{\nu}{\nu\beta^2+\mu^2\tau^2}}\right| > \delta_n\right).\end{aligned}$$

Thus, from (3.1) and (3.7) we get

$$\begin{aligned}d_K\big(T_n(Z_1), T(Z_1, Z_2)\big) &\leqslant d_K\left(\frac{N_n - EN_n}{\sqrt{V(N_n)}}, Z_2\right) + C_8 n^{-\delta/2} + C_9\delta_n\\ &+ P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\frac{\nu}{\nu\beta^2+\mu^2\tau^2}}\right| > \delta_n\right).\end{aligned}$$

Hence

$$d_K(T_n, Z^*) < \epsilon_n + C_{10} n^{-\min(\theta,\delta/2)} + P\left(\left|\sqrt{\frac{N_n}{V(S_{N_n})}} - \sqrt{\frac{\nu}{\nu\beta^2+\mu^2\tau^2}}\right| > \delta_n\right),$$

where ϵ_n is given in (2.2). ■

REMARK 3.1. 1. Işlak [6] proved the random central limit theorem part of the above theorem for the particular case when N_n is the sum of n independent non-negative integer-valued r.v.s with a common distribution having finite variance τ^2. In that case, $\epsilon_n = n^{-1/2}$.

2. Shang [12] proved the random central limit theorem for stationary m-dependent variables. Shang's condition on the random index N_n is weaker than ours but we do not need the maximal inequality condition that Shang [12] assumed. Incidentally, some of the questions raised by Shang [12] in the concluding remarks are already answered in Sreehari [13].

Acknowledgments. Work of the first author was supported under the scheme "Ramanujan Chair Professor" by grants from the Ministry of Statistics and Programme Implementation, Government of India (M 12012/15(170)/2008-SSD dated 8/9/09), the Government of Andhra Pradesh, India (6292/Plg.XVIII dated 17/1/08) and the Department of Science and Technology, Government of India (SR/S4/MS: 516/07 dated 21/4/08) at the CR Rao Advanced Institute of Mathematics, Statistics and Computer Science, Hyderabad, India.

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Received on 13.1.2015;
revised version on 6.5.2015