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INTERMEDIATE EFFICIENCY OF TESTS UNDER HEAVY-TAILED ALTERNATIVES

BY

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Abstract. We show that for local alternatives which are not square integrable the intermediate (or Kallenberg) efficiency of the Neyman–Pearson test for uniformity with respect to the classical Kolmogorov–Smirnov test is infinite. By contrast, for square integrable local alternatives the intermediate efficiency is finite and can be explicitly calculated.

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1. INTRODUCTION AND TESTING PROBLEM

We consider the classical problem of testing for uniformity. We compare the Neyman–Pearson (NP) test with the classical Kolmogorov–Smirnov (KS) test for uniformity for a class of local unbounded alternatives in terms of asymptotic relative efficiency (ARE). By ARE we mean Kallenberg's intermediate efficiency which is a limit of the ratio of sample sizes which guarantee the same precision for both tests (the same significance level tending to 0 less than exponentially fast and the same asymptotically nondegenerate power).

Our main issue is that for alternatives which are not square integrable the efficiency of the NP test with respect to the KS test cannot be finite. In particular, we apply the simplest variant of the intermediate efficiency notion recently elaborated in Inglot et al. [8] and called pathwise intermediate efficiency. We show that this efficiency for the NP test with respect to the KS test for a class of alternatives approaching the null distribution, which are not square integrable, is equal to ∞ (Theorem 1).

Recall that the notion of intermediate efficiency was introduced originally by Kallenberg [9]. Then it was developed and applied to some testing problems and several tests in a series of papers in the last two decades e.g. Inglot [3], Inglot and Ledwina [5], [6], [7], Mason and Eubank [10], Mirakhmedov [11] or recently Inglot et al. [8] and Ćmiel et al. [1]. For more detailed discussions and up-to-date remarks and comments we refer the reader to Inglot et al. [8]. Note that, by the definition, this efficiency notion involves asymmetric requirements for the tests being compared.

Among other results, Inglot and Ledwina [7] found the intermediate efficiency of the KS test with respect to the NP test for sequences of bounded alternatives approaching the null distribution. In the present paper, as a byproduct, we extend that result to unbounded square integrable alternatives in the reverse formulation, i.e. taking the KS test as a benchmark procedure and comparing the NP test with it.

Since we consider both a simple testing problem and very regular statistics, and to make the paper self-contained, we do not refer to general results and technical tools elaborated in Inglot et al. [8]. Instead, we present all auxiliary results and all proofs directly.

Let $X_1, \ldots, X_n$ be independent random variables with values in $[0,1]$ and the same distribution P with continuous distribution function. We denote by P_0 the uniform distribution over the interval $[0,1]$. We test the simple null hypothesis

$$H_0 : P = P_0$$

against

$$H_1 : P \neq P_0.$$

To compare the tests we consider local alternatives with densities (with respect to P_0) of the form

$$p_{\theta_n}(t) = 1 - \theta_n + \theta_n f(t), \qquad t \in (0,1), \tag{1.1}$$

where $\theta_n \in (0,1)$, $\theta_n \to 0$ as $n \to \infty$ and f is a fixed alternative density. We denote by P_{θ_n} the distribution with density $p_{\theta_n}(t)$. Moreover, $P_0^n,\ P_{\theta_n}^n$ will denote the n-fold products of P_0 and P_{θ_n}, respectively.

For each n consider the standardized NP test statistic

$$V_n = \frac{1}{\sqrt{n}\,\sigma_{0n}} \sum_{i=1}^{n} (\log p_{\theta_n}(X_i) - e_{0n}) \tag{1.2}$$

for testing H_0 against the simple hypothesis $H_{1n} : P = P_{\theta_n}$. Here

$$e_{0n} = \int_0^1 \log p_{\theta_n}(t)\,dt, \qquad \sigma_{0n}^2 = \int_0^1 \log^2 p_{\theta_n}(t)\,dt - e_{0n}^2$$

are the first two moments of $\log p_{\theta_n}(X_1)$ under P_0, which are finite due to the integrability of f. Additionally denote by

$$e_n = \int_0^1 p_{\theta_n}(t) \log p_{\theta_n}(t)\,dt, \qquad \sigma_n^2 = \int_0^1 p_{\theta_n}(t) \log^2 p_{\theta_n}(t)\,dt - e_n^2$$

the corresponding moments under P_{θ_n}, which we assume to be finite. For example, this is the case if $f \in L_q[0,1]$ for some $q > 1$.

Now, set

$$b_n = \sqrt{n}\,(e_n - e_{0n})/\sigma_{0n}. \tag{1.3}$$

The sequence b_n will play the role of an asymptotic shift of V_n under P_{θ_n}.

For each n and any fixed $x \in \mathbb{R}$ set

$$\alpha_n = \alpha_n(x) = P_0^n(V_n \geqslant x + b_n), \tag{1.4}$$

the significance level of the NP test corresponding to the critical value $x + b_n$. Since V_n is bounded in probability under P_0, we have $\alpha_n \to 0$ whenever $b_n \to \infty$.

Let

$$K_n = \sqrt{n} \sup_{t \in (0,1)} |\hat{F}_n(t) - t|,$$

where $\hat{F}_n(t)$ is the empirical distribution function of $X_1, \ldots, X_n$, be the classical unweighted KS test statistic. For each n and every $N \geqslant n$ let $u_{N,n}$ be the exact critical value of the KS test at the level α_n defined by (1.4) and for the sample size N, i.e.

$$P_0^N(K_N \geqslant u_{N,n}) = \alpha_n.$$

For each n let N_n be the minimal sample size such that for all $k \geqslant 0$,

$$P_{\theta_n}^{N_n+k}(K_{N_n+k} \geqslant u_{N_n+k,n}) \geqslant P_{\theta_n}^n(V_n \geqslant x + b_n), \tag{1.5}$$

i.e. the minimal sample size beginning from which the power of the KS test under P_{θ_n} and at level α_n is no smaller than that for the NP test at the same level and for the sample size n. Obviously, $N_n \geqslant n$. The limit of the ratio N_n/n, if it exists, is called the intermediate efficiency of the NP test with respect to the KS test provided the asymptotic power of V_n is nondegenerate (cf. Inglot et al. [8]). We study the asymptotic behaviour of the ratio N_n/n for two cases, when f is heavy tailed or square integrable, and show that they lead to qualitatively different answers.

The paper is organized as follows. In Section 2 we consider local alternatives which are not square integrable, while in Section 3 we consider square integrable ones. In Section 4 we present outcomes of a simulation study nicely illustrating our theoretical results. All proofs are deferred to Sections 5–8.

We shall use the following notation: for sequences x_n, y_n of positive numbers, $x_n \asymp y_n$ means that for some positive constants c_1, c_2 one has $c_1 \leqslant x_n/y_n \leqslant c_2$ for all n, while $x_n \sim y_n$ means that $x_n/y_n \to 1$ as $n \to \infty$.

2. HEAVY-TAILED CASE

For any $\eta, C_0 > 0$ and $\zeta \in \mathbb{R}$ define a decreasing function $H(y) = C_0 \int_y^\infty u^{-1-\eta} \log^\zeta u \, du$, $y \in [2, \infty)$, and set $h(t) = H^{-1}(t)$, $t \in (0, H(2)]$.

Assume that a density f in (1.1) satisfies the following condition: for some $\eta \in (1,2)$, $C_0 > 0$ and $\zeta \in \mathbb{R}$ there exist $0 < \delta < \min\{1, H(2)\}$ and positive constants $C_1 \leqslant C_2$ such that

$$C_1 h(t) \leqslant f(t) \leqslant C_2 h(t) \text{ for } t \in (0,\delta], \quad f(t) \leqslant C_2 \text{ for } t \in [\delta, 1]. \tag{2.1}$$

We may always assume that $C_2 > 1$ and $C_1 h(\delta) > 2$ (by taking smaller δ if necessary). This implies $f(t) > 2$ on $(0,\delta]$. Observe that f's satisfying (2.1) are not square integrable. We call such densities *heavy-tailed*.

REMARK 1. The assumption (2.1) obviously covers the case $h(t) = t^{-r}$, $r \in (1/2, 1)$. Then $\eta = 1/r$ and $\zeta = 0$. Another important example is the Gaussian scale model. Suppose the null distribution Q_0 is the standard normal distribution and the alternative Q is the mean zero normal distribution with variance $\sigma^2 > 2$. For a contamination model $Q_n = (1-\theta_n)Q_0 + \theta_n Q$, after the transformation on the unit interval by the standard normal distribution function Φ, we get (1.1) with

$$f(t) = \frac{1}{\sigma}\exp\left\{\frac{\sigma^2 - 1}{2\sigma^2}(\Phi^{-1}(t))^2\right\}, \quad t \in (0,1).$$

Then $f_1(t) = 2f(t)\mathbf{1}_{(0,1/2]}(t)$, where $\mathbf{1}_E(t)$ denotes the indicator of the set E, satisfies (2.1) with $C_0 = \sqrt{\sigma^2/4\pi(\sigma^2-1)}$, $\eta = \sigma^2/(\sigma^2-1)$, $\zeta = -1/2$, $C_1 = 2/\sigma$ and any $C_2 > 1$. By the symmetry of $f(t)$ with respect to $1/2$ the statement of Theorem 1 below holds for $f_1(t)$ as well as for $f(t)$ itself (cf. Remark 3 below).

First we describe the asymptotic behaviour of b_n, defined in (1.3). To this end set $\kappa_n^2 = \theta_n^\eta \log^\zeta(1/\theta_n)$.

PROPOSITION 1. *If f satisfies* (2.1) *then*

$$b_n \asymp \sqrt{n}\,\kappa_n. \tag{2.2}$$

Proposition 1 follows immediately from Lemma 3 proved in Section 6. The next proposition is a simple consequence of (2.2) and is proved in Section 6.

PROPOSITION 2. *Let $p_{\theta_n}(t)$ be a sequence of densities given by* (1.1) *with f satisfying* (2.1) *and $\theta_n \to 0$ such that $n\kappa_n^2 \to \infty$. Then for every $x \in \mathbb{R}$,*

$$0 < \liminf_{n\to\infty} P_{\theta_n}^n(V_n \geqslant x + b_n) \leqslant \limsup_{n\to\infty} P_{\theta_n}^n(V_n \geqslant x + b_n) < 1. \tag{2.3}$$

THEOREM 1. *Let $p_{\theta_n}(t)$ be a sequence of densities given by* (1.1) *with f satisfying* (2.1) *and $\theta_n \to 0$ such that $n\theta_n^\eta \log^\zeta(1/\theta_n) \to \infty$. Then for any $x \in \mathbb{R}$ and the significance levels defined by* (1.4) *we have, for N_n defined by* (1.5),

$$\lim_{n\to\infty} \frac{N_n}{n} = \infty. \tag{2.4}$$

The proof of Theorem 1 is given in Section 5.

REMARK 2. In terms of intermediate efficiency (as defined in Inglot et al. [8]) Theorem 1 says that for f satisfying (2.1) this efficiency of the NP test with respect to the KS test is equal to ∞. The efficiency notion requires a nondegenerate asymptotic power of a test being compared, here the NP test (with respect to a benchmark procedure, here the KS test). It is essential in the proof of Theorem 1 and is ensured by our Proposition 2. In the proof of Theorem 1 we directly show that the intermediate slope of the KS test is equal to $2n\theta_n^2\|A\|_\infty^2$ without introducing such terminology and without referring to regularity conditions (I.1) and (I.2) in Inglot et al. [8]. For the NP test the regularity condition (II.2) (ibid.) can be deduced from the proofs of Proposition 2 and Lemma 4. Moreover, it is enough to show a weaker property than the regularity condition (II.1) (ibid.) saying that an expression which may be considered as the intermediate slope of the NP test is at least of order $n\kappa_n^2$. Anyway, here we prove (2.4) in the simplest possible way. Obviously, the statement (2.4) remains true for any test for uniformity which has positive and finite intermediate efficiency with respect to the KS test and can be chosen as a benchmark procedure. For some further comments see Section 2 in Inglot et al. [8].

REMARK 3. The assumption that f is unbounded at the left end of $(0,1)$ is not essential. Obviously, our result is valid for f unbounded at the right end of $(0,1)$ or at both ends (not necessarily symmetrically) or at some interior point of $(0,1)$, provided a condition analogous to (2.1) is satisfied.

REMARK 4. In Ćmiel et al. [1] the intermediate efficiency of some weighted goodness of fit tests has been investigated. In particular, from the results of that paper it follows that, in contrast to the statement of Theorem 1, for $f \in L_q[0,1]$, $q > 1$, the intermediate efficiency of the integral Anderson–Darling test with respect to the KS test is finite, with an explicit formula for calculating it. Also, for $f \in L_q[0,1]$, $q > 2$, the intermediate efficiencies of the classical Anderson–Darling (weighted supremum) test and its truncated version with respect to the KS test exist, with explicit formulae (cf. Remark 4, ibid.).

3. SQUARE INTEGRABLE CASE

Suppose that f in (1.1) belongs to $L_2[0,1]$. Set

$$a(t) = \frac{1}{c}(f(t)-1),$$

where $c^2 = \int_0^1 (f(t)-1)^2\,dt$. Then by rescaling θ_n we may rewrite (1.1) in the equivalent form

$$p_{\theta_n}(t) = 1 + \theta_n a(t), \qquad t \in (0,1).$$

In the present setting the asymptotic behaviour of b_n (cf. (1.3)), stated below, is an immediate corollary of Lemma 5 proved in Section 8.

PROPOSITION 3. *If* $f \in L_2[0,1]$ *then*

$$b_n \sim \sqrt{n}\,\theta_n \quad \textit{and} \quad \sigma_{0n} \sim \theta_n. \tag{3.1}$$

The following result plays the same role as Proposition 2 in the heavy-tailed case.

PROPOSITION 4. *Let* $p_{\theta_n}(t)$ *be a sequence of densities given by* (1.1) *with* $f \in L_2[0,1]$ *and* $\theta_n \to 0$ *such that* $n\theta_n^2 \to \infty$. *Then for every* $x \in \mathbb{R}$,

$$0 < \liminf_{n\to\infty} P_{\theta_n}^n(V_n \geqslant x + b_n) \leqslant \limsup_{n\to\infty} P_{\theta_n}^n(V_n \geqslant x + b_n) < 1. \tag{3.2}$$

The proof of Proposition 4 is given in Section 8. Now, we state our second main result. Its proof is provided in Section 7.

THEOREM 2. *Let* $p_{\theta_n}(t)$ *be a sequence of densities given by* (1.1) *with* $f \in L_2[0,1]$ *and* $\theta_n \to 0$ *such that* $n\theta_n^2 \to \infty$. *Then for any* $x \in \mathbb{R}$ *and for the significance levels defined by* (1.4) *we have, for* N_n *defined by* (1.5),

$$\lim_{n\to\infty} \frac{N_n}{n} = \frac{1}{4\|A\|_\infty^2} = \mathcal{E}(a), \tag{3.3}$$

where

$$A(t) = \int_0^t a(u)\,du$$

and $\|\cdot\|_\infty$ *denotes the supremum norm on* $[0,1]$.

REMARK 5. Theorem 2 says that the intermediate efficiency (as defined in Inglot et al. [8]) of the NP test with respect to the KS test for convergent square integrable sequences of alternatives exists and equals $1/(4\|A\|_\infty^2)$. Thus it extends Corollary 6.2 of Inglot and Ledwina [7] to the case of unbounded square integrable alternatives. Note that Corollary 6.2 was stated equivalently in terms of the intermediate efficiency of the KS test with respect to the NP test. Note also that in the proof of Theorem 2 we find that the intermediate slopes of the tests being compared are equal to $2n\theta_n^2\|A\|_\infty^2$ and $n\theta_n^2/2$, respectively, under the assumptions of this theorem, without introducing that terminology.

EXAMPLE. For $r \in (0, 1/2)$ let $f_r(t) = (1-r)t^{-r}$, $t \in (0,1)$, and consequently $a_r(t) = (\sqrt{1-2r}/r)((1-r)t^{-r} - 1)$. Then

$$\mathcal{E}(a_r) = \frac{(1-r)^{2-2/r}}{4(1-2r)}. \tag{3.4}$$

Observe that $\mathcal{E}(a_r) \to \infty$ as $r \to 1/2$, which nicely agrees with the statement of Theorem 1.

4. SIMULATION RESULTS

Below, we present results of a small simulation study showing how (2.4) and (3.3) are reflected empirically for a particular density

$$f_r(t) = (1-r)t^{-r}, \quad t \in (0,1).$$

We take the significance level $\alpha = 0.05$, select some small values of $\theta_n = \theta$ and keep powers separated from 0 and 1. We take heavy-tailed alternatives by choosing two values of r greater than 1/2 and square integrable alternatives represented by two values of r smaller than 1/2. In the last two cases the formula (3.4) can be applied. The results are shown in Tables 1–4.

TABLE 1. Empirical powers (in %) of the NP and KS tests for the alternative f_r, small values of θ and several n, with $\alpha = 0.05$, $r = 0.7$

n	$\theta = 0.1$		n	$\theta = 0.05$		n	$\theta = 0.02$	
	KS	NP		KS	NP		KS	NP
7	4	20	11	4	15	39	4	15
16	5	30	20	4	20	70	5	20
28	6	40	42	5	30	155	5	30
41	7	50	70	6	40	250	5	40
60	9	60	105	7	50	3350	15	99
80	11	70	150	7	60	4900	20	100
195	20	92	540	15	94	7700	30	100
300	30	98	750	20	98	10100	40	100
410	40	100	1200	30	100			
520	50	100	1600	40	100			
640	60	100	2080	50	100			
780	70	100	2500	60	100			

TABLE 2. Empirical powers (in %) of the NP and KS tests for the alternative f_r, small values of θ and several n, with $\alpha = 0.05$, $r = 0.6$

n	$\theta = 0.1$		n	$\theta = 0.05$		n	$\theta = 0.02$	
	KS	NP		KS	NP		KS	NP
16	5	20	27	5	15	110	4	15
35	6	30	48	5	20	205	5	20
60	7	40	105	6	30	460	6	30
92	9	50	180	7	40	5400	15	95
127	10	60	260	8	50	7800	20	99
175	13	70	370	9	60	12000	30	100
300	20	86	800	15	85			
480	30	96	1200	20	94			
640	40	99	1940	30	99			
840	50	100	2600	40	100			
1040	60	100	3300	50	100			
1280	70	100	4100	60	100			

TABLE 3. Empirical powers (in %) of the NP and KS tests for the alternative f_r, small values of θ and several n, with $\alpha = 0.05$, $r = 0.4$, $\mathcal{E}(a_{0.4}) = 5.787$

n	$\theta = 0.2$		n	$\theta = 0.1$		n	$\theta = 0.05$	
	KS	NP		KS	NP		KS	NP
15	5	15	54	5	15	205	6	15
28	6	20	100	6	20	380	6	20
62	9	30	220	8	30	810	8	30
100	11	40	360	11	40	2400	15	59
148	14	50	510	13	50	3450	20	72
153	15	51	600	15	55	5600	30	88
200	18	60	700	16	60			
225	20	64	880	20	68			
270	24	70	1430	30	84			
350	30	79	1950	40	93			
490	40	90	2500	50	97			
640	50	95	3160	60	99			
790	60	97						
970	70	99						

TABLE 4. Empirical powers (in %) of the NP and KS tests for the alternative f_r, small values of θ and several n, with $\alpha = 0.05$, $r = 0.3$, $\mathcal{E}(a_{0.3}) = 3.302$

n	$\theta = 0.2$		n	$\theta = 0.1$		n	$\theta = 0.05$	
	KS	NP		KS	NP		KS	NP
40	6	15	160	5	15	640	6	15
75	7	20	290	7	20	1200	7	20
165	10	30	610	10	30	2300	10	30
255	13	40	950	13	40	4800	15	47
300	15	44	1200	15	47	6800	20	59
360	17	50	1340	17	50	11100	30	77
435	20	56	1650	20	58			
480	22	60	1830	21	60			
645	28	70	2800	30	76			
710	30	73	3880	40	87			
980	40	84	5050	50	93			
1260	50	91	6350	60	97			
1600	60	96						
1950	70	98						

Using the results from Tables 1–4 we present in Table 5 the ratios N_n/n for four values of r, some small values of θ and several powers separated from 0 and 1. For a better illustration of our results we present those ratios also for $\alpha = 0.01$ (in bold face).

From Table 5 it is easily seen that for $r > 1/2$ the ratios N_n/n behave unstably and rapidly grow when θ tends to 0, thus confirming the statement of Theorem 1.

By contrast, for $r < 1/2$ the ratios behave stably and take values relatively close to the intermediate efficiency of the NP test with respect to the KS test given by (3.4). For smaller α the ratios become even more stable.

TABLE 5. Ratios N_n/n for the alternative f_r with small θ, several powers separated from 0 and 1 and four values of r, with $\alpha = 0.05$ and $\alpha = 0.01$ (bold face)

r		power in %						
$\mathcal{E}(a_r)$	θ	15	20	30	40	50	60	70
0.7	0.10		27.9	18.8	14.6	12.7	10.7	9.8
	0.05	49.1	37.5	28.6	22.9	19.8	16.7	
	0.02	85.9	70.0	49.7	40.4			
	0.10	**30.4**	**23.7**	**18.2**	**15.0**	**12.9**	**11.0**	**9.7**
	0.05	**44.8**	**36.0**	**27.3**	**22.3**	**19.2**	**16.7**	
	0.02	**80.9**	**63.4**	**47.5**				
0.6	0.10		18.8	13.7	10.7	9.1	8.2	7.3
	0.05	29.6	25.0	18.5	14.4	12.7	11.1	
	0.02	49.1	38.0	26.1				
	0.10	**17.9**	**14.5**	**11.2**	**9.7**	**8.6**	**7.8**	**7.0**
	0.05	**24.4**	**19.9**	**15.5**	**13.4**	**11.6**	**10.5**	
	0.02	**34.7**	**29.4**	**23.1**				
0.4	0.20	10.2	8.0	5.6	4.9	4.3	4.0	3.6
5.787	0.10	11.1	8.8	6.5	5.4	4.9	4.5	
	0.05	11.7	9.1	6.9				
	0.20	**5.8**	**5.0**	**4.4**	**4.0**	**3.8**	**3.6**	**3.4**
	0.10	**6.2**	**5.8**	**5.0**	**4.6**	**4.3**	**4.0**	**3.8**
	0.05	**6.6**	**6.3**	**5.5**				
0.3	0.20	7.5	5.8	4.3	3.8	3.5	3.3	3.0
3.302	0.10	7.5	5.7	4.6	4.1	3.8	3.5	
	0.05	7.5	5.7	4.8				
	0.20	**4.3**	**3.9**	**3.4**	**3.2**	**3.0**	**2.9**	**2.9**
	0.10	**4.3**	**4.0**	**3.6**	**3.4**	**3.3**	**3.1**	
	0.05	**4.4**	**4.0**	**3.7**				

5. PROOF OF THEOREM 1

A key step in the proof of the theorem is a moderate deviation result both for V_n and K_n under the null distribution. Below we state it as two separate propositions. The first one is stated in a weak form but sufficient to prove Theorem 1.

PROPOSITION 5. *If f in* (1.1) *satisfies* (2.1) *then for every sequence x_n of positive numbers such that $x_n = O(\kappa_n)$ we have*

$$- \limsup_{n\to\infty} \frac{1}{nx_n^2} \log P_0^n(V_n \geqslant \sqrt{n}\, x_n) > 0. \tag{5.1}$$

The proof of Proposition 5 is given in Section 6. The pertaining moderate deviation theorem for $K_n$was obtained in Inglot and Ledwina [4]. For completeness we state it below.

PROPOSITION 6. *For every sequence x_n of positive numbers such that $x_n \to 0$ and $nx_n^2 \to \infty$,*

$$- \lim_{n\to\infty} \frac{1}{nx_n^2} \log P_0^n(K_n \geqslant \sqrt{n}\, x_n) = 2. \tag{5.2}$$

Now, we are ready to prove the theorem. Take any $x \in \mathbb{R}$. Proposition 2 says that the sequence of powers of the NP test at the significance level α_n defined by (1.4) is bounded away from 0 and 1. Set $x_n = (x+b_n)/\sqrt{n}$. Then by Proposition 1, $x_n \asymp \kappa_n$, and from Proposition 5 it follows that for some positive constants c, c' and sufficiently large n,

$$- \log \alpha_n = - \log P_0^n(V_n \geqslant x + b_n) \geqslant c' n x_n^2 \geqslant c n \kappa_n^2. \tag{5.3}$$

Set $A(t) = \int_0^t f(u)\, du - t$ and $F_n(t) = t + \theta_n A(t)$. Then by the triangle inequality and for N_n defined by (1.5) we have

$$\begin{aligned} P_{\theta_n}^{N_n}(K_{N_n} \geqslant u_{N_n,n}) &= \Pr(\|e_{N_n} \circ F_n + \sqrt{N_n}\, \theta_n A\|_\infty \geqslant u_{N_n,n}) \\ &\leqslant \Pr(\|e_{N_n}\|_\infty \geqslant u_{N_n,n} - \sqrt{N_n}\, \theta_n \|A\|_\infty), \end{aligned}$$

where $e_N(t)$ denotes the uniform empirical process for the sample of size N while Pr denotes the probability on the underlying probability space. From (1.5), Proposition 2 and the convergence of e_{N_n} in distribution to a Brownian bridge it follows that for some positive C,

$$u_{N_n,n} - \sqrt{N_n}\, \theta_n \|A\|_\infty \leqslant C. \tag{5.4}$$

This implies $u_{N_n,n}/\sqrt{N_n} \to 0$. Since $P_0^{N_n}(K_{N_n} \geqslant u_{N_n,n}) = \alpha_n$ and $\alpha_n \to 0$, we have $u_{N_n,n} \to \infty$ and Proposition 6 applied to $x_n = u_{N_n,n}/\sqrt{N_n}$ gives

$$- \log \alpha_n = 2u_{N_n,n}^2(1 + o(1)). \tag{5.5}$$

This together with (5.3) and (5.4) gives, for sufficiently large n,

$$\begin{aligned} cn\kappa_n^2 \leqslant 2u_{N_n,n}^2(1 + o(1)) &\leqslant 2(C + \sqrt{N_n}\, \theta_n \|A\|_\infty)^2(1 + o(1)) \\ &\leqslant 5C^2 + 5N_n\theta_n^2\|A\|_\infty^2. \end{aligned}$$

As $\theta_n/\kappa_n \to 0$ the above implies $n/N_n \to 0$ and finishes the proof of (2.4). ■

6. PROOFS OF PROPOSITIONS 2 AND 5

6.1. Auxiliary lemmas. For $k = 0, 1$ and integer $m \geqslant 1$ set

$$I_{km}(n) = \int_0^1 [\theta_n g(t)]^k \log^m(1 + \theta_n g(t))\, dt,$$

$$J_{km}(n) = \int_0^1 [1 + \theta_n g(t)]^k |\log(1 + \theta_n g(t)) - e_{0n}|^m\, dt,$$

where $g(t) = f(t) - 1$ while f and θ_n are as in (1.1). The first lemma describes the asymptotic behaviour of $I_{km}(n)$ and $J_{km}(n)$ as $n \to \infty$.

LEMMA 1. *Suppose f satisfies* (2.1). *Then for any $k = 0, 1$ and any integer $m \geqslant 1$ such that $k + m \geqslant 2$ we have*

$$I_{km}(n) \asymp \kappa_n^2. \tag{6.1}$$

Moreover, for any $k = 0, 1$ and $m \geqslant 2$ we have

$$J_{km}(n) \asymp \kappa_n^2. \tag{6.2}$$

The proof of Lemma 1 is based on the following elementary fact.

LEMMA 2. *Suppose f satisfies* (2.1). *Then for any $k = 0, 1$ and any integer $m \geqslant 1$ such that $k + m \geqslant 2$ we have*

$$\mathbb{I}_{km}(n) = \int_0^\delta [\theta_n g(t)]^k \log^m(1 + \theta_n g(t))\, dt \asymp \kappa_n^2 \tag{6.3}$$

and for any $k = 0, 1$ and any integer $m \geqslant 2$ we have

$$\mathbb{J}_{km}(n) = \int_0^\delta [1 + \theta_n g(t)]^k |\log(1 + \theta_n g(t)) - e_{0n}|^m dt \asymp \kappa_n^2. \tag{6.4}$$

Proof of Lemma 2. If $k+m$ is odd then the function $\psi_{km}(y) = y^k \log^m(1+y)$ is increasing on $(-1, \infty)$ while for $k + m$ even $\psi_{km}(y)$ is decreasing on $(-1, 0)$ and increasing on $(0, \infty)$. Hence, by the monotonicity of $\psi_{km}(y)$ on $(0, \infty)$, the relation $f(t) > 2$ on $(0, \delta]$ and the inequality $y/2 \leqslant \log(1+y)$ holding on $(0, 1/2)$, from (2.1) and after the substitution $y = \theta_n(C_1 h(t) - 1)$, the integral in (6.3) can be estimated for n sufficiently large from below by

$$\begin{aligned}
\mathbb{I}_{km}(n) &\geqslant \int_0^\delta [\theta_n(C_1 h(t) - 1)]^k \log^m(1 + \theta_n(C_1 h(t) - 1))\, dt \\
&= C_0 C_1^\eta \theta_n^\eta \int_{\theta_n(C_1 h(\delta)-1)}^\infty y^k \frac{\log^m(1+y) \log^\zeta[(\theta_n + y)/(C_1\theta_n)]}{(\theta_n + y)^{1+\eta}}\, dy \\
&\geqslant 2^{-1} C_0 C_1^\eta \kappa_n^2 \begin{cases} \int_1^\infty y^{k-1-\eta} \log^m(1+y)\, dy & \text{if } \zeta \geqslant 0, \\ 2^{-m} \int_{\theta_n(C_1 h(\delta)-1)}^{1/2} y^{k+m-1-\eta}\, dy & \text{if } \zeta < 0. \end{cases}
\end{aligned}$$

Denote $l_n(y) = \log^\zeta[(\theta_n + y)/(C_2\theta_n)]$ and $\vartheta_n = \theta_n(2C_2 - 1)$. As $h(\delta) \geqslant 2$, for n

sufficiently large the substitution $y = \theta_n(C_2h(t) - 1)$ gives

$$
\begin{aligned}
\text{(6.5)}\qquad \mathbb{I}_{km}(n) &\leqslant \int_0^\delta [\theta_n(C_2h(t)-1)]^k \log^m\big(1+\theta_n(C_2h(t)-1)\big)\,dt\\
&= C_0C_2^\eta\theta_n^\eta \int_{\theta_n(C_2h(\delta)-1)}^{\infty} y^k \frac{\log^m(1+y)l_n(y)}{(\theta_n+y)^{1+\eta}}\,dy\\
&\leqslant C_0C_2^\eta\theta_n^\eta\bigg[\int_1^\infty y^k\frac{\log^m(1+y)l_n(y)}{y^{(3+\eta)/2}(\theta_n+y)^{(\eta-1)/2}}dy + \int_{\vartheta_n}^1 y^{k+m-1-\eta}l_n(y)\,dy\bigg].
\end{aligned}
$$

Since, given η and ζ, the function $z(v) = v^{(1-\eta)/2}\log^\zeta(v/(C_2\theta_n))$ is decreasing on $(1,\infty)$ for n sufficiently large, the first term on the right hand side of (6.5) can be further estimated from above by

$$C_0C_2^\eta\theta_n^\eta\log^\zeta\left(\frac{1}{C_2\theta_n}\right)\int_1^\infty y^k\frac{\log^m(1+y)}{y^{(3+\eta)/2}}\,dy.$$

To deal with the second term in (6.5) we estimate the factor $l_n(y)$ for sufficiently large n by $\log^\zeta(1/\theta_n)$ if $\zeta \geqslant 0$ and by

$$\log^\zeta 2\,\mathbf{1}_{[\vartheta_n,\tau_n]}(y) + \log^\zeta\left(\frac{\tau_n}{C_2\theta_n}\right)\mathbf{1}_{(\tau_n,1]}(y)$$

if $\zeta < 0$, where $\tau_n = [\log(1/\theta_n)]^{\zeta/(k+m-\eta)}$. Consequently, the second term in (6.5) can be estimated from above by

$$\frac{C_0C_2^\eta\theta_n^\eta}{2-\eta}\left[\log^\zeta(1/\theta_n) + (1+\log^\zeta 2)\log^\zeta\left(\frac{1}{C_2\theta_n}\right)\right].$$

Since all terms in the above estimates are of order κ_n^2 the relation (6.3) is proved.

Now, observe that for n sufficiently large

$$\text{(6.6)}\qquad -2\theta_n \leqslant \log(1-\theta_n) \leqslant e_{0n} = \int_0^1 \log p_{\theta_n}(t)\,dt \leqslant \theta_n\int_0^1 (f(t)-1)\,dt = 0.$$

Moreover, by (6.6) and the relation $f(t) > 2$ on $(0,\delta]$ we have

$$\mathbb{I}_{0m}(n) \leqslant \mathbb{J}_{km}(n) \leqslant 2^{m-1}(\mathbb{I}_{0m}(n) + \mathbb{I}_{1m}(n)) + 2^{2m-1}\theta_n^m.$$

Since $m \geqslant 2$, (6.4) follows from (6.3). ∎

Proof of Lemma 1. The monotonicity properties of the functions $\psi_{km}(y)$ defined in the proof of Lemma 2 and boundedness of $f(t)$ on $[\delta,1]$ (cf. (2.1)) imply that for n sufficiently large,

$$|I_{km}(n) - \mathbb{I}_{km}(n)| \leqslant \theta_n^k|\log(1-\theta_n)|^m + [\theta_nC_2]^k\log^m(1+\theta_nC_2) \asymp \theta_n^{k+m},$$

and due to (6.6),

$$|J_{km}(n) - \mathbb{J}_{km}(n)| \leqslant (1+\theta_n C_2)^k (\log(1+\theta_n C_2) - e_{0n})^m \asymp \theta_n^m.$$

Since $k + m \geqslant 2$, (6.1) follows from (6.3), while (6.2) follows from (6.4) and the assumption $m \geqslant 2$. ∎

LEMMA 3. *If f satisfies* (2.1) *then*

$$e_n - e_{0n} \asymp \kappa_n^2 \quad \textit{and} \quad \sigma_{0n}^2 \asymp \kappa_n^2 \asymp \sigma_n^2. \tag{6.7}$$

Proof. Observe that $e_n - e_{0n} = I_{11}(n)$, $\sigma_{0n}^2 = I_{02}(n) - e_{0n}^2$ and $\sigma_n^2 = J_{12}(n) - (e_n - e_{0n})^2$. Hence, (6.7) follows immediately from Lemma 1. ∎

LEMMA 4. *For each $n \geqslant 1$ let $X_1, \ldots, X_n$ be independent random variables each with density $p_{\theta_n}(t)$ and let f satisfy* (2.1). *If $\theta_n \to 0$ and $n\kappa_n^2 \to \infty$ then for every $y \in \mathbb{R}$,*

$$\lim_{n\to\infty} P_{\theta_n}^n \left(\frac{1}{\sqrt{n}\,\sigma_n} \sum_{i=1}^n (\log p_{\theta_n}(X_i) - e_n) \leqslant y \right) = \Phi(y).$$

Proof. Let $Y_{ni} = \log p_{\theta_n}(X_i) - e_n$, $i = 1, \ldots, n$, $n \geqslant 1$, be a triangular array of independent mean 0 random variables. To prove Lemma 4 it is enough to check the Lyapunov condition. We have $E_{\theta_n}|Y_{ni}|^3 \leqslant 4J_{13}(n) + 4(e_n - e_{0n})^3 \asymp \kappa_n^2$ by (6.2) and (6.7). Since $\sigma_n^3 \asymp \kappa_n^3$ by (6.7), the Lyapunov condition holds true due to the assumption $n\kappa_n^2 \to \infty$. ∎

6.2. Proof of Proposition 2. Observe that

$$V_n = \frac{\sigma_n}{\sigma_{0n}} \left[\frac{1}{\sqrt{n}\,\sigma_n} \sum_{i=1}^n (\log p_{\theta_n}(X_i) - e_n) \right] + b_n.$$

So, for $x \in \mathbb{R}$,

$$P_{\theta_n}^n (V_n \geqslant x + b_n) = P_{\theta_n}^n \left(\frac{1}{\sqrt{n}\,\sigma_n} \sum_{i=1}^n (\log p_{\theta_n}(X_i) - e_n) \geqslant x \frac{\sigma_{0n}}{\sigma_n} \right)$$

and (2.3) is an immediate consequence of Lemmas 3 and 4. ∎

6.3. Proof of Proposition 5. We shall apply the following version of the Bernstein inequality (cf. Yurinskii [12]).

THEOREM A. *Let $\xi_1, \ldots, \xi_n$, $n \geqslant 1$, be independent identically distributed random variables with $E\xi_1 = 0$ and $E\xi_1^2 = 1$ such that for some constant $M > 0$,*

$$E|\xi_1|^m \leqslant \frac{m!}{2} M^{m-2} \quad \textit{for every } m \geqslant 3. \tag{6.8}$$

Then for all $x > 0$,

$$P\left(\frac{\xi_1 + \cdots + \xi_n}{\sqrt{n}} \geqslant x \right) \leqslant 2 \exp\left\{ -\frac{x^2}{2(1 + xM/\sqrt{n}\,)} \right\}. \tag{6.9}$$

In Theorem A set $\xi_i = (\log p_{\theta_n}(X_i) - e_{0n})/\sigma_{0n}$, $i = 1, \ldots, n$, where $X_1, \ldots, X_n$ are independent uniformly distributed over $[0, 1]$. Then $E_0|\xi_1|^m = J_{0m}(n)/\sigma_{0n}^m$ for all $m \geqslant 3$. For $m \geqslant 3$ from boundedness of $f(t)$ on $[\delta, 1]$ we have

$$\begin{aligned} J_{0m}(n) &\leqslant 2^{m-1}\Big[\mathbb{I}_{0m}(n) + \int_\delta^1 |\log(1+\theta_n g(t)|^m\,dt + |e_{0n}|^m\Big] \\ &\leqslant 2^m \mathbb{I}_{0m}(n) + 4^m C_2^m \theta_n^m. \end{aligned} \tag{6.10}$$

Since

$$\int_1^\infty \frac{\log^m(1+y)}{y^{(3+\eta)/2}}\,dy \leqslant 2^{(3+\eta)/2}\int_0^\infty \frac{\log^m(1+y)}{(1+y)^{(3+\eta)/2}}\,dy = 2^{(3+\eta)/2}\Big(\frac{2}{1+\eta}\Big)^{m+1} m!,$$

using the estimates of both terms in (6.5) from the proof of Lemma 2, (6.10) and the relation $\sigma_{0n}^2 \asymp \theta_n^\eta \log^\zeta(1/(C_2\theta_n))$ it follows that there exists a constant $D = D(\delta, \eta, \zeta, C_0, C_2) \geqslant 1$ such that for all $m \geqslant 3$,

$$E_0|\xi_1|^m = \frac{J_{0m}}{\sigma_{0n}^m} \leqslant D\frac{m!}{2}\Big(\frac{2}{\sigma_{0n}}\Big)^{m-2}.$$

This implies that (6.8) holds with e.g. $M_n = 2D/\sigma_{0n}$. Applying (6.9) to $x = \sqrt{n}\,x_n$ we get

$$P_0^n(V_n \geqslant \sqrt{n}\,x_n) \leqslant 2\exp\Big\{-\frac{nx_n^2}{2(1+x_nM_n)}\Big\}.$$

By the assumption and Lemma 3 we have $x_nM_n = O(\kappa_n/\sigma_{0n}) = O(1)$ and hence (5.1) follows. ■

7. PROOF OF THEOREM 2

We shall apply the following moderate deviation result for V_n, proved in Section 8.

PROPOSITION 7. *If f in* (1.1) *satisfies $f \in L_2[0,1]$ and $n\theta_n^2 \to \infty$, then for any positive $\delta < 1/2$ and every sequence x_n satisfying $2\delta\sigma_{0n} < x_n < 2(1-\delta)\sigma_{0n}$,*

$$-\lim_{n\to\infty}\frac{1}{nx_n^2}\log P_0^n(V_n \geqslant \sqrt{n}\,x_n) = \frac{1}{2}. \tag{7.1}$$

Now, take any $x \in \mathbb{R}$. Proposition 4 says that the sequence of powers of the NP test at the significance level α_n defined by (1.4) is bounded away from 0 and 1. Set $x_n = (x + b_n)/\sqrt{n}$. Then, by Proposition 3, x_n satisfies the assumption of Proposition 7 for sufficiently large n. Hence

$$-\log\alpha_n = -\log P_0^n(V_n \geqslant x + b_n) = \frac{n\theta_n^2}{2}(1+o(1)). \tag{7.2}$$

Set $F_n(t) = t + \theta_n A(t)$. Recall that here $A(t)$, defined in Theorem 2, corresponds to the normalized function a. Repeating the same argument as in the proof of (5.5) we get

$$-\log \alpha_n = 2u^2_{N_n,n}(1+o(1)). \tag{7.3}$$

This together with (7.2) gives

$$n\theta_n^2/2 = 2u^2_{N_n,n}(1+o(1)) \leqslant 2(C+\sqrt{N_n}\,\theta_n\|A\|_\infty)^2(1+o(1)),$$

which means that

$$\limsup_{n\to\infty} \frac{n}{N_n} \leqslant 4\|A\|^2_\infty. \tag{7.4}$$

On the other hand, using the minimality property of N_n in (1.5) and Proposition 4, a similar argument to the one used to get (5.4) leads to

$$u_{N_n-1,n} - \sqrt{N_n-1}\,\theta_n\|A\|_\infty \geqslant -C \tag{7.5}$$

for some positive constant C. Observe that $u_{N_n-1,n}/\sqrt{N_n-1} \to 0$. Indeed, Proposition 6 applies to $x_n' = \theta_n\sqrt{n/(N_n-1)} \to 0$ and shows that $\log P_0^{N_n-1}(K_{N_n-1} \geqslant x_n'\sqrt{N_n-1}) = -2n\theta_n^2(1+o(1))$, which together with (7.2) and the definition of $u_{N_n-1,n}$ implies that, for n sufficiently large, $u_{N_n-1,n}/\sqrt{N_n-1} \leqslant x_n'$, thus proving our claim. Again applying Proposition 6 to $x_n = u_{N_n-1,n}/\sqrt{N_n-1}$ we obtain

$$-\log \alpha_n = 2u^2_{N_n-1,n}(1+o(1)),$$

and consequently from (7.2) and (7.5),

$$n\theta_n^2/2 = 2u^2_{N_n-1,n}(1+o(1)) \geqslant 2(\sqrt{N_n-1}\,\theta_n\|A\|_\infty - C)^2(1+o(1)).$$

Hence,

$$\liminf_{n\to\infty} \frac{n}{N_n} \geqslant 4\|A\|^2_\infty,$$

which together with (7.4) proves (3.3). ∎

8. PROOFS OF PROPOSITIONS 4 AND 7

Recall some useful simple inequalities:

$$\log^2(1+y) \leqslant y, \qquad y \geqslant 0, \tag{8.1}$$

$$\log^3(1+y) \leqslant \min\{\tfrac{3}{2}y, y^2\}, \qquad y \geqslant 0, \tag{8.2}$$

and for any $0<\varepsilon<1/2$,

$$(1-\varepsilon)y^2 \leqslant y\log(1+y) \leqslant (1+\varepsilon)y^2, \qquad y\in[-\varepsilon,\varepsilon], \tag{8.3}$$

$$(1-\varepsilon)y^2 \leqslant \log^2(1+y) \leqslant (1+2\varepsilon)y^2, \qquad y\in[-\varepsilon,\varepsilon]. \tag{8.4}$$

LEMMA 5. *If $f\in L_2[0,1]$ then $e_n - e_{0n} = \theta_n^2(1+o(1))$, $\sigma_{0n}^2 = \theta_n^2(1+o(1))$ and $\sigma_n^2 = \theta_n^2(1+o(1))$.*

Proof. Taking $\varepsilon = \sqrt{\theta_n}$ in (8.3) and remembering that $a(t) \geqslant -1$ a.s. we have, for sufficiently large n,

$$\begin{aligned} e_n - e_{0n} &= \int\limits_{a<1/\sqrt{\theta_n}} \theta_n a(t)\log(1+\theta_n a(t))\,dt + \int\limits_{a\geqslant 1/\sqrt{\theta_n}} \theta_n a(t)\log(1+\theta_n a(t))\,dt \\ &\leqslant (1+\sqrt{\theta_n})\theta_n^2 \int\limits_{a<1/\sqrt{\theta_n}} a^2(t)\,dt + \theta_n^2 \int\limits_{a\geqslant 1/\sqrt{\theta_n}} a^2(t)\,dt = \theta_n^2(1+o(1)) \end{aligned}$$

and

$$e_n - e_{0n} \geqslant \int\limits_{a<1/\sqrt{\theta_n}} \theta_n a(t)\log(1+\theta_n a(t))\,dt = \theta_n^2(1+o(1)),$$

which proves the first statement.

From (8.4) with $\varepsilon = \sqrt{\theta_n}$ and a similar argument we get

$$\int_0^1 \log^2(1+\theta_n a(t))\,dt = \theta_n^2(1+o(1)). \tag{8.5}$$

Moreover, the obvious inequality $y - y^2 \leqslant \log(1+y)$, $y \in [-1/2, 1/2]$, implies

$$0 \geqslant e_{0n} \geqslant -\theta_n \int\limits_{a\geqslant 1/\sqrt{\theta_n}} a(t)\,dt - \theta_n^2 = o(\theta_n). \tag{8.6}$$

In fact, one may show that $e_{0n} = -(\theta_n^2/2)(1+o(1))$. Combining (8.5) and (8.6) we get the second statement.

To prove the third one, note first that $e_n = (e_n - e_{0n}) + e_{0n} = o(\theta_n)$. Moreover, since $a(t) \geqslant -1$ a.s., from (8.1) we get

$$\begin{aligned} -\theta_n \log^2(1-\theta_n) &\leqslant \int_0^1 \theta_n a(t)\log^2(1+\theta_n a(t))\,dt \\ &\leqslant \theta_n^2 \int\limits_{a\geqslant 1/\sqrt{\theta_n}} a^2(t)\,dt + \int\limits_{0<a<1/\sqrt{\theta_n}} \theta_n a(t)\log^2(1+\theta_n a(t))\,dt \\ &\leqslant \theta_n^2 \int\limits_{a\geqslant 1/\sqrt{\theta_n}} a^2(t)\,dt + \theta_n^{5/2} = o(\theta_n^2). \end{aligned}$$

From (8.5) and the above,

$$\sigma_n^2 = \int_0^1 \log^2(1+\theta_n a(t))\,dt + \int_0^1 \theta_n a(t)\log^2(1+\theta_n a(t))\,dt - e_n^2 = \theta_n^2(1+o(1)). \quad \blacksquare$$

LEMMA 6. *For each $n \geqslant 1$ let $X_1, \ldots, X_n$ be independent random variables with density $p_{\theta_n}(t)$ given by* (1.1) *with $f \in L_2[0,1]$ and $\theta_n \to 0$ such that $n\theta_n^2 \to \infty$. Then for every $y \in \mathbb{R}$,*

$$\lim_{n\to\infty} P_{\theta_n}^n\Big(\frac{1}{\sqrt{n}\,\sigma_n}\sum_{i=1}^n \big(\log(1+\theta_n a(X_i)) - e_n\big) \leqslant y\Big) = \Phi(y).$$

Proof. Denote $Y_{ni} = \log(1+\theta_n a(X_i)) - e_n$, $i = 1, \ldots, n$, $n \geqslant 1$, a triangular array of independent mean 0 random variables. It is enough to check the Lyapunov condition. Indeed, from (8.2), (8.6) and Lemma 5, for sufficiently large n we have

$$\begin{aligned}E_{\theta_n}|Y_{ni}|^3 &\leqslant 4e_n^3 + 4 \int\limits_{a\geqslant 1/\sqrt{\theta_n}} (1+\theta_n a(t))\log^3(1+\theta_n a(t))\,dt \\ &\quad + 4 \int\limits_{a<1/\sqrt{\theta_n}} (1+\theta_n a(t))|\log(1+\theta_n a(t))|^3\,dt \\ &\leqslant 4e_n^3 + 10\theta_n^2 \int\limits_{a\geqslant 1/\sqrt{\theta_n}} a^2(t)\,dt + 4\sqrt{\theta_n}\,(\sigma_n^2+e_n^2) = o(\theta_n^2).\end{aligned}$$

By Lemma 5 and the assumption $n\theta_n^2 \to \infty$ the Lyapunov condition holds. ∎

It is easily seen that with the use of Lemma 6 the proof of Proposition 4 goes exactly in the same way as that of Proposition 2.

The proof of Proposition 7 is based on the following moderate deviation result of Ermakov [2].

THEOREM B. *Let $Y_{n1},\dots,Y_{nn}$, $n\geqslant 1$, be a triangular array of independent identically distributed random variables with $EY_{n1}=0$, $\operatorname{Var} Y_{n1}=1$ and for some sequence $h_n>0$, $h_n\to 0$, $nh_n^2\to\infty$ and some $C>0$ we have*

(i) $Ee^{h_nY_{n1}} < C$;

(ii) $E|Y_{n1}|^3 \leqslant C\omega_n/h_n$ *for some* $\omega_n > 0$ (*possibly depending on* h_n).

Then for all x such that $\varepsilon h_n \leqslant x \leqslant (1-\varepsilon)h_n$ for some $\varepsilon>0$,

$$\log P\biggl(\frac{Y_{n1}+\dots+Y_{nn}}{\sqrt{n}} \geqslant \sqrt{n}\,x\biggr) = -\frac{nx^2}{2} + O(nh_n^2\omega_n). \tag{8.7}$$

In Theorem B set $Y_{ni} = (\log(1+\theta_n a(X_i)) - e_{0n})/\sigma_{0n}$ and $h_n = 2\sigma_{0n}$. Then $nh_n^2\to\infty$ by the assumption and Lemma 5. Moreover, for sufficiently large n,

$$E_0 e^{h_nY_{n1}} = e^{-2e_{0n}} \int\limits_0^1 (1+\theta_n a(t))^2\,dt < 2,$$

which proves condition (i) in the theorem. From (8.2) and (8.6) we have

$$\begin{aligned}E_0|Y_{n1}|^3 &\leqslant \frac{4|e_{0n}|^3}{\sigma_{0n}^3} + \frac{4}{\sigma_{0n}^3} \int\limits_{a\geqslant 1/\sqrt{\theta_n}} \log^3(1+\theta_n a(t))\,dt \\ &\quad + \frac{4\sqrt{\theta_n}}{\sigma_{0n}^3} \int\limits_{a<1/\sqrt{\theta_n}} \log^2(1+\theta_n a(t))\,dt \\ &\leqslant \frac{4}{\sqrt{\theta_n}}(1+o(1)) + \frac{4}{\theta_n} \int\limits_{a\geqslant 1/\sqrt{\theta_n}} a^2(t)\,dt.\end{aligned}$$

So, condition (ii) of Theorem B holds with $\omega_n = \max\{\sqrt{\theta_n}, \int_{a\geqslant 1/\sqrt{\theta_n}} a^2(t)\,dt\}$ tending to 0 and (7.1) follows from (8.7) by inserting x_n in place of x. ∎

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