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ON THE RATE OF CONVERGENCE IN THE CENTRAL LIMIT THEOREM FOR FUNCTIONS OF THE AVERAGE OF INDEPENDENT RANDOM VARIABLES

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Abstract. We give the rate of convergence in the central limit theorem and the random central limit theorem for functions belonging to the class $\mathscr{G}$ of all real differentiable functions g such that $g' \in L(1)$.

1. Introduction and notation. Let $\{X_k, k \geqslant 1\}$ be a sequence of independent random variables and put $S_n = \sum X_k$, $k = 1, 2, \ldots, n$. The asymptotical normality of $\{g(S_n/n), n \geqslant 1\}$, where g is a real function, were considered for instance in [1] (Theorem 4.2.5, p. 76), [8] (Theorem 9.3.1, p. 259), [5], [6], and in [3] for random elements of a Hilbert space. We are interested in the rate convergence in law of the normalized sequence $\{g(S_n/n), n \geqslant 1\}$.

Throughout this paper we shall use the following notation:

$\mathscr{G}$ – the class of all real, differentiable functions g such that g' satisfies the Lipschitz condition, i.e.

$$(1) \qquad |g'(x) - g'(y)| < L|x-y|,$$

where L is a positive constant;

Φ – the class of all functions φ defined on R for which

(a) φ is nonnegative, even, and nondecreasing on $[0, \infty]$,

(b) $x/\varphi(x)$ is defined for all x and nondecreasing $[0, \infty)$;

$\mathscr{D}$ – the class of all sequences $\{d_n, n \geqslant 1\}$ of positive numbers such that $d_n \to \infty$, $n \to \infty$,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

C denotes a positive constant.

Moreover, we shall often use the following results.

LEMMA 1.1 ([7], p. 28). *Assume that X and Y are random variables and $F(x) = P[X < x]$, $G(x) = P[X+Y < x]$. Then, for any $\varepsilon > 0$, $x \in \mathbf{R}$, and any distribution function H,*

$$\text{(i)}\quad |G(x)-H(x)| \leqslant \sup_x |F(x)-H(x)| + \max\{|H(x-\varepsilon)-H(x)|, |H(x+\varepsilon)-H(x)|\} + P[|Y| \geqslant \varepsilon].$$

From (i) we get

COROLLARY 1.1. *For any given $\varepsilon > 0$*

$$\text{(ii)}\quad \sup_x |G(x)-\Phi(x)| \leqslant \sup_x |F(x)-\Phi(x)| + \varepsilon/\sqrt{2\pi} + P[|Y| \geqslant \varepsilon].$$

2. Uniform estimates. In what follows we need the following

LEMMA 2.1. *Let Z be a random variable, and let $b, c \in \mathbf{R}$, $c \neq 0$. Then for every $d > 0$ and every $g \in \mathcal{G}$ with $g'(b/c) \neq 0$*

$$(2)\quad \sup_x \left|P\left\{\frac{c}{g'(b/c)}[g(Z/c)-g(b/c)] < x\right\} - \Phi(x)\right| \leqslant 5\sup_x |P[Z-b < x] - \Phi(x)| + \frac{4}{d\sqrt{2\pi}}\exp\{-d^2/2\} + \frac{Ld^2}{|cg'(b/c)|\sqrt{2\pi}}.$$

Proof. Put

$$h(x) = \begin{cases} \dfrac{g(x)-g(b/c)}{(x-b/c)g'(b/c)} & \text{if } x \neq b/c, \\ 1 & \text{if } x = b/c. \end{cases}$$

We see that

$$\frac{c}{g'(b/c)}\left[g\left(\frac{Z}{c}\right) - g(b/c)\right] = (Z-b)\,h\left(\frac{Z}{c}\right).$$

Hence, by (ii), for any given $\varepsilon > 0$, we have

$$(3)\quad \sup_x \left|P\left\{\frac{c}{g'(b/c)}[g(Z/c)-g(b/c)] < x\right\} - \Phi(x)\right| = \sup_x |P\{Z-b+(Z-b)(h(Z/c)-1) < x\} - \Phi(x)| \leqslant \sup_x |P[Z-b < x] - \Phi(x)| + \varepsilon/\sqrt{2\pi} + P[|(Z-b)(h(Z/c)-1)| \geqslant \varepsilon].$$

Note now that for any $d > 0$

$$(4)\quad P[|(Z-b)(h(Z/c)-1)| \geqslant \varepsilon] \leqslant P[|Z-b| > d] + P[|h(Z/c)-1| \geqslant \varepsilon/d] \leqslant 2\sup_x |P[Z-b < x] - \Phi(x)| + 2(1-\Phi(d)) + P[|h(Z/c)-1| \geqslant \varepsilon/d].$$

Taking into account the definition of h and (1), we get

$$(5)\quad P[|h(Z/c)-1| > \varepsilon/d] = P\left[\left|\frac{g(Z/c)-g(b/c)}{(Z/c-b/c)g'(b/c)}-1\right| \geqslant \varepsilon/d\right]$$

$$= P\left[\left|\frac{g'(b/c+\theta(Z/c-b/c))}{g'(b/c)}-1\right| \geqslant \varepsilon/d\right] \leqslant P\left[|Z-b| \geqslant (\varepsilon/d)L^{-1}|cg'(b/c)|\right]$$

$$\leqslant 2\sup_x |P[Z-b<x]-\Phi(x)| + 2\left(1-\Phi\left((\varepsilon/d)L^{-1}|cg'(b/c)|\right)\right)$$

as $0<\theta<1$.

Combining (3)-(5) we obtain

$$(6)\quad \sup_x \left|P\left\{\frac{c}{g'(b/c)}[g(Z/c)-g(b/c)] < x\right\}-\Phi(x)\right|$$

$$\leqslant 5\sup_x |P[Z-b<x]-\Phi(x)| + 2(1-\Phi(d))+$$

$$+2\left(1-\Phi\left((\varepsilon/d)L^{-1}|cg'(b/c)|\right)\right)+\varepsilon/\sqrt{2\pi}.$$

Putting, in (6), $\varepsilon = Ld^2/|cg'(b/c)|$ we get (2).

COROLLARY 2.1. *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of random variables, and let* $S_n = \sum X_k$ $(k = 1, 2, \ldots, n)$. *Suppose that* $\{a_k, k \geqslant 1\}$, $\{b_k, k \geqslant 1\}$ *and* $\{c_k, k \geqslant 1\}$ *are sequences of real numbers such that* $a_k > 0$, $c_k \neq 0$, $k \geqslant 1$. *Then for every* $d > 0$ *and every* $g \in \mathscr{G}$ *with* $g'(b_k/c_k) \neq 0$, $k \geqslant 1$,

$$(7)\quad \sup_x \left|P\left\{\frac{c_n}{g'(b_n/c_n)}\left[g\left(\frac{S_n}{a_n c_n}\right)-g\left(\frac{b_n}{c_n}\right)\right] < x\right\}-\Phi(x)\right|$$

$$\leqslant 5\sup_x \left|P\left[\frac{S_n}{a_n}-b_n < x\right]-\Phi(x)\right| + \frac{4}{d\sqrt{2\pi}}\exp\{-d^2/2\} + \frac{Ld^2}{|c_n g'(b_n/c_n)|\sqrt{2\pi}}.$$

COROLLARY 2.2. *Let* $\{X_k, k \geqslant 1\}$ *be a sequence of independent random variables with finite expectations* EX_k *and variances* $\sigma^2 X_k$, $k \geqslant 1$. *Then for every* $d > 0$ *and every* $g \in \mathscr{G}$ *with* $g'(\mu_n) \neq 0$, *where* $\mu_n = n^{-1}\sum EX_k$ $(k = 1, 2, \ldots, n)$,

$$(8)\quad \sup_x \left|P\left\{\frac{n}{s_n g'(\mu_n)}\left[g\left(\frac{S_n}{n}\right)-g(\mu_n)\right] < x\right\}-\Phi(x)\right|$$

$$\leqslant 5\sup_x \left|P\left[\frac{S_n-ES_n}{s_n} < x\right]-\Phi(x)\right| + \frac{Ld^2 s_n}{n|g'(\mu_n)|} + \frac{4}{d\sqrt{2\pi}}\exp\{-d^2/2\},$$

$$s_n^2 = \sum_{k=1}^{n} \sigma^2 X_k.$$

COROLLARY 2.3. *Let $\{X_k, k \geq 1\}$ be a sequence of independent identically distributed random variables with $EX_1 = \mu$, $\sigma^2 X_1 = \sigma^2 < \infty$. Then for every $d > 0$ and every $g \in \mathcal{G}$ with $g'(\mu) \neq 0$*

$$(9)\quad \sup_x \left| P\left\{\frac{\sqrt{n}}{g'(\mu)\sigma}\left[g\left(\frac{S_n}{n}\right) - g(\mu)\right] < x\right\} - \Phi(x)\right| \leq 5\sup_x \left| P\left[\frac{S_n - n\mu}{\sigma\sqrt{n}} < x\right] - \Phi(x)\right| + \frac{Ld^2\sigma}{\sqrt{n}|g'(\mu)|} + \frac{4}{d\sqrt{2\pi}}\exp(-d^2/2).$$

Put

$$\mu_n = n^{-1}\sum_{k=1}^{n} EX_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma^2 X_k, \quad X_k^0 = X_k - EX_k, \quad k \geq 1.$$

Estimates (7)-(9) and the known estimates the convergence rate in the central limit theorem allow to obtain, among other things, the following results.

THEOREM 2.4. *Let $\{X_k, k \geq 1\}$ be a sequence of independent random variables such that $E(X_k^0)^2\varphi(X_k^0) < \infty$, $k \geq 1$, for some $\varphi \in \Phi$.*

Then for every $g \in \mathcal{G}$ with $g'(\mu_n) \neq 0$, $n \geq 1$, and any sequence $\{d_n, n \geq 1\} \in \mathcal{D}$

$$(10)\quad \sup_x \left| P\left\{\frac{n}{s_n g'(\mu_n)}\left[g\left(\frac{S_n}{n}\right) - g(\mu_n)\right] < x\right\} - \Phi(x)\right| = O\left(\frac{\sum_{k=1}^{n} E(X_k^0)^2\varphi(X_k^0)}{s_n^2\varphi(s_n)} + \frac{s_n d_n^2}{n|g'(\mu_n)|} + d_n^{-1}\exp\{-d_n^2/2\}\right).$$

If $E|X_k^0|^3 < \infty$, $k \geq 1$, then for every $g \in \mathcal{G}$ with $g'(\mu) \neq 0$, $n \geq 1$, and any sequence $\{d_n, n \geq 1\} \in \mathcal{D}$

$$(11)\quad \sup_x \left| P\left\{\frac{n}{s_n g'(\mu_n)}\left[g\left(\frac{S_n}{n}\right) - g(\mu_n)\right] < x\right\} - \Phi(x)\right| = O\left(\frac{\sum_{k=1}^{n} E|X_k^0|^3}{s_n^3} + \frac{s_n d_n^2}{n|g'(\mu_n)|} + d_n^{-1}\exp\{-d_n^2/2\}\right).$$

COROLLARY 2.5. *If $\{X_k, k \geq 1\}$ is a sequence of independent identically distributed random variables, then under the assumptions of Theorem 2.4 for every $g \in \mathcal{G}$ with $g'(\mu) \neq 0$, and any sequence $\{d_n, n \geq 1\} \in \mathcal{D}$, we have*

$$(10')\quad \sup_x \left| P\left\{\frac{\sqrt{n}}{\sigma g'(\mu)}\left[g\left(\frac{S_n}{n}\right) - g(\mu)\right] < x\right\} - \Phi(x)\right| = O\left(\frac{1}{\varphi(\sigma\sqrt{n})} + \frac{d_n^2}{\sqrt{n}} + d_n^{-1}\exp\{-d_n^2/2\}\right),$$

$$(11') \quad \sup_x \left| P\left\{ \frac{\sqrt{n}}{\sigma g'(\mu)} \left[g\left(\frac{S_n}{n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O\left(\frac{d_n^2}{\sqrt{n}} + d_n^{-1} \exp\{-d_n^2/2\} \right),$$

respectively.

Note now that putting in (10)

$$d_n = \left\{ 2 \ln \left(1 + \frac{s_n^2 \varphi(s_n)}{\sum_{k=1}^{n} E(X_k^0)^2 \varphi(X_k^0)} \right) \right\}^{1/2},$$

and in (11)

$$d_n = \left\{ 2 \ln \left(1 + \frac{s_n^3}{\sum_{k=1}^{n} E|X_k^0|^3} \right) \right\}^{1/2},$$

one can get the following estimates:

Corollary 2.6. *Under the assumptions of Theorem 2.4 for every $g \in \mathscr{G}$ with $g'(\mu_n) \neq 0$, $n \geqslant 1$,*

$$(13) \quad \sup_x \left| P\left\{ \frac{n}{s_n g'(\mu_n)} \left[g\left(\frac{S_n}{n}\right) - g(\mu_n) \right] < x \right\} - \Phi(x) \right| = O\left(\frac{\sum_{k=1}^{n} E(X_k^0)^2 \varphi(X_k^0)}{s_n^2 \varphi(s_n)} + \frac{s_n \ln \varphi(s_n)}{n |g'(\mu_n)|} \right),$$

and

$$(14) \quad \sup_x \left| P\left\{ \frac{n}{s_n g'(\mu_n)} \left[g\left(\frac{S_n}{n}\right) - g(\mu_n) \right] < x \right\} - \Phi(x) \right| = O\left(\frac{\sum_{k=1}^{n} E|X_k^0|^3}{s_n^3} + \frac{s_n \ln s_n}{n |g'(\mu_n)|} \right),$$

respectively.

From (13) and (14) we get

Corollary 2.7. *Under the assumptions of Corollary 2.5 we have*

$$(10'') \quad \sup_x \left| P\left\{ \frac{\sqrt{n}}{\sigma g'(\mu)} \left[g\left(\frac{S_n}{n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O\left(\frac{1}{\varphi(\sigma\sqrt{n})} + \frac{\ln \varphi(\sigma\sqrt{n})}{\sqrt{n}} \right),$$

$$(11'') \quad \sup_x \left| P\left\{ \frac{\sqrt{n}}{\sigma g'(\mu)} \left[g\left(\frac{S_n}{n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O\left(\frac{\ln n}{\sqrt{n}} \right).$$

The estimate (9) allows us to give a generalization of a result given in paper [2]:

THEOREM 2.8. *Let $\{X_k, k \geqslant 1\}$ be a sequence of independent identically distributed random variables with $EX_1 = \mu$, $\sigma^2 X_1 = \sigma^2 < \infty$, and $E|X_1|^{2+\delta} < \infty$, $0 < \delta < 1$.*

Then for every $g \in \mathcal{G}$ with $g'(\mu) \neq 0$

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x \left| P\left\{ \frac{\sqrt{n}}{\sigma g'(\mu)} \left[g\left(\frac{S_n}{n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right| < \infty. \tag{15}$$

If $E(X_1-\mu)^2 \log(1+|X_1-\mu|^2) < \infty$, then (15) *converges with $\delta = 0$.*

Proof. From (9) with $d = \sqrt{\ln n}$, we get

$$\sup_x \left| P\left\{ \frac{\sqrt{n}}{\sigma g'(\mu)} \left[g\left(\frac{S_n}{n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right|$$
$$\leqslant C \left\{ \sup_x \left| P\left[\frac{S_n - n\mu}{\sigma \sqrt{n}} < x \right] - \Phi(x) \right| + \frac{\ln n}{\sqrt{n}} \right\}.$$

Moreover, we know [2] that

$$\sum_{n=1}^{\infty} n^{-1+\delta/2} \sup_x \left| P\left[\frac{S_n - n\mu}{\sigma \sqrt{\pi}} < x \right] - \Phi(x) \right| < \infty,$$

which together with the obvious fact

$$\sum_{n=1}^{\infty} \left(n^{-1+\delta/2} (\ln n)/\sqrt{n}\right) < \infty, \qquad 0 \leqslant \delta < 1,$$

allow us to obtain (15).

3. Partial sums with random indices. Following the consideration of Section 1 one can prove the following

LEMMA 3.1. *Let $\{X_k, k \geqslant 1\}$ be a sequence of independent identically distributed random variables with $EX_1 = \mu$, $\sigma^2 X_1 = \sigma^2 < \infty$. Suppose that $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables. Then for every $d > 0$, $\varepsilon > 0$, and every $g \in \mathcal{G}$ with $g'(\mu) \neq 0$*

$$\sup_x \left| P\left\{ \frac{\sqrt{N_n}}{g'(\mu)\sigma} \left[g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right| \tag{16}$$
$$\leqslant 3 \sup_x \left| P\left[\frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} < x \right] - \Phi(x) \right| + P\left\{ \left| \frac{S_{N_n} - N_n \mu}{\sigma \sqrt{N_n}} \right| \geqslant \frac{|g'(\mu)|}{\sigma L} (\varepsilon/d) \sqrt{N_n} \right\} +$$
$$+ \frac{2}{d\sqrt{2\pi}} \exp\{-d^2/2\} + \varepsilon/\sqrt{2\pi}.$$

Using Lemma 4.1 we can give the following results:

THEOREM 3.2. *Let $\{X_k, k \geqslant 1\}$ be a sequence of independent identically distributed random variables with $EX_1 = \mu$, $\sigma^2 X_1 = \sigma^2$, and $E|X_1|^3 < \infty$. Suppose that $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables such that*

$$P\left[\left|\frac{N_n}{na} - 1\right| \geqslant \varepsilon_n\right] = O(\sqrt{\varepsilon_n}), \tag{17}$$

where a is a positive constant, and $1/n \leqslant \varepsilon_n \to 0$, $n \to \infty$.

Then for every $g \in \mathscr{G}$ with $g'(\mu) \neq 0$, and any sequence $\{d_n, n \geqslant 1\}$

$$\sup_x \left|P\left\{\frac{\sqrt{N_n}}{\sigma g'(\mu)}\left[g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu)\right] < x\right\} - \Phi(x)\right| = O(\sqrt{\varepsilon_n} + d_n^2/\sqrt{n} + d_n^{-1}\exp\{-d_n^2/2\}). \tag{18}$$

Proof. Following the considerations of the proof of Lemma 2.1 and using (16) together with assumption (17) one can get

$$\sup_x \left|P\left\{\frac{\sqrt{N_n}}{\sigma g'(\mu)}\left[g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu)\right] < x\right\} - \Phi(x)\right|$$
$$\leqslant C\left\{\sup_x \left|P\left[\frac{S_{N_n} - N_n\mu}{\sigma\sqrt{N_n}} < x\right] - \Phi(x)\right| + d_n^2/\sqrt{n} + d_n^{-1}\exp\{-d_n^2/2\}\right\}$$

for any sequence $\{d_n, n \geqslant 1\} \in \mathscr{D}$, where C is a positive constant. But it has been proved in [4] that

$$\sup_x \left|P\left[\frac{S_{N_n} - N_n\mu}{\sigma\sqrt{N_n}} < x\right] - \Phi(x)\right| = O(\sqrt{\varepsilon_n}),$$

hence we obtain (18).

COROLLARY 3.3. *Under the assumptions of Theorem 3.2*

$$\sup_x \left|P\left\{\frac{\sqrt{N_n}}{\sigma g'(\mu)}\left[g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu)\right] < x\right\} - \Phi(x)\right| = O\left(\sqrt{\varepsilon_n} + \frac{\ln n}{\sqrt{n}}\right).$$

COROLLARY 3.4. *If (17) hold with $\varepsilon_n = (\ln^2 n)/n$, then*

$$\sup_x \left|P\left\{\frac{\sqrt{N_n}}{\sigma g'(\mu)}\left[g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu)\right] < x\right\} - \Phi(x)\right| = O\left(\frac{\ln n}{\sqrt{n}}\right).$$

THEOREM 3.5. *Let $\{X_n, n \geqslant 1\}$ be a sequence of independent identically distributed random variables such that $EX_1 = \mu$, $\sigma^2 X_1 = \sigma^2$, $E|X_1|^3 < \infty$, and $\{\eta_n, n \geqslant 1\}$ be a sequence with $n^{-1} \leqslant \eta_n \to \infty$, $n \to \infty$. Suppose that $\{N_n, n \geqslant 1\}$ is a sequence of positive integer-valued random variables such that there* *exist positive constants* c_1, c_2 *for which*

$$P\left[\left|\frac{N_n}{[\lambda n]}-1\right|>c_1\eta_n\right]=O(\sqrt{\eta_n}), \tag{19}$$

$$P\left[\lambda<\frac{c_2}{n\eta_n}\right]=O(\sqrt{\eta_n}), \tag{20}$$

λ *being a random variable taking values in* $(0, \infty)$ *and independent of* $\{X_k, k \geqslant 1\}$.

Then for every $g \in \mathcal{G}$ *with* $g'(\mu) \neq 0$ *and any sequence* $\{d_n, n \geqslant 1\} \in \mathcal{D}$

$$\sup_x\left|P\left\{\frac{\sqrt{N_n}}{\sigma g'(\mu)}\left[g\left(\frac{S_{N_n}}{N_n}\right)-g(\mu)\right]<x\right\}-\Phi(x)\right| = O(\sqrt{\eta_n}\,d_n^2+d_n^{-1}\exp\{-d_n^2/2\}). \tag{21}$$

Proof. From (16) we have

$$\sup_x\left|P\left\{\frac{\sqrt{N_n}}{\sigma g'(\mu)}\left[g\left(\frac{S_{N_n}}{N_n}\right)-g(\mu)\right]<x\right\}-\Phi(x)\right| \leqslant 3\sup_x\left|P\left[\frac{S_{N_n}-N_n\mu}{\sigma\sqrt{N_n}}<x\right]-\Phi(x)\right|+ \tag{22}$$

$$+P\left[\left|\frac{S_{N_n}-N_n\mu}{\sigma\sqrt{N_n}}\right|\geqslant\left|\frac{|g'(\mu)|}{\sigma L}\right|\geqslant(\varepsilon_n/d_n)\sqrt{N_n}\right]+\frac{2}{d_n\sqrt{2\pi}}\exp\{-d_n^2/2\}+\varepsilon_n/\sqrt{2\pi}$$

for any given $\varepsilon_n > 0$ and $\{d_n, n \geqslant 1\} \in \mathcal{D}$.

Note now that by (19) and (20) we have

$$P\left[\left|\frac{S_{N_n}-N_n\mu}{\sigma\sqrt{N_n}}\right|>\frac{|g'(\mu)|}{\sigma L}(\varepsilon_n/d_n)\sqrt{N_n}\right] \tag{23}$$

$$\leqslant C\left\{P\left[\left|\frac{S_{N_n}-N_n\mu}{\sigma\sqrt{N_n}}\right|\geqslant\frac{|g'(\mu)|}{\sigma L}(\varepsilon_n\sqrt{(1-c_1\eta_n)[c_2/\eta_n]}/d_n)\right]+\sqrt{\eta_n}\right\}$$

$$\leqslant C\left\{2\sup_x\left|P\left[\frac{S_{N_n}-N_n\mu}{\sigma\sqrt{N_n}}<x\right]-\Phi(x)\right|+\right.$$

$$\left.+2\left(1-\Phi\left(\frac{|g'(\mu)|}{\sigma L}\varepsilon_n\sqrt{(1-c_1\eta_n)[c_2/\eta_n]}/d_n\right)\right)+\sqrt{\eta_n}\right\}.$$

Putting

$$\varepsilon_n = d_n^2/\left(\frac{|g'(\mu)|}{\sigma L}\sqrt{(1-c_1\eta_n)[c_2/\eta_n]}\right)$$

and combining (22) and (23), we obtain

$$\sup_x \left| P\left\{ \frac{\sqrt{N_n}}{\sigma g'(\mu)} \left[g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right|$$
$$\leqslant C\left\{ \sup_x \left| P\left[\frac{S_{N_n} - N_n \mu}{\sigma\sqrt{N_n}} < x \right] - \Phi(x) \right| + \sqrt{\eta_n}\, d_n^2 + d_n^{-1} \exp\{-d_n^2/2\} \right.$$

Using now [4] the estimate

$$\sup_x \left| P\left[\frac{S_{N_n} - N_n \mu}{\sigma\sqrt{N_n}} < x \right] - \Phi(x) \right| = O(\sqrt{\eta_n}),$$

we obtain (21)

COROLLARY 3.6. *Under the assumptions of Theorem* 3.5 *for every* $g \in \mathscr{G}$ *with* $g'(\mu) \neq 0$

$$\sup_x \left| P\left\{ \frac{\sqrt{N_n}}{\sigma g'(\mu)} \left[g\left(\frac{S_{N_n}}{N_n}\right) - g(\mu) \right] < x \right\} - \Phi(x) \right| = O\left(\sqrt{\eta_n} \ln(1/\eta_n)\right).$$

REFERENCES

[1] T. W. Anderson, *An introduction to multivariate statistical analysis*, Wiley, New York 1958.

[2] C. C. Heyde, *On the influence of moments on the rate of convergence to the normal distribution*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 8 (1967), p. 12-18.

[3] J. H. Kellermeier, *A limit theorem for functions of asymptotically Gaussian random elements of a Hilbert space*, Bull. Inst. Math. Acad. Sinica 9 (3) (1981), p. 341-345.

[4] D. Landers and L. Rogge, *The exact approximation order in the Central-Limit-Theorem for random summation*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 36 (1977), p. 269-283.

[5] K. W. Morris and D. Szynal, *On the limiting behaviour of some functions of the average of independent random variables*, Ann. Univ. Mariae Curie--Skłodowska, Sect. A, 31 (12) (1977), p. 85-95.

[6] – *On the convergence rate in the central limit theorem of some functions of the average of independent random variables*, PAMS 3 (1) (1982), p. 85-95.

[7] V. V. Petrov, *Sums of independent random variables* (in Russian), Izdat. Nauka, Moscow 1972.

[8] S. S. Wilks, *Mathematical statistics*, New York 1963.

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