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AN EXPLICIT CHARACTERIZATION OF ADMISSIBLE LINEAR ESTIMATORS OF FIXED AND RANDOM EFFECTS IN BALANCED RANDOM MODELS

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Abstract. A necessary and sufficient conditions for a linear estimator of a linear function of fixed and random effects in a balanced random model to be admissible are given. The formulae for admissible estimators depend on certain coefficients from the interval $[0, 1]$, as in well-known results for other models (see e.g. Cohen [3]).

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1. INTRODUCTION

The problem of simultaneous linear estimation of fixed and random effects was initiated by Henderson [7], who was interested in estimating genetic parameters in animal breeding. Further results on unbiased estimation of both kinds of effects were obtained, among others, by Goldberger [4], Henderson ([8], [9]), Harville [6], Rao [19], Jiang [10], Liu et al. [16] and Tian [27]. Synówka-Bejenka and Zontek ([25], [26]) have dealt with admissibility of linear estimators of fixed and random effects in some linear models. To get a characterization, they have shown that the problem of admissibility for a linear function of fixed and random effects is equivalent to the problem of admissibility for a linear function of the expected value only, in another properly defined linear model (called the dual model; see Section 3 for more details). This reduction allows the use of the results concerning admissibility of linear estimators of linear functions of expected value in a general linear model. Using a duality, Synówka-Bejenka and Zontek [25] obtained a characterization of linear admissible estimators of a linear function of fixed and random effects in the multi-way balanced nested classification random model and in the multi-way balanced crossed classification random model (see also Shiqing et al. [21]). They have proved, using a stepwise procedure elaborated by LaMotte [13], that limits of ULBEs are admissible.

The problem of admissibility of linear estimators has received considerable attention in the literature. Despite this, explicit characterizations have been obtained only for special cases. Basing on algebraic properties of matrices Cohen [3] has described all admissible linear estimators of the mean vector in a Gauss–Markov model with identity covariance matrix. Further generalizations have been given, among others, by Rao [18], Mathew et al. [17], Stępniak [22], Zontek [28], Klonecki and Zontek [12], Baksalary and Markiewicz [1], Baksalary et al. [2], Groß and Markiewicz [5] and Stępniak [24]. The problem of characterizing admissible linear estimators of a linear function of expected value was also considered in terms of connection between the closure of the set of unique locally best estimators (ULBE) and the set of admissible linear estimators. This approach was applied, among others, by Stępniak [23], Zontek [29] and LaMotte [15]. LaMotte has shown that every admisible linear estimator is the limit of linear estimators that are uniquely best at points in the minimal closed convex cone containing the original parameter set. On the other hand, he gave an example of a model in which a limit of ULBEs may not be admissible (see LaMotte [14]).

Basing on LaMotte's results [13] Synówka-Bejenka and Zontek [26] have proved that for linear models with finitely generated parameter space every limit of a sequence of ULBEs is admissible. To prove that, they applied a stepwise procedure of LaMotte [13]. Thus they showed that for such models the class of all admissible linear estimators consists of all ULBEs and their limits. For example, they described this class in a model dual to a random linear model for spatially located sensors measuring intensity of a source of signals at discrete instants of time. A special case of this model is the so called one-way balanced random model.

In this paper we note that for any model dual to a balanced random model the parameter space is finitely generated. In such models it is enough to present formulas for ULBEs in the form for which their limits can be characterized. We will use this approach to give explicit formulas for linear admissible estimators of a linear function of fixed and random effects in a balanced random model.

Throughout this paper, $\mathcal{M}_{m\times q}$ denotes the space of $m \times q$ real matrices. The symbols A' and $\mathcal{R}(A)$ stand for the transpose and column space of $A \in \mathcal{M}_{m\times q}$, respectively. For $A_1 \in \mathcal{M}_{m_1\times q_1}$ and $A_2 \in \mathcal{M}_{m_2\times q_2}$ the symbols $A_1 \otimes A_2$ and $\mathrm{diag}(A_1, A_2)$ denote the Kronecker product and the matrix whose diagonal consists of A_1 and A_2, respectively. The minimal closed convex cone containing a set $A \subset \mathcal{M}_{m\times m} \times \mathcal{M}_{m\times m}$ will be denoted by $[A]$. We write I_m and $\mathbf{1}_m$ for the $m\times m$ identity matrix and the m-vector of ones, respectively.

2. BALANCED RANDOM MODELS

Let $Y_{i_1,\ldots,i_r}$, where $i_j = 1, \ldots, n_j$ for $j = 1, \ldots, r$, be a random variable having the following structure (compare Khuri et al. [11]):

(2.1) $$Y_{i_1,i_2,\dots,i_r} = \beta + u_{1\psi_1} + u_{2\psi_2} + \cdots + u_{k\psi_k} + e_{i_1,i_2,\dots,i_r},$$

where β is an unknown parameter (fixed effect), the term $u_{i\psi_i}$ denotes the ith random effect and the symbol ψ_i will be identified with the set of subscripts for the ith effect. The last term in (2.1) denotes a random experimental error. We assume that $u_{1\psi_1},\dots,u_{k\psi_k}$ and $e_{i_1,\dots,i_r}$ are uncorrelated random variables with zero mean and variances $\sigma_1^2,\dots,\sigma_k^2$ and σ_{k+1}^2, respectively. To simplify the notation let $\psi_0 = \emptyset$ and $\psi_{k+1} = \{i_1,\dots,i_r\}$. Arranging the $Y_{i_1,\dots,i_r}$ in lexicographic order into an n-vector Y, where $n = \prod_{j=1}^r n_j$, we get

(2.2) $$Y = Z_0\beta + Z_1 u_1 + \cdots + Z_k u_k + Z_{k+1} e,$$

where the matrix Z_i is the Kronecker product

(2.3) $$Z_i = \bigotimes_{j=1}^{r} N_{ij}, \quad i = 0, 1, \dots, k+1,$$

where

$$N_{ij} = \begin{cases} I_{n_j} & \text{when } i_j \in \psi_i, \\ \mathbf{1}_{n_j} & \text{otherwise.} \end{cases}$$

Of course $Z_0 = \mathbf{1}_n$ and $Z_{k+1} = I_n$. Clearly,

$$\mathrm{E}(Y) = Z_0\beta \quad \text{and} \quad \mathrm{cov}(Y) = \sum_{i=1}^{k} \sigma_i^2 Z_i Z_i' + \sigma_{k+1}^2 I_n.$$

This will be schematically written as

(2.4) $$Y \sim \Big(Z_0\beta, \sum_{i=1}^{k+1} \sigma_i^2 Z_i Z_i'\Big).$$

This structure covers the well-known examples of balanced random models, e.g. the multi-way nested classification model or the multi-way crossed classification model (with or without interactions). Such models are widely used in various areas of scientific research. Many examples of their applications are described by Sahai and Ojeda [20].

To obtain a characterization of admissible linear estimators of fixed and random effects in the model (2.4), we use some properties of balanced models. Directly from (2.3) we have

(2.5) $$(Z_i Z_i')^2 = p_i Z_i Z_i', \quad i = 0, 1, \dots, k+1,$$

where

$$p_i = \begin{cases} \prod_{i_l \notin \psi_i} n_l & \text{when } \psi_i \neq \{i_1,\dots,i_r\}, \\ 1 & \text{when } \psi_i = \{i_1,\dots,i_r\}. \end{cases}$$

Note that (2.5) implies that $\frac{1}{p_i}Z_iZ_i'$ is an orthogonal projector on $\mathcal{R}(Z_i)$. Moreover, an orthogonal and idempotent basis of the space generated by the matrices $Z_0Z_0', Z_1Z_1', \ldots, Z_{k+1}Z_{k+1}'$ has the form

$$(E_0, E_1, \ldots, E_{k+1})' = ((P\Delta)^{-1} \otimes I_n)(Z_0Z_0', Z_1Z_1', \ldots, Z_{k+1}Z_{k+1}')', \tag{2.6}$$

where

$$P = \operatorname{diag}\{p_0, p_1, \ldots, p_{k+1}\},$$
$$\Delta = (\delta_{ij}) = \begin{cases} 1 & \text{when } \psi_j \subset \psi_i, \\ 0 & \text{when } \psi_j \not\subset \psi_i. \end{cases}$$

If $Z_1Z_1', \ldots, Z_kZ_k'$ are ordered in such a way that

$$\mathcal{R}\Big(\sum_{j=1}^{i} Z_jZ_j'\Big) \subsetneq \mathcal{R}\Big(\sum_{j=1}^{i+1} Z_jZ_j'\Big), \quad i = 1, \ldots, k-1,$$

then Δ is a lower triangular matrix. Putting $\Lambda = (\lambda_{ij}) = \Delta^{-1}$ we get $E_i = \sum_{j=0}^{i} \frac{\lambda_{ij}}{p_j} Z_jZ_j'$ for $i = 0, 1, \ldots, k+1$.

From (2.6) one can check the following relationships between the matrices Z_iZ_i' and E_i, $i = 0, 1, \ldots, k+1$ (compare Khuri et al. [11]):

(i) $Z_iZ_i' = p_i \sum_{j=0}^{i} \delta_{ij}E_j$, $i = 0, 1, \ldots, k+1$,

(ii) $E_jZ_iZ_i' = p_i\delta_{ij}E_j$, $i, j = 0, 1, \ldots, k+1$.

To obtain a characterization of linear admissible estimators of a linear function of fixed and random effects in a balanced random model we use our earlier results on some duality rule. Section 3 describes background for balanced random models.

3. BACKGROUND

Let Y be a random n-vector for which the expected value $\mathrm{E}\,Y$ and the covariance $\operatorname{cov}(Y)$ are given by (2.4). We are interested in admissible estimation of

$$\theta = [(K'Z_0\beta)', (Q_1'Z_1u_1)', \ldots, (Q_k'Z_ku_k)']' \tag{3.1}$$

in the class of linear estimators

$$L'Y = (L_0, L_1, \ldots, L_k)'Y, \tag{3.2}$$

where $K, L_0 \in \mathcal{M}_{n\times t_0}$; $Q_1, L_1 \in \mathcal{M}_{n\times t_1}$; $\ldots$; $Q_k, L_k \in \mathcal{M}_{n\times t_k}$. To compare estimators we use the ordinary quadratic risk function

$$\mathrm{E}[(L'Y - \theta)'(L'Y - \theta)].$$

Synówka-Bejenka and Zontek [25] have shown that the quadratic risk function of the estimator $L'Y$ of θ in the model (2.4) is equal to the quadratic risk function of the linear estimator $\boldsymbol{L}'\boldsymbol{Y}$ of $\boldsymbol{K}'\boldsymbol{X}\beta$ in the dual model, which is defined by

$$\boldsymbol{Y} = [Y', (Z_1u_1)', \dots, (Z_ku_k)']' \sim \Big(\boldsymbol{X}\beta, \sum_{i=1}^{k+1} \sigma_i^2 \boldsymbol{V}_i\Big), \tag{3.3}$$

where

$$\begin{aligned} \boldsymbol{X} &= (Z_0', \mathbf{0}, \dots, \mathbf{0})', \\ \boldsymbol{V}_i &= (v_1 + v_{i+1})(v_1 + v_{i+1})' \otimes Z_iZ_i', \qquad i = 1, \dots, k, \\ \boldsymbol{V}_{k+1} &= v_1v_1' \otimes I_n, \end{aligned}$$

while v_i is the ith versor in $\mathbb{R}^{k+1}$. This means that a linear estimator $L'Y$ of θ is admissible in the model (2.4) if and only if the corresponding estimator

$$\boldsymbol{L}'\boldsymbol{Y} = \begin{bmatrix} L_0 & L_1 & \cdots & L_k \\ \mathbf{0} & -Q_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -Q_k \end{bmatrix}' \begin{bmatrix} Y \\ Z_1u_1 \\ \vdots \\ Z_ku_k \end{bmatrix} \tag{3.4}$$

of

$$\boldsymbol{K}' \mathrm{E}\, \boldsymbol{Y} = \begin{bmatrix} K & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}' \begin{bmatrix} Z_0\beta \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

is admissible in the model (3.3).

Note that the class of linear estimators of $\boldsymbol{K}' \mathrm{E}\, \boldsymbol{Y}$ considered in the model (3.3) is restricted to the set

$$\boldsymbol{\mathcal{E}}_0 = \{\boldsymbol{L}'\boldsymbol{Y} : \boldsymbol{L} \in \mathcal{L}_0\},$$

where $\mathcal{L}_0$ is the affine set given by

$$\mathcal{L}_0 = \{\boldsymbol{L}_0 + \boldsymbol{\Pi}_0\boldsymbol{M} : \boldsymbol{M} \in \mathcal{M}_{(k+1)n\times(t_0+\cdots+t_k)}\},$$

while $\boldsymbol{L}_0 = \operatorname{diag}(\mathbf{0}, -Q_1, \dots, -Q_k)$ and $\boldsymbol{\Pi}_0 = v_1v_1' \otimes I_n$. Following LaMotte [13], consider the set

$$\mathcal{T} = \{(\operatorname{cov}(\boldsymbol{Y}), \mathrm{E}\,\boldsymbol{Y}(\mathrm{E}\,\boldsymbol{Y})') : \beta \in \mathbb{R},\ \sigma_1^2 \geqslant 0, \dots, \sigma_{k+1}^2 \geqslant 0\,\} \tag{3.5}$$

as a new space of parameters and a point $(W_1, W_2) \in [\mathcal{T}]$ as an argument of an extended quadratic risk function of $\boldsymbol{L}'\boldsymbol{Y}$, i.e.,

$$\mathrm{R}(\boldsymbol{L}'\boldsymbol{Y}; (W_1, W_2)) = \operatorname{tr}[\boldsymbol{L}'W_1\boldsymbol{L} + (\boldsymbol{L} - \boldsymbol{K})'W_2(\boldsymbol{L} - \boldsymbol{K})].$$

Recall that an estimator $\boldsymbol{L}'\boldsymbol{Y}$ with $\boldsymbol{L} \in \mathcal{L}$, where $\mathcal{L}$ is an affine subset of $\mathcal{L}_0$, is called *locally best among* $\mathcal{L}$ at a point $(W_1, W_2) \in [\mathcal{T}]$ if

$$\mathrm{R}(\boldsymbol{L}'\boldsymbol{Y}; (W_1, W_2)) \leqslant \mathrm{R}(\boldsymbol{N}'\boldsymbol{Y}; (W_1, W_2))$$

for every $\boldsymbol{N} \in \mathcal{L}$. LaMotte [13] has shown that an estimator $\boldsymbol{L}'\boldsymbol{Y}$ is locally best among $\mathcal{L}$ at $(W_1, W_2) \in [\mathcal{T}]$ iff

$$\boldsymbol{\Pi}'(W_1 + W_2)\boldsymbol{L} = \boldsymbol{\Pi}' W_2 \boldsymbol{K},$$

where $\boldsymbol{\Pi}$ is a $((k+1)n \times (k+1)n)$-matrix such that

$$\mathcal{L} = \{\boldsymbol{L} + \boldsymbol{\Pi}\boldsymbol{M} : \boldsymbol{M} \in \mathcal{M}_{(k+1)n\times(t_0+\cdots+t_k)}\}.$$

Note that the parameter space given by (3.5) corresponding to the dual model (3.3) is a finitely generated closed convex cone, i.e.,

$$\mathcal{T} = [\mathcal{T}] = \Big\{\sum_{i=0}^{k+1} t_i(W_{1i}, W_{2i}) : t_0 \geqslant 0, \ldots, t_{k+1} \geqslant 0\Big\}, \tag{3.6}$$

where

$$\begin{aligned}(W_{10}, W_{20}) &= (\boldsymbol{0}, \boldsymbol{X}\boldsymbol{X}'),\\ (W_{1i}, W_{2i}) &= (\boldsymbol{V}_i, \boldsymbol{0}), \qquad i = 1, \ldots, k+1.\end{aligned}$$

To avoid trivialities we also assume that

$$\mathcal{R}\Big(\sum_{i=0}^{k+1} \boldsymbol{\Pi}'(W_{1i} + W_{2i})\boldsymbol{\Pi}\Big) = \mathcal{R}(\boldsymbol{\Pi}') \tag{3.7}$$

and

$$\boldsymbol{\Pi}'(W_{1i} + W_{2i})\boldsymbol{\Pi} \neq 0 \quad \text{for } i = 0, \ldots, k+1.$$

In the next section we apply this approach to obtain an explicit characterization of admissible linear estimators of fixed and random effects in the model (2.4).

4. MAIN RESULT

To characterize admissible estimators $\boldsymbol{L}'\boldsymbol{Y}$ for $\boldsymbol{K}'\,\mathrm{E}\,\boldsymbol{Y}$ among $\mathcal{L}_0$ in the model (3.3) corresponding to (2.4) we prove the following lemma, which gives necessary and sufficient conditions for an estimator $\boldsymbol{L}'\boldsymbol{Y}$ to be ULBE at $(W_1, W_2) \in \mathcal{T}$.

LEMMA 4.1. *An estimator* $\boldsymbol{L}'\boldsymbol{Y}$ *is ULBE at* $(\sum_{i=1}^{k+1} s_i\boldsymbol{V}_i, s_0\boldsymbol{X}\boldsymbol{X}')$ *in* $\mathcal{T}$ *among* $\mathcal{L}_0$ *in the model* (3.3) *corresponding to* (2.4) *if and only if* $s_0 \geqslant 0, \ldots, s_k \geqslant 0, s_{k+1} > 0$ *and*

$$\boldsymbol{L} = \begin{bmatrix} L_0 & L_1 & \cdots & L_k \\ \boldsymbol{0} & -Q_1 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & -Q_k \end{bmatrix},$$

where

$$L_0 = \frac{s_0 p_0}{w_0} E_0 K,$$
$$L_i = \frac{s_i p_i}{w_i} \Big(\sum_{j=0}^{i} \delta_{ij} \frac{w_i}{w_j} E_j \Big) Q_i \quad \textit{for } i = 1, \ldots, k,$$

while

$$w_i = \sum_{l=i}^{k+1} s_l p_l \delta_{li} \quad \textit{for } i = 0, \ldots, k+1.$$

Proof. An estimator $\boldsymbol{L}'\boldsymbol{Y}$ is locally best at $(\sum_{i=1}^{k+1} s_i \boldsymbol{V}_i, s_0 \boldsymbol{X}\boldsymbol{X}')$ in $\mathcal{T}$ among $\mathcal{L}_0$ iff $s_j \geqslant 0$ for $j = 0, \ldots, k+1$ and

$$\boldsymbol{\Pi}_0 \Big(\sum_{i=1}^{k+1} s_i \boldsymbol{V}_i + s_0 \boldsymbol{X}\boldsymbol{X}' \Big) \boldsymbol{L} = s_0 \boldsymbol{\Pi}_0 \boldsymbol{X}\boldsymbol{X}'\boldsymbol{K}.$$

In more detail, this equation can be written as

$$W L_0 = s_0 X X' K,$$
$$W L_i - s_i Z_i Z_i' Q_i = \mathbf{0} \quad \text{for } i = 1, \ldots, k,$$

where $W = \sum_{i=0}^{k+1} s_i Z_i Z_i'$. Of course, the above equations have only one solution $L_0, \ldots, L_k$ iff the matrix W is nonsingular, that is, $s_{k+1} > 0$. Moreover, since by (i),

$$W = \sum_{i=0}^{k+1} s_i Z_i Z_i' = \sum_{i=0}^{k+1} s_i \Big(p_i \sum_{j=0}^{i} \delta_{ij} E_j \Big) = \sum_{i=0}^{k+1} w_i E_i,$$

and since $E_i E_j = 0$ for $i \neq j$, we see that $W^{-1} = \sum_{i=0}^{k+1} \frac{1}{w_i} E_i$. Hence and by (ii) we get

$$L_i = s_i W^{-1} Z_i Z_i' Q_i = s_i \Big(\sum_{j=0}^{k+1} \frac{1}{w_j} E_j \Big) Z_i Z_i' Q_i = s_i \Big(\sum_{j=0}^{k+1} \frac{p_i \delta_{ij}}{w_j} E_j \Big) Q_i$$
$$= \frac{s_i p_i}{w_i} \Big(\sum_{j=0}^{i} \delta_{ij} \frac{w_i}{w_j} E_j \Big) Q_i. \ \blacksquare$$

Explicit formulas for linear admissible estimators $\boldsymbol{L}'\boldsymbol{Y}$ will be given in terms of certain coefficients a_i. Let

$$a = (a_0, a_1, \ldots, a_{k+1})' = \Big(\frac{s_0 p_0}{w_0}, \frac{s_1 p_1}{w_1}, \ldots, \frac{s_{k+1} p_{k+1}}{w_{k+1}} \Big)'.$$

Note that for any fixed $s_{i+1} \geqslant 0, \ldots, s_k \geqslant 0$ and $s_{k+1} > 0$ the parameter a_i runs over $[0, 1)$ when $s_i \in [0, +\infty)$ for $i = 0, \ldots, k$ and $a_{k+1} = 1$. Further, taking

$$w = \Delta' P s,$$

where $w = (w_0, w_1, \ldots, w_{k+1})'$ and $s = (s_0, s_1, \ldots, s_{k+1})'$, we have

$$a = (\operatorname{diag}\{w_0, w_1, \ldots, w_{k+1}\})^{-1} P s = (\operatorname{diag}\{w_0, w_1, \ldots, w_{k+1}\})^{-1} \Lambda' w.$$

In more detail the last equation can be written as

$$\begin{cases} a_0 &= \lambda_{00} + \lambda_{10}\frac{w_1}{w_0} + \lambda_{20}\frac{w_2}{w_0} + \ldots + \lambda_{k0}\frac{w_k}{w_0} + \lambda_{k+1,0}\frac{w_{k+1}}{w_0}, \\ a_1 &= \lambda_{11} + \lambda_{21}\frac{w_2}{w_1} + \ldots + \lambda_{k1}\frac{w_k}{w_1} + \lambda_{k+1,1}\frac{w_{k+1}}{w_1}, \\ a_2 &= \lambda_{22} + \ldots + \lambda_{k2}\frac{w_k}{w_2} + \lambda_{k+1,2}\frac{w_{k+1}}{w_2}, \\ \vdots & \qquad \ddots \quad \vdots \qquad \vdots \\ a_k &= \lambda_{kk} + \lambda_{k+1,k}\frac{w_{k+1}}{w_k}, \\ a_{k+1} &= \lambda_{k+1,k+1}. \end{cases}$$

Note that we can express the above system in the form

$$\mathbf{0} = (\Lambda' - \mathrm{diag}\{a_0, a_1, \ldots, a_{k+1}\})w = (I - \Delta'\mathrm{diag}\{a_0, a_1, \ldots, a_{k+1}\})w.$$

Since $\delta_{ii} = 1$ for $i = 0, 1, \ldots, k+1$, we get

$$\mathbf{0} = Aw, \tag{4.1}$$

where

$$A = \begin{bmatrix} 1-a_0 & -a_1\delta_{10} & -a_2\delta_{20} & \cdots & -a_k\delta_{k,0} & -a_{k+1}\delta_{k+1,0} \\ 0 & 1-a_1 & -a_2\delta_{21} & \cdots & -a_k\delta_{k,1} & -a_{k+1}\delta_{k+1,1} \\ 0 & 0 & 1-a_2 & \cdots & -a_k\delta_{k,2} & -a_{k+1}\delta_{k+1,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1-a_k & -a_{k+1}\delta_{k+1,k} \\ 0 & 0 & 0 & \cdots & 0 & 1-a_{k+1} \end{bmatrix}.$$

LEMMA 4.2. *The system* (4.1) *has a non-trivial solution*

$$w_i = \frac{w_{k+1}}{1-a_i}\Bigg(1 + \sum_{l_1=i+1}^{k} \frac{a_{l_1}\delta_{l_1 i}}{1-a_{l_1}} + \sum_{l_1=i+1}^{k}\sum_{l_2=l_1+1}^{k} \frac{a_{l_1}\delta_{l_1 i}}{1-a_{l_1}}\frac{a_{l_2}\delta_{l_2 l_1}}{1-a_{l_2}} + \sum_{l_1=i+1}^{k}\sum_{l_2=l_1+1}^{k}\sum_{l_3=l_2+1}^{k} \frac{a_{l_1}\delta_{l_1 i}}{1-a_{l_1}}\frac{a_{l_2}\delta_{l_2 l_1}}{1-a_{l_2}}\frac{a_{l_3}\delta_{l_3 l_2}}{1-a_{l_3}} + \cdots + \sum_{l_1=i+1}^{k}\sum_{l_2=l_1+1}^{k}\cdots\sum_{l_{k-i}=l_{k-i-1}+1}^{k} \frac{a_{l_1}\delta_{l_1 i}}{1-a_{l_1}}\frac{a_{l_2}\delta_{l_2 l_1}}{1-a_{l_2}}\cdots\frac{a_{l_{k-i}}\delta_{l_{k-i}l_{k-i-1}}}{1-a_{l_{k-i}}}\Bigg) \tag{4.2}$$

for $i = 0, 1, \ldots, k$.

Proof. Since $0 \leqslant a_i < 1$ for $i = 0, 1, \ldots, k$ and $a_{k+1} = 1$ we find that $\mathrm{rank}(A) = k+1$. Moreover, the vector $w = (w_0, w_1, \ldots, w_{k+1})'$, where $w_0, w_1, \ldots, w_k$ are given by (4.2), can be easily checked to satisfy $\mathbf{0} = Aw$. ∎

THEOREM 4.1. *For an estimator $\boldsymbol{L}'\boldsymbol{Y}$ of $\boldsymbol{K}'\boldsymbol{X}\beta$ to be admissible among $\mathcal{L}_0$ in the model* (3.3) *corresponding to* (2.4) *it is necessary and sufficient that $\boldsymbol{L}$ belongs to the closure of the set*

(4.3)
$$\left\{\begin{bmatrix} L_0(a) & L_1(a) & \cdots & L_k(a) \\ \mathbf{0} & -Q_1 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & -Q_k \end{bmatrix} : a = (a_0, a_1, \ldots, a_k)' \in [0,1)^{k+1}\right\},$$

where

$$L_0 = a_0 E_0 K,$$
$$L_i = a_i\Big(\sum_{j=0}^{i} \delta_{ij}\frac{w_i}{w_j}E_j\Big)Q_i \quad \textit{for } i = 1, \ldots, k,$$

while w_i and w_j are given by (4.2).

Proof. Necessity. Let $\boldsymbol{L}$ belong to the set (4.3) with $a_i \in [0,1)$ for $i = 0, 1, \ldots, k$. Using Lemma 4.1 it can be checked that $\boldsymbol{L}'\boldsymbol{Y}$ is an ULBE at $W = (\sum_{i=1}^{k+1} s_i\boldsymbol{V}_i, s_0\boldsymbol{X}\boldsymbol{X}')$ in $\mathcal{T}$ given by

$$s_i = \frac{a_i}{p_i}\frac{w_i}{w_{k+1}}s_{k+1} \quad \text{for } i = 0, 1, \ldots, k,$$
$$s_{k+1} > 0.$$

Since for any fixed $a_{i+1} \in [0,1), \ldots, a_k \in [0,1)$ the value s_i runs over $[0, +\infty)$ when $a_i \in [0,1)$ for $i = 0, \ldots, k$, the set given by (4.3) is the closure of

$$\{\boldsymbol{M} : \boldsymbol{M}'\boldsymbol{Y} \text{ is an ULBE at a point of } \mathcal{T} \text{ among } \mathcal{L}_0\}.$$

So the first part of the proof is completed by using a result of LaMotte [15] that each linear estimator of $\boldsymbol{K}'\,\mathrm{E}\,\boldsymbol{Y}$ admissible among $\mathcal{L}_0$ is a limit of members of $\mathcal{L}_0$ that are uniquely best among $\mathcal{L}_0$ in $\mathcal{T}$.

Sufficiency follows straightforwardly from a result of Synówka-Bejenka and Zontek [26] that for a model with finitely generated parameter space each limit of members of $\mathcal{L}_0$ is admissible. ∎

5. EXAMPLE

Consider a two-factor study where factors F_1 and F_2 have n_1 and n_2 levels, respectively, and there are n_3 replications in each cell. We say that the factors *interact* if the differences between the mean response at different levels of factor F_1 tend to vary over the different levels of factor F_2. The two-way crossed classification random model with interaction, a special case of the model (2.1), is given by

(5.1)
$$Y_{i_1i_2i_3} = \beta + u_{1i_1} + u_{2i_2} + u_{3i_1i_2} + e_{i_1i_2i_3},$$
$$i_1 = 1, \ldots, n_1;\; i_2 = 1, \ldots, n_2;\; i_3 = 1, \ldots, n_3,$$

where $Y_{i_1 i_2 i_3}$ is the i_3th observation corresponding to the (i_1, i_2)th cell, β is a constant, the u_{1i_1} are effects due to factor F_1, the u_{2i_2} are effects due to factor F_2, and $u_{3i_1i_2}$ is the interaction between u_{1i_1} and u_{2i_2}. For this model we have

$$
\begin{aligned}
&\psi_0 = \emptyset, \quad \psi_1 = \{i_1\}, \quad \psi_2 = \{i_2\}, \quad \psi_3 = \{i_1, i_2\}, \quad \psi_4 = \{i_1, i_2, i_3\},\\
&Z_0 = \mathbf{1}_{n_1} \otimes \mathbf{1}_{n_2} \otimes \mathbf{1}_{n_3}, \quad Z_1 = I_{n_1} \otimes \mathbf{1}_{n2} \otimes \mathbf{1}_{n_3}, \quad Z_2 = \mathbf{1}_{n_1} \otimes I_{n2} \otimes \mathbf{1}_{n_3},\\
&Z_3 = I_{n_1} \otimes I_{n2} \otimes \mathbf{1}_{n_3}, \quad Z_4 = I_{n_1} \otimes I_{n2} \otimes I_{n_3},\\
&p_0 = n_1 n_2 n_3, \quad p_1 = n_2 n_3, \quad p_2 = n_1 n_3, \quad p_3 = n_3, \quad p_4 = 1.
\end{aligned}
$$

Hence

$$
\Delta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix},
$$

while

$$
\begin{aligned}
w_3 &= \frac{w_4}{1-a_3},\\
w_2 &= \frac{w_4}{1-a_2}\left(1 + \frac{a_3\delta_{32}}{1-a_3}\right) = \frac{w_4}{1-a_2}\left(1 + \frac{a_3}{1-a_3}\right) = \frac{w_4}{(1-a_2)(1-a_3)},\\
w_1 &= \frac{w_4}{1-a_1}\left(1 + \frac{a_2\delta_{21}}{1-a_2} + \frac{a_3\delta_{31}}{1-a_3} + \frac{a_2\delta_{21}a_3\delta_{32}}{(1-a_2)(1-a_3)}\right) = \frac{w_4}{(1-a_1)(1-a_3)},\\
w_0 &= \frac{w_4}{1-a_0}\left(1 + \frac{a_1\delta_{10}}{1-a_1} + \frac{a_2\delta_{20}}{1-a_2} + \frac{a_3\delta_{30}}{1-a_3}\right.\\
&\qquad + \frac{a_1\delta_{10}a_2\delta_{21}}{(1-a_1)(1-a_2)} + \frac{a_1\delta_{10}a_3\delta_{31}}{(1-a_1)(1-a_3)} + \frac{a_2\delta_{20}a_3\delta_{32}}{(1-a_2)(1-a_3)}\\
&\qquad \left. + \frac{a_1\delta_{10}a_2\delta_{21}a_3\delta_{32}}{(1-a_1)(1-a_2)(1-a_3)}\right)\\
&= \frac{w_4}{1-a_0}\left(1 + \frac{a_1}{1-a_1} + \frac{a_2}{1-a_2} + \frac{a_3}{1-a_3}\right.\\
&\qquad \left. + \frac{a_1a_3}{(1-a_1)(1-a_3)} + \frac{a_2a_3}{(1-a_2)(1-a_3)}\right)\\
&= \frac{w_4}{1-a_0}\,\frac{1-a_1a_2}{(1-a_1)(1-a_2)(1-a_3)}.
\end{aligned}
$$

After appropriate transformations we obtain the following explicit formulas for ULBEs:

$$
\begin{aligned}
L_0(a) &= a_0 E_0 K,\\
L_1(a) &= a_1\left[\delta_{10}\frac{w_1}{w_0}E_0 + E_1\right]Q_1 = a_1\left[(1-a_0)\frac{1-a_2}{1-a_1a_2}E_0 + E_1\right]Q_1,
\end{aligned}
$$

$$\begin{aligned} L_2(a) &= a_2\left[\delta_{20}\frac{w_2}{w_0}E_0 + \delta_{21}\frac{w_2}{w_1}E_1 + E_2\right]Q_2 \\ &= a_2\left[(1-a_0)\frac{1-a_1}{1-a_1a_2}E_0 + E_2\right]Q_2, \\ L_3(a) &= a_3\left[\delta_{30}\frac{w_3}{w_0}E_0 + \delta_{31}\frac{w_3}{w_1}E_1 + \delta_{32}\frac{w_3}{w_2}E_2 + E_3\right]Q_3 \\ &= a_3\left[(1-a_0)\frac{(1-a_1)(1-a_2)}{1-a_1a_2}E_0 + (1-a_1)E_1 + (1-a_2)E_2 + E_3\right]Q_3. \end{aligned}$$

For a complete characterization of admissible linear estimators $\boldsymbol{L}'\boldsymbol{Y}$ of $\boldsymbol{K}'\boldsymbol{X}\beta$, it is enough to give a form of limits of ULBEs. Note that L_i, $i = 1, 2, 3$, can be expressed as

$$\begin{aligned} L_1(a) &= a_1\left[\delta_{10}\frac{w_1}{w_0}E_0 + E_1\right]Q_1 = a_1[E_1 + (1-a_0)A_1E_0]Q_1, \\ L_2(a) &= a_2\left[\delta_{20}\frac{w_2}{w_0}E_0 + \delta_{21}\frac{w_2}{w_1}E_1 + E_2\right]Q_2 = a_2[E_2 + (1-a_0)A_2E_0]Q_2, \\ L_3(a) &= a_3\left[\delta_{30}\frac{w_3}{w_0}E_0 + \delta_{31}\frac{w_3}{w_1}E_1 + \delta_{32}\frac{w_3}{w_2}E_2 + E_3\right]Q_3 \\ &= a_3[E_3 + (1-a_0)(A_1 + A_2 - 1)E_0 + (1-a_1)E_1 + (1-a_2)E_2]Q_3, \end{aligned}$$

where

$$A_1 = \frac{\frac{1}{1-a_1}}{\frac{1}{1-a_1} + \frac{1}{1-a_2} - 1}, \qquad A_2 = \frac{\frac{1}{1-a_2}}{\frac{1}{1-a_1} + \frac{1}{1-a_2} - 1} \quad \text{for } a_1, a_2 \in [0, 1). \tag{5.2}$$

Of course for A_1 and A_2 defined by (5.2) we have $1 < A_1 + A_2 \leqslant 2$. Since $\frac{1}{1-a_1} + \frac{1}{1-a_2} \to +\infty$ as $a_1 \to 1$ or $a_2 \to 1$, we have $A_1 + A_2 \to 1$ and $A_i \to 0$ for $i = 1, 2$. So each linear estimator of $\boldsymbol{K}' \operatorname{E} \boldsymbol{Y}$ admissible among $\mathcal{L}_0$ is a limit of members of $\mathcal{L}_0$ that are uniquely best among $\mathcal{L}_0$ in $\mathcal{T}$.

6. SIMULATIONS

In simulation studies we considered the following model:

$$Y_{i_1i_2} = \beta + u_{1i_1} + e_{i_1i_2}, \quad i_1 = 1, \ldots, n_1,\ i_2 = 1, \ldots, n_2. \tag{6.1}$$

For this model we have

$$\begin{aligned} &\psi_0 = \emptyset, \quad \psi_1 = \{i_1\}, \quad \psi_2 = \{i_1, i_2\}, \\ &Z_0 = 1_{n_1} \otimes 1_{n_2}, \quad Z_1 = I_{n_1} \otimes 1_{n_2}, \quad Z_2 = I_{n_1} \otimes I_{n_2}, \\ &p_0 = n_1n_2, \quad p_1 = n_2, \quad p_2 = 1. \end{aligned}$$

We choose $K = \frac{1}{p_0} Z_0$ and $Q_1 = \frac{1}{p_1} Z_1$, so that the estimated parameter vector θ is

$$\theta = (\beta, u_{11}, u_{12}, \dots, u_{1n_1})'.$$

An admissible estimator of θ takes the form

$$L'Y = (a_0 E_0 K, a_1 [E_1 + (1 - a_0) E_0] Q_1)' Y, \tag{6.2}$$

where $a_0, a_1 \in [0, 1]$, while $E_0 = \frac{1}{p_0} Z_0 Z_0'$ and $E_1 = \frac{1}{p_1} Z_1 Z_1' - E_0$.

If we decide to compare estimators using the quadratic risk function, then from the theoretical point of view each admissible estimator is a good choice. But admissibility does not guarantee good effects of estimation from the practical point of view. For example, the estimator (6.2) with $a_0 = a_1 = 0$ is perfect as an estimator of θ but only for $\mu = 0$ and $\sigma_1^2 = 0$. When $\mu \neq 0$ and $\sigma_1^2 \neq 0$, it cannot be recommended. Simulations presented below suggest that a satisfactory result is obtained for a_0 very close to (or equal to) 1 and for a_1 close to 1.

For chosen $a_0, a_1 \in [0, 1]$ and a fixed number N of generated data denote by

$$(\hat{\beta}_j, \hat{u}_{11j}, \hat{u}_{12j}, \dots, \hat{u}_{1n_1j})', \quad j = 1, \dots, N,$$

values of estimators of θ.

As a result of simulations, in the tables below we present the means of estimates (the first row for a_1 given in the first column), that is,

$$\bar{\beta} = \frac{1}{N} \sum_{j=1}^{N} \hat{\beta}_j, \quad \bar{u}_{1i_1} = \frac{1}{N} \sum_{j=1}^{N} \hat{u}_{1i_1j}, \quad i_1 = 1, \dots, n_1,$$

and the average square of the difference between the estimated values and corresponding estimates (the second row for given a_1, in brackets), i.e.

$$\frac{1}{N} \sum_{j=1}^{N} (\hat{\beta}_j - \beta)^2, \quad \frac{1}{N} \sum_{j=1}^{N} (\hat{u}_{1i_1j} - u_{1i_1j})^2, \quad i_1 = 1, \dots, n_1.$$

In simulation we used the following parameters:

$n_1 = 4$, $n_2 = 10$, $\beta = 10$, $u_{1i_1} \sim N(0, 1)$, $e_{i_1i_2} \sim N(0, 1)$ and $N = 10000$.

TABLE 1. Simulation results for $a_0 = 0.90$ and selected a_1.

0.85	9.015	0.848	0.848	0.854	0.856
	(1.782)	(1.646)	(1.645)	(1.647)	(1.637)
0.90	9.015	0.898	0.898	0.904	0.907
	(1.782)	(1.692)	(1.690)	(1.694)	(1.686)
0.95	9.015	0.948	0.947	0.954	0.957
	(1.782)	(1.759)	(1.755)	(1.762)	(1.755)
1.00	9.015	0.998	0.997	1.004	1.007
	(1.782)	(1.845)	(1.841)	(1.849)	(1.844)

TABLE 2. Simulation results for $a_0 = 0.95$ and selected a_1.

0.85	9.515	0.422	0.422	0.428	0.431
	(1.138)	(1.192)	(1.188)	(1.190)	(1.187)
0.90	9.515	0.447	0.447	0.453	0.456
	(1.138)	(1.178)	(1.173)	(1.175)	(1.174)
0.95	9.515	0.472	0.472	0.478	0.481
	(1.138)	(1.180)	(1.173)	(1.178)	(1.178)
1.00	9.515	0.497	0.496	0.503	0.507
	(1.138)	(1.198)	(1.190)	(1.196)	(1.199)

TABLE 3. Simulation results for $a_0 = 1.00$ and selected a_1.

0.85	10.016	−0.003	−0.004	0.002	0.005
	(1.001)	(1.105)	(1.098)	(1.098)	(1.102)
0.90	10.016	−0.004	−0.004	0.002	0.005
	(1.001)	(1.074)	(1.066)	(1.067)	(1.073)
0.95	10.016	−0.004	−0.004	0.002	0.005
	(1.001)	(1.058)	(1.048)	(1.051)	(1.059)
1.00	10.016	−0.004	−0.004	0.003	0.006
	(1.001)	(1.057)	(1.046)	(1.050)	(1.060)

REFERENCES

[1] J. K. Baksalary and A. Markiewicz, *Admissible linear estimators in the general Gauss–Markov model*, J. Statist. Plann. Inference 19 (1988), 349–359.

[2] J. K. Baksalary, A. Markiewicz and C. R. Rao, *Admissible linear estimation in the general Gauss–Markov model with respect to an arbitrary quadratic risk function*, J. Statist. Plann. Inference 44 (1995), 341–347.

[3] A. Cohen, *All admissible estimates of the mean vector*, Ann. Math. Statist. 37 (1966), 458–463.

[4] A. S. Goldberger, *Best linear unbiased prediction in the generalized linear regression model*, J. Amer. Statist. Assoc. 57 (298) (1962), 369–375.

[5] J. Groß and A. Markiewicz, *Characterization of admissible linear estimators in the linear model*, Linear Algebra Appl. 388 (2004), 239–248.

[6] D. A. Harville, *Extension of the Gauss–Markov theorem to include the estimation of random effects*, Ann. Statist. 2 (1976), 384–395.

[7] C. R. Henderson, *Estimation of genetic parameters (abstract)*, Ann. Math. Statist. 21 (1950), 309–310.

[8] C. R. Henderson, *Selection index and expected genetic advance*, in: Statistical Genetics and Plant Breeding, NAS-NRC 982, Washington, 1963, 141–163.

[9] C. R. Henderson, *Best linear unbiased estimation and prediction under a selection model*, Biometrics 31 (1975), 423–447.

[10] J. Jiang, *A derivation of BLUP-Best linear unbiased predictor*, Statist. Probab. Lett. 32 (1997), 321–324.

[11] A. I. Khuri, T. Mathew and B. K. Sinha, *Statistical Tests for Mixed Linear Models*, Wiley, New York, 1998.

[12] W. Klonecki and S. Zontek, *On the structure of admissible linear estimators*, J. Multivariate Anal. 24 (1988), 11–30.

[13] L. R. LaMotte, *Admissibility in linear estimation*, Ann. Statist. 10 (1982), 245–255.

[14] L. R. LaMotte, *Admissibility, unbiasedness, and nonnegativity in the balanced, random, one-way ANOVA model*, in: T. Caliński and W. Klonecki (eds.), Linear Statistical Inference (Poznań, 1984), Springer, Berlin, 1985, 184–199.
[15] L. R. LaMotte, *On limits of uniquely best linear estimators*, Metrika 45 (1997), 197–211.
[16] X. Q. Liu, J. Y. Rong and X. Y. Liu, *Best linear unbiased prediction for linear combinations in general mixed linear models*, J. Multivariate Anal. 99 (2008), 1503–1517.
[17] T. Mathew, C. R. Rao and B. K. Sinha, *Admissible linear estimation in singular linear models*, Comm. Statist. Theory Methods 13 (1984), 3033–3045.
[18] C. R. Rao, *Estimation of parameters in a linear model*, Ann. Statist. 4 (1976), 1023–1037.
[19] C. R. Rao, *Estimation in linear models with mixed effects: a unified theory*, in: Proc. Second International Tampere Conference in Statistics, Dept. Math. Sci., Univ. of Tampere, 1987, 73–98.
[20] H. Sahai and M. M. Ojeda, *Analysis of Variance for Random Models. Volume 1: Balanced Data, Theory, Methods, Applications and Data Analysis*, Birkhäuser, Boston, MA, 2004.
[21] W. Shiqing, M. Ying and F. Zhijun, *Integral expression form of admissible linear estimators of effects in linear mixed models*, in: Proc. 2010 International Conference on Computing, Control and Industrial Engineering, IEEE, Wuhan, 2010, 56–60.
[22] C. Stępniak, *On admissible estimators in a linear model*, Biometrical J. 26 (1984), 815–816.
[23] C. Stępniak, *A complete class for linear estimation in a general linear model*, Ann. Inst. Statist. Math. A 39 (1987), 563–573.
[24] C. Stępniak, *Admissible invariant esimators in a linear model*, Kybernetika 50 (2014), 310–321.
[25] E. Synówka-Bejenka and S. Zontek, *A characterization of admissible linear estimators of fixed and random effects in linear models*, Metrika 68 (2008), 157–172.
[26] E. Synówka-Bejenka and S. Zontek, *On admissibility of linear estimators in models with finitely generated parameter space*, Kybernetika 52 (2016), 724–734.
[27] Y. Tian, *A new derivation of BLUPs under random-effects model*, Metrika 78 (2015), 905–918.
[28] S. Zontek, *On characterization of linear admissible estimators: an extension of a result due to C. R. Rao*, J. Multivariate Anal. 23 (1987), 1–12.
[29] S. Zontek, *Admissibility of limits of the unique locally best linear estimators with application to variance components models*, Probab. Math. Statist. 9 (1988), 29–44.

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