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INTERPOLATION ERROR OPERATOR FOR HILBERT SPACE VALUED STATIONARY STOCHASTIC PROCESSES

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Abstract. In the paper a characterization of interpolation error operator for Hilbert space valued stationary stochastic processes is obtained.

1. Introduction. Let $\boldsymbol{Z}$ be the set of integers and let $X_n = (X_n^1, \ldots, X_n^q)$, $n \in \boldsymbol{Z}$, be a q-variate stationary stochastic process over $\boldsymbol{Z}$. Suppose that all random variables X_n, $n \neq 0$, are known. Then the linear predictor of X_0 is given by the formula

$$\hat{X}_0 = P_{M_0(X)} X_0,$$

where $M_0 \overset{\text{df}}{=} \overline{\text{sp}} \{X_n^i\colon n \neq 0, i = 1, \ldots, q\}$, P_M denotes the orthogonal projection operator onto M, and $P_M(X^1, \ldots, X^q) = (P_M X^1, \ldots, P_M X^q)$. An important problem in the prediction theory of such processes is to obtain a formula for the interpolation error matrix

$$\Sigma_0 \overset{\text{df}}{=} [(X_0^i - \hat{X}_0^i, X_0^j - \hat{X}_0^j)]_{i,j=1}^q.$$

A complete solution of this problem for $q = 1$ was given by A. N. Kolmogorov in 1941. In 1960 Masani [4] extended Kolmogorov's result to the case of minimal full rank q-variate processes and later (1967) Salehi [8] proved that for any q-variate minimal process ($q < \infty$) we have

$$\Sigma_0 = \left[\frac{1}{2\pi} \int_0^{2\pi} P_J \left(\frac{dF^a}{dt}\right)^{\#} (t) P_J dt\right]^{\#},$$

where F^a is the absolutely continuous part of the spectral measure of the process and J is a range of Σ_0. Finally, in 1976, Makagon and Weron [3] obtained a full description of the space $J = \text{range}\,\Sigma_0$.

In the present paper a characterization of the interpolation error operator for Hilbert space valued stationary processes is given.

2. Preliminaries. Let H and K be complex Hilbert spaces. Unless otherwise stated, the following conventions and notation will remain fixed in this paper:

$L(H, K)$ – the set of all linear and continuous operators from H into K;

$L^+(H)$ – the set of all selfadjoint operators $T \in L(H, H)$ such that $(Tx, x) \geqslant 0$ for every $x \in H$; we write $T \geqslant S$ if $T - S \in L^+(H)$, $T, S \in L(H, H)$;

$\mathcal{N}(T)$ – the null space of an operator T, i.e.,

$$\mathcal{N}(T) = \{x \in H: \ Tx = 0\}, \qquad T \in L(H, K);$$

$\mathcal{R}(T)$ – the range of T, i.e., $\mathcal{R}(T) = \{y \in K: \ y = Tx, \ x \in H\}$;

$T^{\#}$ – the generalized inverse operator of an operator $T \in L(H, K)$, i.e., a linear mapping from $\mathcal{R}(T)$ onto $\mathcal{N}(T)^{\perp}$ such that $T^{\#} y = x$ if and only if $Tx = y$ and $x \in \mathcal{N}(T)^{\perp}$ (we observe that $T^{\#}$ is closed because $T^{\#} = (T|\mathcal{N}(T)^{\perp})^{-1}$;

P_M – the orthogonal projection operator onto M;

$M^{\perp}$ – the orthogonal complement of M;

$\overline{\text{sp}}\, L$ – the minimal closed linear subspace containing the set L;

G – a discrete Abelian group;

Γ – the dual group of G;

$\langle g, \gamma \rangle$ – the value of $\gamma \in \Gamma$ at a point $g \in G$;

$\mathcal{B}(\Gamma)$ – the Borel σ-algebra of Γ;

$\varkappa$ – the normed Haar measure of Γ;

$L^2(\Gamma, \varkappa, K)$ – the Hilbert space of all K-valued $\varkappa$-square (Bochner) integrable functions on Γ;

$\mathbf{C}$ – the set of all complex numbers.

2.1. Definition. A function $X = \{X_g: \ g \in G\}$ from G into $L(H, K)$ is said to be an *H-valued stationary stochastic process* (SSP) if its correlation $X_h^* X_g = K(g-h)$ depends only on $g-h$.

If X is an H-valued SSP, then there exists a unique Borel regular $L^+(H)$-valued measure (countably additive in the weak operator topology) such that for every $x \in H$

$$\big(K(g)x, x\big) = \int_{\Gamma} \langle g, \gamma \rangle \big(F(d\gamma)x, x\big).$$

F is called the *spectral measure* of the process X. By the *spectral density* of the process X we mean an $L^+(H)$-valued function $F'(\cdot)$ on Γ such that, for every $x \in H$, $\big(F'(\cdot)x, x\big)$ is $\varkappa$-integrable and

$$\big(F(\varDelta)x, x\big) = \int_{\varDelta} \big(F'(\gamma)x, x\big)\varkappa(d\gamma).$$

If H is separable and F' exists, then it is unique up to the $\varkappa$-a.e. equality.

2.2. LEMMA. *Let H be a separable Hilbert space and let F' be the spectral density of an H-valued SSP X. Then*

(1) $[F'(\cdot)]^{1/2} x \in L^2(\Gamma, \varkappa, H)$ *for every* $x \in H$;

(2) $\overline{\text{sp}}\, \{\langle g, \cdot \rangle F'(\cdot)^{1/2} x\colon\ g \in G, x \in H\} = \{f \in L^2(\Gamma, \varkappa, H)\colon\ f(\gamma) \in \overline{\mathscr{R}(F'(\gamma))}$ $\varkappa$-*a.e.*$\}$;

(3) *for every $\varkappa$-measurable function $f\colon \Gamma \to H$ such that $f(\gamma) \in \mathscr{R}(F'(\gamma)^{1/2})$ $\varkappa$-a.e. the function $\varphi(\gamma) = [F'(\gamma)^{1/2}]^{\#} f(\gamma)$ is $\varkappa$-measurable.*

Proof. The first part follows from the definition of F' and the construction of an operator square root (see, e.g., [5], p. 183).

To prove (2) it suffices to show that the right-hand side (RHS) of (2) is included in the left one (LHS) (the converse inclusion is trivial). Suppose that $f \in$ RHS and $f \perp$ LHS and let $\{x_k\colon k = 1, 2, \ldots\}$ be a countable dense set in H. Then for every $k = 1, 2, \ldots$ and $g \in G$ we have

$$\int_\Gamma \langle g, \gamma \rangle \big(F'(\gamma)^{1/2} x_k, f(\gamma)\big) \varkappa(d\gamma) = 0.$$

Hence there exists a set $\Delta \in \mathscr{B}(\Gamma)$ such that $\varkappa(\Delta) = 0$ and

$$f(\gamma) \perp \overline{\mathscr{R}(F'(\gamma)^{1/2})} = \overline{\mathscr{R}(F'(\gamma))} \quad \text{for every } \gamma \notin \Delta.$$

Thus $f = 0$ $\varkappa$-a.e. since $f \in$ RHS.

Part (3) is a special case of Lemma 2.7 (2) in [2].

For any H-valued SSP X and $A \subset G$ we put

$$M_A(X) = \overline{\text{sp}}\, \{X_g x\colon\ x \in H, g \in A\}, \qquad M(X) = M_g(X),$$
$$N_A(X) = M(X) \ominus M_A(X).$$

The following lemma was observed by Rosanov (see [7]):

2.3. LEMMA. *Let H be separable and let X be an H-valued SSP. Suppose that the spectral density F' of the process X exists. Then the mapping V defined by the formula*

$$V(X_g x)(\cdot) = \langle g, \cdot \rangle F'(\cdot)^{1/2} x, \qquad x \in H, g \in G,$$

extends to an isometry from $M(X)$ onto

$$L(F) \stackrel{\text{df}}{=} \{f \in L^2(\Gamma, \varkappa, H)\colon f(\gamma) \in \overline{\mathscr{R}(F'(\gamma))}\ \varkappa\text{-}a.e.\}.$$

Moreover,

$$V(N_{G-\{0\}}(X)) = \{f \in L(F)\colon\ F'(\gamma)^{1/2} f(\gamma) = \text{const}\ \ \varkappa\text{-}a.e.\}.$$

Proof. From Lemma 2.2 it follows that $Y = \{Y_g\colon\ g \in G\}$, where

$$(Y_g x)(\cdot) \stackrel{\text{df}}{=} \langle g, \cdot \rangle\ F'(\cdot)^{1/2} x, \ \ Y_g \in L(H, L^2(\Gamma, \varkappa, H)),$$

is an H-valued SSP with the same spectral measure as the process X. Moreover, $M(Y) = L(F)$. A simple calculation shows that the mapping V defined by $V(X_g x) = Y_g x$ extends linearly to an isometry from $M(X)$ onto $M(Y)$. To prove the remaining part of the lemma it suffices to observe that $f \in V(N_{G-\{0\}}(X))$ if and only if

$$\int_{\Gamma} \overline{\langle g, \gamma \rangle} (F'(\gamma)^{1/2} f(\gamma), x) \varkappa(d\gamma) = (f, V(X_g x)) = (V^{-1} f, X_g x) = 0$$

for every $g \in G - \{0\}$ and $x \in H$. The last equality holds if and only if $F'(\gamma)^{1/2} f(\gamma) = \mathrm{const}$ $\varkappa$-a.e.

3. The main theorem. Let X be an H-valued SSP over an Abelian group G, $A \subset G$, and let g be a fixed element of G. Then the predictor X_g^A of X_g and the prediction error operator Σ_g^A are defined by the following formulas:

$$X_g^A = P_{M_A(X)} X_g \quad \text{and} \quad \Sigma_g^A = (X_g - X_g^A)^* (X_g - X_g^A).$$

First, we describe the prediction error operator using the Yaglom idea (cf. [9], p. 175). For convenience we introduce the following definition:

3.1. Definition. Let $\mathscr{A}$ be a subset of $L^+(H)$. An operator $T \in L^+(H)$ is said to be the *maximal element* of $\mathscr{A}$ (we write $T = \max \mathscr{A}$) if $T \in \mathscr{A}$ and $T \geqslant S$ for every $S \in \mathscr{A}$.

3.2. LEMMA. *Suppose that X is an H-valued SSP over an Abelian group G, $A \subset G$, and $g \in G$. Let $\mathscr{A}_g^A$ be the set of all operators $T \in L(H, K)$ such that*

(i) $$\mathscr{R}(T) \subset N_A(X),$$

(ii) $$T^* T = X_g^* T$$

and let

$$\mathscr{D}_g^A \stackrel{\mathrm{df}}{=} \{T^* T \colon T \in \mathscr{A}_g^A\}.$$

Then $\Sigma_g^A = \max \mathscr{D}_g^A$. Moreover, if $\Sigma_g^A = T^* T$ and $T \in \mathscr{A}_g^A$, then $T = X_g - X_g^A$.

Proof. First, we note that $Y_g^A \stackrel{\mathrm{df}}{=} X_g - X_g^A$ satisfies (i) and (ii). Hence $\Sigma_g^A \in \mathscr{D}_g^A$. Let now $T \in \mathscr{A}_g^A$. Since $(Y_g^A)^* T = X_g^* T = T^* T$, we have

$$\Sigma_g^A = (Y_g^A)^* Y_g^A = ((Y_g^A - T) + T)^* ((Y_g^A - T) + T) = (Y_g^A - T)^* (Y_g^A - T) + T^* T \geqslant T^* T.$$

Thus Σ_g^A is the maximal element of $\mathscr{D}_g^A$. Moreover, $\Sigma_g^A = T^* T$ if and only if $T = Y_g^A$.

We will be interested only in the case whence G is a discrete Abelian group, $A = G - \{0\}$, and $g = 0$. For simplicity we write

$$\hat{X}_0 = X_0^{G-\{0\}} \quad \text{and} \quad \Sigma_0 = \Sigma_0^{G-\{0\}}.$$

3.3. LEMMA. *Suppose that H is separable, X is an H-valued SSP, and that F' is its spectral density. Then $S\in\mathscr{D}_0^{G-\{0\}}$* (see Lemma 2.2) *if and only if $S\in L(H, H)$ and for every $x\in H$:*

(i) $$Sx\in\mathscr{R}\big(F'(\gamma)^{1/2}\big)\ \varkappa\text{-a.e.},$$

(ii) $$\int_\Gamma \big([F'(\gamma)^{\#}]^{1/2}Sx,\ [F'(\gamma)^{1/2}]^{\#}Sx\big)\varkappa(d\gamma) = (Sx, x).$$

Proof. Let $\mathscr{D}_0$ denote the set of all operators $S\in L(H, H)$ for which (i) and (ii) are satisfied. First we show that $\mathscr{D}_0^{G-\{0\}}\subset\mathscr{D}_0$. Let V denote the isomorphism defined in Lemma 2.3. Suppose that $S = T^*T$, where $T\in\mathscr{A}_0^{G-\{0\}}$. Then

$$\begin{aligned}(Sx, y) &= (T^*Tx, y) = (X_0^*Tx, y) = (VTx, VX_0y)\\ &= \int_\Gamma \big((VTx)(\gamma), F'(\gamma)^{1/2}y\big)\varkappa(d\gamma)\\ &= \int_\Gamma \big(F'(\gamma)^{1/2}(VTx)(\gamma), y\big)\varkappa(d\gamma)\\ &= \big(F'(\gamma)^{1/2}(VTx)(\gamma), y\big)\ \varkappa\text{-a.e.}\end{aligned}$$

since $VTx\in V\big(N_{G-\{0\}}(X)\big)$ and, by 2.3, $F'(\gamma)^{1/2}(VTx)(\gamma) = \text{const}$ $\varkappa$-a.e. Thus $Sx = F'(\gamma)^{1/2}(VTx)(\gamma)$ $\varkappa$-a.e. and (i) is satisfied. From the above equation it follows that

$$(VTx)(\gamma) = [F'(\gamma)^{1/2}]^{\#}Sx\ \ \varkappa\text{-a.e.}$$

Therefore

$$(Sx, y) = (VTx, VTy) = \int_\Gamma \big([F'(\gamma)^{1/2}]^{\#}Sx,\ [F'(\gamma)^{1/2}]^{\#}Sy\big)\varkappa(d\gamma).$$

It remains to prove that $\mathscr{D}_0\subset\mathscr{D}_0^{G-\{0\}}$. Let $S\in\mathscr{D}_0$. Then $S\geqslant 0$ and from Lemmas 2.3 and 2.2 we deduce that $[F'(\gamma)^{1/2}]^{\#}Sx$ is a function defined $\varkappa$-a.e. and belonging to $L(F)$ for every $x\in H$ (for the definition of $L(F)$ see Lemma 2.3). Let us note that by the Closed Graph Theorem a mapping R defined by the formula

$$H\ni x\xrightarrow{R}[F'(\cdot)^{1/2}]^{\#}Sx\in L^2(\Gamma, \varkappa, H)$$

is continuous. In fact, let $x_n\to x_0$ in H and $[F'(\cdot)^{1/2}]^{\#}Sx_n\to g(\cdot)$ in $L^2(\Gamma, \varkappa, H)$. Without loss of generality one can assume that

$$[F'(\gamma)^{1/2}]^{\#}Sx_n\to g(\gamma)\ \ \varkappa\text{-a.e.}$$

By (i) there exists a set $\Delta\in\mathscr{B}(\Gamma)$ of $\varkappa$-measure zero such that $Sx_n\in\mathscr{R}\big(F'(\gamma)^{1/2}\big)$ for all $n = 0, 1, \ldots$ and

$$g(\gamma) = \lim\,[F'(\gamma)^{1/2}]^{\#}Sx_n$$

provided $\gamma \notin \Delta$. Since $Sx_n \to Sx_0$ and $[F'(\gamma)^{1/2}]^{\#}$ is a closed operator for every $\gamma \in \Gamma$, we obtain $g(\gamma) = [F'(\gamma)^{1/2}]^{\#} Sx_0$ for every $\gamma \notin \Delta$. Since $F'(\gamma)^{1/2}(Rx)(\gamma) = Sx$ $\varkappa$-a.e., Lemma 2.3 implies that

$$(1) \qquad \mathscr{R}(R) \subset V\big(N_{G-\{0\}}(X)\big).$$

Moreover, we have

$$(2) \qquad (R^* Rx, x) = \int_\Gamma \|[F'(\gamma)^{1/2}]^{\#} Sx\|^2 \varkappa(d\gamma) = (Sx, x) \qquad (\text{i.e., } R^* R = S),$$

$$(3) \qquad \begin{aligned}(R^* Rx, y) &= (Sx, y) = \big(F'(\gamma)^{1/2}(Rx)(\gamma), y\big) \\ &= \int_\Gamma \big((Rx)(\gamma), F'(\gamma)^{1/2} y\big)\varkappa(d\gamma) \\ &= (Rx, VX_0 y) = \big((VX_0)^* Rx, y\big) \ \varkappa\text{-a.e.}\end{aligned}$$

Let us put $T = V^{-1} R$, where V is the isomorphism defined in Lemma 2.3. Then, by (1)-(3), we have $T \in \mathscr{A}_0^{G-\{0\}}$, and hence $T^* T = R^* R = S \in \mathscr{D}_0^{G-\{0\}}$.

Now we state the main result of this paper.

3.4. Theorem. *Let $X = \{X_g: g \in G\}$ be an H-valued SSP over a discrete Abelian group G and let F be its spectral measure. Suppose that H is separable and the process X has the operator spectral density F' (with respect to $\varkappa$). Let $\mathscr{D}_0$ be the set of all those operators $S \in L^+(H)$ for which*

$$Sx \in \mathscr{R}\big(F'(\gamma)^{1/2}\big) \ \varkappa\text{-a.e.}$$

and

$$(Sx, x) = \int_\Gamma \big([F'(\gamma)^{1/2}]^{\#} Sx, [F'(\gamma)^{1/2}]^{\#} Sx\big)\varkappa(d\gamma)$$

for every $x \in H$.

Then the prediction error operator Σ_0 and the predictor $\hat{X}_0$ are given by the following formulas:

$$(1) \qquad \Sigma_0 = \max \mathscr{D}_0,$$

$$(2) \qquad \hat{X}_0 x = V^{-1}\big(F'(\cdot)^{1/2} x - [F'(\cdot)^{1/2}]^{\#} \Sigma_0 x\big), \qquad x \in H.$$

Proof. Part (1) follows immediately from Lemmas 3.2 and 3.3. To prove (2) it suffices to observe that if $T = V^{-1} R$, where $(Rx)(\cdot) = [F'(\cdot)^{1/2}]^{\#} \Sigma_0 x$, $x \in H$, then according to the proof of Lemma 3.3 we have $T \in \mathscr{A}_0^{G-\{0\}}$ and $T^* T = \Sigma_0$. Thus, by Lemma 3.2, $T = X_0 - \hat{X}_0$, which completes the proof.

4. Final remarks.

4.1. First, let us note that for a finite-dimensional Hilbert space Theorem 3.4 contains the results stated in the Introduction. Suppose $H = \mathbf{C}^q$ $(q < \infty)$.

Then, by Theorem 3.4, Σ_0 is the maximal element of the set $\mathscr{D}_0$ of all $q\times q$ Hermitian non-negative matrices S for which

(i) $$\mathscr{R}(S)\subset\mathscr{R}\big(F'(\gamma)^{1/2}\big)=\mathscr{R}\big(F'(\gamma)\big)\ \varkappa\text{-a.e.},$$

(ii) $$S=\int_\Gamma SF'(\gamma)^{\#}S\varkappa(d\gamma)$$

(here $A^{\#}$ denotes the generalized inverse matrix of a matrix A in the sense of Penrose; for its properties we refer to [6]). In that case one can find an explicit form of Σ_0.

4.2. COROLLARY (cf. [8] and [3], Theorem 4.5). *Assume that* $\{X_g=(X_g^1,\ldots,X_g^q):g\in G\}$ *is a* q*-variate stationary stochastic process over* G *and let* F' *be its spectral density. Let* J *be the space of all* $x\in\mathbf{C}^q$ *(the elements of* $\mathbf{C}^q$ *are regarded as column vectors) such that*

$$x\in\mathscr{R}\big(F'(\gamma)\big)\ \varkappa\text{-a.e.},\qquad \int_\Gamma x^*F'(\gamma)^{\#}x\varkappa(d\gamma)<\infty.$$

Then

$$\Sigma_0=\Big[\int_\Gamma P_JF'(\gamma)^{\#}P_J\varkappa(d\gamma)\Big]^{\#}.$$

Proof. Write

$$S_0=\Big[\int_\Gamma P_JF'(\gamma)^{\#}P_J\varkappa(d\gamma)\Big]^{\#}$$

and observe that, according to the results of [6], S_0 satisfies (i) and (ii) of 4.1. In fact, we have

(i) $$\mathscr{R}(S_0)=\mathscr{N}\Big(\int_\Gamma P_JF'(\gamma)^{\#}P_J\varkappa(d\gamma)\Big)^{\perp}$$
$$=\mathscr{R}\Big(\int_\Gamma P_JF'(\gamma)^{\#}P_J\varkappa(d\gamma)\Big)\subset J\subset\mathscr{R}\big(F'(\gamma)\big)\ \varkappa\text{-a.e.},$$

(ii) $$S_0=S_0S_0^{\#}S_0=S_0\Big(\int_\Gamma P_JF'(\gamma)^{\#}P_J\,d\varkappa\Big)S_0=\int_\Gamma S_0F'(\gamma)^{\#}S_0\varkappa(d\gamma).$$

We shall prove that $S_0\geqslant S$ for every S satisfying (i) and (ii) of 4.1. Since

$$S=\Big[\int P_{\mathscr{R}(S)}F'(\gamma)^{\#}P_{\mathscr{R}(S)}\,d\varkappa\Big]^{\#}=\Big[P_{\mathscr{R}(S)}\Big(\int P_JF'(\gamma)^{\#}P_J\,d\varkappa\Big)P_{\mathscr{R}(S)}\Big]^{\#}$$
$$=\big[P_{\mathscr{R}(S)}S_0^{\#}P_{\mathscr{R}(S)}\big]^{\#},$$

the proof will be completed if we show the following fact:

If S_0 *is a* $q\times q$ *Hermitian non-negative (complex) matrix and* M *is a subspace of* $\mathscr{R}(S_0)$*, then* $S_0\geqslant(P_MS_0^{\#}P_M)^{\#}$.

To see this we put $S=(P_MS_0^{\#}P_M)^{\#}$ and $T=(S_0^{1/2})^{\#}S$. Since

$$\mathscr{R}(S)=\mathscr{R}(P_MS_0^{\#}P_M)\subset M\quad\text{and}\quad S^{\#}=P_MS_0^{\#}P_M,$$

we have

$$(S_0^{1/2}-T)^* T = S_0^{1/2} S_0^{1/2\#} S - SS_0^{\#} S = P_{\mathscr{R}(S)} S - (SP_M) S_0^{\#} (P_M S)$$
$$= S - S(P_M S_0^{\#} P_M) S = S - SS^{\#} S = 0.$$

Thus

$$S_0 = (S_0^{1/2})(S_0^{1/2}) = [(S_0^{1/2}-T)+T]^* [(S_0^{1/2}-T)+T]$$
$$= (S_0^{1/2}-T)^* (S_0^{1/2}-T) + T^* T \geqslant T^* T = SS_0^{\#} S = S.$$

Finally, we state a generalization of Theorem 3.4 to the case of processes without spectral densities. The following proposition will take place of Lemma 2.3 in our considerations.

4.3. PROPOSITION ([1], see also [2]). *Let X be an H-valued SSP over G with the spectral measure F and let $H_{2,F}$ denote the set of all H-valued measures m on $\mathscr{B}(\Gamma)$ such that*

$$m(\Delta) \in \mathscr{R}(F(\Delta)^{1/2}) \quad \text{for every } \Delta \in \mathscr{B}(\Gamma),$$
$$\sup_{\sigma \in \mathscr{F}} \sum_{\Delta \in \sigma} \|[F(\Delta)^{1/2}]^{\#} m(\Delta)\|^2 \overset{\text{df}}{=} \|m\|_F^2 < \infty,$$

where $\mathscr{F}$ is the set of all finite Borel partitions of Γ. Then

(a) *$(H_{2,F}, \|\cdot\|_F)$ is a Hilbert space;*

(b) *the mapping U defined by*

$$U(X_g x)(\Delta) = \int_{\Delta} \langle g, \gamma \rangle F(d\gamma)\, x, \qquad \Delta \in \mathscr{B}(\Gamma),$$

extends linearly to an isometry from $M(X)$ onto $H_{2,F}$;

(c) *for every $A \subset G$*

$$U(N_A(X)) = \{m \in H_{2,F}\colon \hat{m}(g) \overset{\text{df}}{=} \int_{\Gamma} \overline{\langle g, \gamma \rangle}\, m(d\gamma) = 0 \text{ for all } g \in A\}.$$

Using this proposition and the same arguments as in Theorem 3.4 one can show the following

4.4. THEOREM. *Suppose that X is an H-valued SSP with the spectral measure F. Then Σ_0 is the maximal element of the set $\mathscr{D}_0$ of all operators $S \in L^+(H)$ for which*

$$Sx\varkappa(\Delta) \in \mathscr{R}(F(\Delta)^{1/2}) \quad \text{for every } \Delta \in \mathscr{B}(\Gamma),$$
$$\sup_{\sigma \in \mathscr{F}} \sum_{\Delta \in \sigma} \|[F(\Delta)^{1/2}]^{\#} Sx\varkappa(\Delta)\|^2 = (Sx, x)$$

for every $x \in H$.

We point out that according to the results of [1] (or [2]) a Hilbert space H in Theorems 3.4 and 4.4 can be replaced by any linear topological space (under adequate definitions of $[F'(\cdot)]^{1/2}$ and $H_{2,F}$).

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