Abstract. For a centered self-similar Gaussian process \( \{Y(t) : t \in [0, \infty)\} \) and \( R \geq 0 \) we analyze the asymptotic behavior of

\[
\mathcal{H}_Y^R(T) = \mathbf{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2} Y(t) - (1 + R) \sigma_Y^2(t) \right) \right)
\]

as \( T \to \infty \). We prove that \( \mathcal{H}_Y^R = \lim_{T \to \infty} \mathcal{H}_Y^R(T) \in (0, \infty) \) for \( R > 0 \) and

\[
\mathcal{H}_Y = \lim_{T \to \infty} \frac{\mathcal{H}_Y^0(T)}{T^\gamma} \in (0, \infty)
\]

for suitably chosen \( \gamma > 0 \). Additionally, we find bounds for \( \mathcal{H}_Y^R, R > 0 \), and a surprising relation between \( \mathcal{H}_Y \) and the classical Pickands constants.

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1. INTRODUCTION

For a centered Gaussian process \( \{Y(t) : t \in [0, \infty)\} \) with a.s. continuous sample paths, \( \text{Var}(Y(t)) = \sigma_Y^2(t) \) and \( Y(0) = 0 \) a.s., let

\[
(1.1) \quad \mathcal{H}_Y^R(T) = \mathbf{E} \exp \left( \sup_{t \in [0, T]} \left( \sqrt{2} Y(t) - (1 + R) \sigma_Y^2(t) \right) \right),
\]

where \( R \geq 0 \) and let \( \mathcal{H}_Y(T) := \mathcal{H}_Y^0(T) \).

The functionals \( \mathcal{H}_Y^R(T), \mathcal{H}_Y(T) \) play an important role in many areas of probability theory. For example, consider a fractional Brownian motion \( \{B_\kappa(t) : t \in [0, \infty)\} \) with Hurst parameter \( \kappa/2 \in (0, 1] \), i.e. a centered Gaussian process with stationary increments, continuous sample paths a.s. and variance function \( \text{Var}(B_\kappa(t)) = t^\kappa \). Then, for \( \kappa \in (0, 2] \), the Pickands constants \( \mathcal{H}_{B_\kappa} \) defined
as
\[ \mathcal{H}_{B_\kappa} = \lim_{T \to \infty} \frac{\mathcal{H}_{B_\kappa}(T)}{T}, \]
and the Piterbarg constants \( \mathcal{H}_{B_\kappa}^R \), for \( R > 0 \), defined as
\[ \mathcal{H}_{B_\kappa}^R = \lim_{T \to \infty} \mathcal{H}_{B_\kappa}^R(T) \]
play a key role in the extreme value theory of Gaussian processes; see, e.g., [25], [26], [27] or more recent contributions [18], [24]. In [6] it was observed that the notion of Pickands and Piterbarg constants can be extended to generalized Pickands and Piterbarg constants, defined as
\[ \mathcal{H}_\eta = \lim_{T \to \infty} \frac{\mathcal{H}_\eta(T)}{T} \quad \text{and} \quad \mathcal{H}_\eta^R = \lim_{T \to \infty} \mathcal{H}_\eta^R(T) \]
respectively, where \( R > 0 \) and \( \{ \eta(t) : t \in (0, \infty) \} \) is a centered Gaussian process with stationary increments. We refer to [2], [3], [5], [12], [14], [16] for properties and other representations of \( \mathcal{H}_{B_\kappa} \), \( \mathcal{H}_{B_\kappa}^R \) and generalized Pickands–Piterbarg constants, and to [10], [11] for multidimensional analogs of Pickands–Piterbarg constants.

Recently (see e.g. [12]), it was found that for general Gaussian processes \( Y \) (satisfying some regularity conditions) the functionals (1.1) appear in the formulas for exact asymptotics of suprema of some Gaussian processes (see Proposition 2.1). The interest in (1.1) also stems from an important contribution [16] which established a direct connection between Pickands constants and max-stationary stable processes (see also [7], [8], [9]).

The constants \( \mathcal{H}_Y(T) \) also appear in the context of convex geometry where they are known as Wills functionals (see [28]).

In this contribution we analyze the properties of \( \mathcal{H}_Y^R(T) \) and \( \mathcal{H}_Y(T) \) for a class of general self-similar Gaussian processes \( Y \) with non-stationary increments. In particular, we find analogs of limits (1.2), (1.3) and give some bounds for them. Surprisingly, it appears that, up to some explicitly given constant, \( \mathcal{H}_Y \) is equal to the classical \( \mathcal{H}_{B_\kappa} \) for some appropriately chosen \( \kappa \).

2. NOTATION AND PRELIMINARY RESULTS

Let \( \{ Y(t) : t \geq 0 \} \) be a centered Gaussian process with a.s. continuous sample paths and let
\[ V_Y(s,t) := \text{Var}(Y(s) - Y(t)), \quad R_Y(s,t) := \text{Cov}(Y(s), Y(t)). \]

We say that a stochastic process \( Y(\cdot) \) is self-similar with index \( H > 0 \) if for all \( a > 0 \),
\[ \{ Y(at) : t \geq 0 \} \overset{D}{=} \{ a^H Y(t) : t \geq 0 \}. \]
A straightforward consequence of (2.1) is that for self-similar Gaussian processes, 
\( \sigma^2_Y(t) = \sigma^2_Y(1)t^{2H} \) for \( t \geq 0 \).

We write \( Y \in S(\alpha, \kappa, c_Y) \) if

\[ S1. \quad Y(\cdot) \text{ is self-similar with index } \alpha/2 > 0 \text{ and } \sigma^2_Y(1) = 1; \]

\[ S2. \quad \text{there exist } \kappa \in (0, 2] \text{ and } c_Y > 0 \text{ such that} \]

\[ \text{Var}(Y(1) - Y(1 - h)) = c_Y|h|^\kappa + o(|h|^\kappa) \text{ as } h \to 0. \]

It is well known (see Lamperti [21]) that \( \{Y(t) : t \geq 0\} \) is a self-similar Gaussian process with index \( \alpha/2 \) if and only if its Lamperti transform \( X(t) = e^{-\alpha/2}tY(e^t) \) is a stationary Gaussian process. Thus, there is a unique correspondence between self-similar Gaussian processes and stationary Gaussian processes. In fact condition \( S2 \) relates to regularity of the covariance function of the stationary counterpart of \( Y \). More precisely, let \( \{X(t) : t \in \mathbb{R}\} \) be a stationary Gaussian process such that \( R_X(t, 0) = 1 - a|t|^\kappa + o(|t|^\kappa) \) as \( t \to 0 \) with \( \kappa \in (0, 2], a > 0 \). Then one can check that the self-similar process \( Y(t) := t^{\alpha/2}X(\log t) \) for \( \alpha \in (0, 2] \) is \( S(\alpha, \kappa, c_Y) \) with

\[ c_Y = \begin{cases} 
2a & \text{for } \kappa < 2, \\
\alpha^2/4 + 2a & \text{for } \kappa = 2.
\end{cases} \]

Below we specify some important classes of self-similar Gaussian processes that satisfy \( S1–S2 \).

\( \diamond \) **Fractional Brownian motion** \( B_\alpha \) is in \( S(\alpha, \alpha, 1) \) with \( \alpha/2 \in (0, 1] \).

\( \diamond \) **Bifractional Brownian motion** \( \{Y^{(1)}(t) : t \geq 0\} \) with parameters \( \alpha \in (0, 2) \) and \( K \in (0, 1] \) is a centered Gaussian process with covariance function

\[ R_{Y^{(1)}}(t,s) = \frac{1}{2^K}((t^\alpha + s^\alpha)^K - |t - s|^\alpha K) \]

(see e.g. [19], [22]). We have \( Y^{(1)} \in S(\alpha K, \alpha K, 2^{1-K}) \).

\( \diamond \) **Sub-fractional Brownian motion** \( \{Y^{(2)}(t) : t \geq 0\} \) with parameter \( \alpha \in (0, 2) \) is a centered Gaussian process with covariance function

\[ R_{Y^{(2)}}(t,s) = \frac{1}{2-2\alpha-1}\left(t^\alpha + s^\alpha - \frac{(t+s)^\alpha + |t-s|^\alpha}{2}\right) \]

(see [4], [17]). Then \( Y^{(2)} \in S(\alpha, \alpha, (2 - 2^{\alpha-1})^{-1}) \).
\[ Y^{(3),1}(t) = \sqrt{\alpha + 2} \int_0^t B_\alpha(s) \, ds, \]
\[ Y^{(3),k}(t) = \sqrt{\frac{k(\alpha + 2k)(\alpha + k - 1)}{\alpha + 2k - 2}} \int_0^t Y^{(3),k-1}(s) \, ds \quad \text{for } k \geq 2. \]

Then \( Y^{(3),k} \in \mathcal{S}(\alpha + 2k, 2, \frac{k(\alpha + 2k)(\alpha + k - 1)}{\alpha + 2k - 2}) \) for \( k \geq 1 \).

\[ Y^{(4)}(t) = \sqrt{\alpha + 2} \frac{1}{t} \int_0^t B_\alpha(s) \, ds. \]

Its covariance function is
\[ R_{Y^{(4)}}(t, s) = \frac{(\alpha + 2)(s^{\alpha+1}t + st^{\alpha+1}) + |t - s|^{\alpha+2} - t^{\alpha+2} - s^{\alpha+2}}{2(\alpha + 1)ts} \]
and we have \( Y^{(4)} \in \mathcal{S}(\alpha, 2, 1) \).

\[ Y^{(5)}(t) = t^{\alpha+1} \sqrt{\frac{2}{\Gamma(\alpha + 1)}} \int_0^\infty B_\alpha(s)e^{-st} \, ds \]
(see [23]). We have
\[ R_{Y^{(5)}}(t, s) = \frac{t^{\alpha}s + s^{\alpha}t}{t + s} \]
and \( Y^{(5)} \in \mathcal{S}(\alpha, 2, \alpha/2) \).

In the rest of the paper, \( \overline{X}(s) := X(s)/\sigma_X(s) \), and \( \Psi(\cdot) \) denotes the tail distribution function of the standard normal random variable.

The following proposition plays a key role in the proofs of our main results, confirming also that the functionals \( H_Y(\cdot) \) and \( H_{Y^B}(\cdot) \) for \( Y \in \mathcal{S}(\alpha, \kappa, c_Y) \) appear in the asymptotics of extremes of Gaussian processes.

**Proposition 2.1.** Let \( Y \in \mathcal{S}(\alpha, \kappa, c_Y) \) and let \( \{X(t) : t \geq 0\} \) be a centered Gaussian process with \( R_X(t, s) = \exp(-aV_Y(t,s)) \) for \( a > 0 \) and \( \sigma_X(t) = \frac{1}{1+bt^\beta} \) for \( b > 0, \beta > 0 \).
(i) \( \alpha = \beta \), then as \( u \to \infty \),

\[
P\left( \sup_{t \in [0,Tu^{-2/\alpha}]} X(t) > u \right) = \mathcal{H}^{h/\alpha}_Y (a^{1/\alpha} T) \Psi(u)(1 + o(1)).
\]

(ii) \( \alpha < \beta \), then as \( u \to \infty \),

\[
P\left( \sup_{t \in [0,Tu^{-2/\alpha}]} X(t) > u \right) = \mathcal{H}_Y (a^{1/\alpha} T) \Psi(u)(1 + o(1)).
\]

The proof of Proposition 2.1 is given in Section 4.1.

3. Pickands–Piterbarg Constants for Self-Similar Gaussian Processes

The aim of this section is to find analogs of Pickands and Piterbarg constants for self-similar Gaussian processes \( Y \in S(\alpha, \kappa, c_Y) \).

3.1. Piterbarg Constants. For \( R > 0 \) and \( Y \in S(\alpha, \kappa, c_Y) \) let us introduce an analog of the Piterbarg constant \( H_{RB_1} \) as follows:

\[
H_{RB} := \lim_{T \to \infty} H_{RB}(T) = \lim_{T \to \infty} E \exp\left( \sup_{t \in [0,T]} \left( \sqrt{2} Y(t) - (1 + R)t^\alpha \right) \right).
\]

In the next theorem we prove that \( H_{RB} \) is well-defined and we compare it with the classical Piterbarg constants.

Theorem 3.1. Let \( Y \in S(\alpha, \kappa, c_Y) \). Then, for any \( R > 0 \),

\[
\mathcal{H}_Y^R \in (0, \infty).
\]

Furthermore

\[
\mathcal{H}_B^{R/c_1} \leq \mathcal{H}_Y^R \leq \mathcal{H}_B^{R/c_2},
\]

where

\[
c_1 = \inf_{x \in (0,1)} \frac{V_Y(1, x^{\kappa/\alpha})}{|1-x|^\kappa} \quad \text{and} \quad c_2 = \sup_{x \in (0,1)} \frac{V_Y(1, x^{\kappa/\alpha})}{|1-x|^\kappa}.
\]

The proof of Theorem 3.1 is given in Section 4.2.

Proposition 3.2. Let \( Y \in S(\alpha, \kappa, c_Y) \). Then

\[
\mathcal{H}_Y^R \geq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right).
\]

The proof of Proposition 3.2 is postponed to Section 4.3.

The following corollary follows from Theorem 3.1 combined with the fact that \( \mathcal{H}_{B_1}^R = 1 + 1/R \) (see, e.g., [13]) and \( \mathcal{H}_{B_2}^R = \frac{1}{2} (1 + \sqrt{1 + 1/R}) \) (see, e.g., [20]).
COROLLARY 3.3. Let $Y \in S(\alpha, \kappa, c_Y)$.

(i) If $\kappa = 1$, then

\[ 1 + \frac{1}{R} \left( \inf_{x \in [0,1]} \frac{V_Y(1, x^{1/\alpha})}{1 - x} \right) \leq \mathcal{H}_Y^R \leq 1 + \frac{1}{R} \left( \sup_{x \in [0,1]} \frac{V_Y(1, x^{1/\alpha})}{1 - x} \right). \]

(ii) If $\kappa = 2$, then

\[ \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right) \leq \mathcal{H}_Y^R \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4k(\alpha + k - 1)}{R(\alpha + 2k)(\alpha + 2k - 2)}} \right). \]

In the following example we specify Corollary 3.3 for some particular self-similar processes introduced in Section 2.

EXAMPLE 3.1. The following bounds hold:

\begin{itemize}
  \item $k$-fold integrated fractional Brownian motion $Y^{(3),k}$:
  \[ \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right) \leq \mathcal{H}_Y^{R, (3), k} \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4k(\alpha + k - 1)}{R(\alpha + 2k)(\alpha + 2k - 2)}} \right). \]
  \item Time-average of fractional Brownian motion $Y^{(4)}$ with parameter $\alpha \in (0, 2]$:
  \[ \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right) \leq \mathcal{H}_Y^{R, (4)} \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4}{R(\alpha + 2)}} \right). \]
\end{itemize}

The above bounds improve the results obtained in [15] for the constants

\[ \mathcal{F}_\alpha = \lim_{T \to \infty} \mathbb{E} \exp \left( \sup_{t \in (0,T]} \frac{1}{t} \int_0^t \sqrt{2} B_\alpha(s) - s^\alpha \, ds \right) \]

leading to

\[ \frac{1}{2} (1 + \sqrt{2 + \alpha}) \leq \mathcal{F}_\alpha \leq \frac{1}{2} (1 + \sqrt{1 + 4(\alpha + 1)/\alpha^2}), \]

while in [15] it was proved that $\mathcal{F}_\alpha \leq 2 + \alpha$ for $\alpha \in [1, 2]$.

\item Dual fractional Brownian motion $Y^{(5)}$ with parameter $\alpha \in (0, 2]$:
  \[ \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{R}} \right) \leq \mathcal{H}_Y^{R, (5)} \leq \frac{1}{2} \left( 1 + \sqrt{1 + \frac{2}{R\alpha}} \right). \]
3.2. Pickands constants. In this section we focus on an analog of Pickands constants for $Y \in S(\alpha, \kappa, c_Y)$. Let

$$
\mathcal{H}_Y := \lim_{T \to \infty} \frac{\mathcal{H}_Y(T)}{T^{\alpha/\kappa}} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0,T]} (\sqrt{2} Y(t) - t^\alpha))}{T^{\alpha/\kappa}}.
$$

We observe that for $Y(t) = B_\kappa(t)$ the above definition agrees with the notion of the classical Pickands constant $\mathcal{H}_{B_\kappa}$, since $\alpha = \kappa$ in this case.

In the following theorem we show that $\mathcal{H}_Y$ is well-defined and find a surprising relation between $\mathcal{H}_Y$ and $\mathcal{H}_{B_\kappa}$.

**Theorem 3.4.** Let $Y \in S(\alpha, \kappa, c_Y)$. Then $\mathcal{H}_Y \in (0, \infty)$ and

$$
\mathcal{H}_Y = \frac{\kappa}{\alpha} (c_Y)^{1/\kappa} \mathcal{H}_{B_\kappa}.
$$

A complete proof of Theorem 3.4 is presented in Section 4.4.

The following corollary is an immediate consequence of Theorem 3.4 and the fact that $\mathcal{H}_{B_1} = 1$ and $\mathcal{H}_{B_2} = 1/\sqrt{\pi}$.

**Corollary 3.5.** Let $Y \in S(\alpha, \kappa, c_Y)$.

(i) If $\kappa = 1$, then

$$
\mathcal{H}_Y = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0,T]} (\sqrt{2} Y(t) - t^\alpha))}{T^\alpha} = \frac{c_Y}{\alpha}.
$$

(ii) If $\kappa = 2$, then

$$
\mathcal{H}_Y = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0,T]} (\sqrt{2} Y(t) - t^\alpha))}{T^{\alpha/2}} = \frac{2}{\alpha} \sqrt{\frac{c_Y}{\pi}}.
$$

In the following example we specify the findings of this section for self-similar Gaussian processes introduced in Section 2.

**Example 3.2.** The following equalities hold:

- Bifractional Brownian motion with parameters $\alpha \in (0, 2)$ and $K \in (0, 1]$:

  $$
  \mathcal{H}_{Y^{(1)}} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0,T]} (\sqrt{2} Y^{(1)}(t) - t^{\alpha K}))}{T} = 2^{1-K/\alpha} \mathcal{H}_{B_{\alpha K}}.
  $$

- Sub-fractional Brownian motion with parameter $\alpha \in (0, 2)$:

  $$
  \mathcal{H}_{Y^{(2)}} = \lim_{T \to \infty} \frac{\mathbb{E} \exp(\sup_{t \in [0,T]} (\sqrt{2} Y^{(2)}(t) - t^\alpha))}{T} = (2 - 2^{\alpha-1})^{-1/\alpha} \mathcal{H}_{B_{\alpha}}.
  $$
k-fold integrated fractional Brownian motion with parameters $k \in \mathbb{N}$ and $\alpha \in (0, 2]$:
\[
\mathcal{H}_{Y^{(3)},k} = \lim_{T \to \infty} \mathbb{E}\exp\left(\sup_{t \in [0,T]} \left( \sqrt{2} Y^{(3),k}(t) - t^{k+\alpha/2} \right) \right) / T^{k+\alpha/2} = \sqrt{4k(\alpha + k - 1)} / \pi(\alpha + 2k)(\alpha + 2k - 2).
\]

Time-average of fractional Brownian motion with parameter $\alpha \in (0, 2]$:
\[
\mathcal{H}_{Y^{(4)}} = \lim_{T \to \infty} \mathbb{E}\exp\left(\sup_{t \in [0,T]} \left( \sqrt{2} Y^{(4)}(t) - t^\alpha \right) \right) / T^{\alpha/2} = \frac{2}{\sqrt{\pi} \alpha}.
\]

Dual fractional Brownian motion with parameter $\alpha \in (0, 2]$:
\[
\mathcal{H}_{Y^{(5)}} = \lim_{T \to \infty} \mathbb{E}\exp\left(\sup_{t \in [0,T]} \left( \sqrt{2} Y^{(5)}(t) - t^\alpha \right) \right) / T^{\alpha/2} = \sqrt{2 / \pi \alpha}.
\]

4. PROOFS

In the rest of the paper we use the notation $v_Y(t) := V_Y(1, t)$. We begin with the following lemma, skipping its straightforward proof.

**Lemma 4.1.** Let $Y \in S(\alpha, \kappa, c_Y)$ and $\hat{Y}_{\alpha_1} = Y(t^{\alpha_1})$ for some $\alpha_1 > 0$. Then, for any $R \geq 0$ and $T > 0$,
\begin{enumerate}[label=(i)]
  \item $\mathcal{H}_Y^R(T) = \mathcal{H}_Y^R(c^{2/\alpha}T)$ for any $c > 0$;
  \item $\mathcal{H}_{\hat{Y}_{\alpha_1}}^R(T) = \mathcal{H}_Y^R(T^{\alpha_1})$.
\end{enumerate}

**4.1. Proof of Proposition 2.1.** In the next lemma we present a useful bound on $V_Y(\cdot, \cdot)$ for $Y \in S(\alpha, \kappa, c_Y)$.

**Lemma 4.2.** Let $Y \in S(\alpha, \kappa, c_Y)$. Then there exists a positive constant $C$ such that for $\gamma = \min(\alpha, \kappa)$, $T > 0$ and all $t, s \in [0, T]$,
\[
V_Y(t, s) \leq CT^{\alpha-\gamma}|t-s|^{\gamma}.
\]

**Proof.** For $t = s$ the conclusion is obvious. Suppose that $0 \leq s < t \leq T$ and let $\epsilon \in (0, 1)$ be such that for $\delta \in (0, 1)$,
\[
(1 - \epsilon)c_Y|1 - x|^\kappa \leq V_Y(1, x) \leq (1 + \epsilon)c_Y|1 - x|^\kappa.
\]
for all $x \in [\delta, 1]$ (due to S2). For $s/t \geq \delta$ we have

$$V_Y(t, s) = t^\alpha V_Y(1, s/t) \leq t^\alpha (1 + \epsilon) c_Y |1 - s/t|^{\kappa}$$

$$\leq t^\alpha (1 + \epsilon) c_Y |1 - s/t|^{\min(\alpha, \kappa)}$$

$$= t^{\alpha - \gamma} (1 + \epsilon) c_Y |t - s|^{\gamma} \leq T^{\alpha - \gamma} (1 + \epsilon) c_Y |t - s|^{\gamma}.$$

For $s/t \leq \delta$ we have $|1 - \delta|^{\gamma} \leq |1 - s/t|^{\gamma}$. Hence, $t^{\gamma} |1 - \delta|^{\gamma} \leq |t - s|^{\gamma}$. Then

$$V_Y(t, s) = t^\alpha V_Y(1, s/t)$$

$$\leq t^{\gamma t^{\alpha - \gamma} |1 - \delta|^{\gamma}} \max_{x \in [0, \delta]} V_Y(1, x) \leq T^{\alpha - \gamma} \max_{x \in [0, \delta]} V_Y(1, x) |1 - \delta|^{\gamma}.$$ 

Hence the proof is completed with $C = \max((1 + \epsilon) c_Y, \max_{x \in [0, \delta]} V_Y(1, x) |1 - \delta|^{\gamma})$. ■

**Proof of Proposition 4.2** Since for any Gaussian process $Y(\cdot, \cdot)$, the variogram function $V_Y(\cdot, \cdot)$ is negative definite, by the Schoenberg theorem the function $\exp(-V_Y(\cdot, \cdot))$ is positive definite. Thus there exists a Gaussian process $\{X(t) : t \geq 0\}$ with $R_X(t, s) = \exp(-V_Y(t, s))$.

The rest of the proof follows straightforwardly from [12] Theorem 2.1 and Lemma 4.2 applied to $X_u(t) = X(t u^{-2/\alpha})$. ■

### 4.2. Proof of Theorem 3.1

**Lemma 4.3.** Let $Y \in S(\alpha, \kappa, c_Y)$ and define $\hat{Y}(t) = Y(t^{\kappa/\alpha})$. Then $\hat{Y} \in S(\kappa, \kappa, c_Y(\kappa/\alpha)^\kappa)$ and there exist finite and positive constants

$$c_1 = \inf_{x \in [0, 1]} \frac{V_Y(1, x^{\kappa/\alpha})}{|1 - x|^\kappa} = \inf_{x \in [0, 1]} \frac{V_Y(1, x)}{|1 - x|^\kappa},$$

$$c_2 = \sup_{x \in [0, 1]} \frac{V_Y(1, x^{\kappa/\alpha})}{|1 - x|^\kappa} = \sup_{x \in [0, 1]} \frac{V_Y(1, x)}{|1 - x|^\kappa}.$$ 

Moreover for all $t, s \geq 0$,

$$c_1 |t - s|^\kappa \leq V_Y(t^{\kappa/\alpha}, s^{\kappa/\alpha}) = V_{\hat{Y}}(t, s) \leq c_2 |t - s|^\kappa.$$

**Proof.** Observe that $\hat{Y} \in S(\kappa, \kappa, c_Y(\kappa/\alpha)^\kappa)$ with $\hat{V}_Y(t, s) = V_Y(t^{\kappa/\alpha}, s^{\kappa/\alpha})$.

Consider the function $f(x) = \frac{V_Y(1, x)}{|1 - x|^\kappa}$ for $x \in [0, 1)$. Due to S2, $\lim_{x \to 1^-} f(x) = c_\hat{Y} > 0$, $f(0) = 1$ and $f(x) = 0$ only for $x = 1$. Hence, $c_1, c_2 > 0$ exist.

Moreover, for all $t \geq s > 0$,

$$c_1 |t - s|^\kappa = c_1 t^\kappa |1 - s/t|^\kappa \leq t^\kappa V_Y(1, s/t) \leq c_2 t^\kappa |1 - s/t|^\kappa = c_2 |t - s|^\kappa.$$ 

This completes the proof. ■
Proof of Theorem 3.1 Let $R,T > 0$. Define $\hat{Y}(t) = Y(t^{\kappa/\alpha})$. Then $\hat{Y} \in S(\kappa, \kappa, c\hat{Y}(\kappa/\alpha))$ with $V_{\hat{Y}}(1, x) = V_Y(1, x^{\kappa/\alpha})$ and $c_1, c_2 > 0$ exist. Let $\{X(t) : t \geq 0\}, \{X_1(t) : t \geq 0\}, \{X_2(t) : t \geq 0\}$ be centered Gaussian processes with $R_{X_1}(t, s) = \exp(-V_{\hat{Y}}(t, s))$, $R_{X_2}(t, s) = \exp(-c_1V_{B_\kappa}(t, s))$ and $\sigma_X(t) = \frac{1}{\sqrt{1+RT}}$. Then, for all $t, s \geq 0$,

$$R_{X_1}(t, s) = \exp(-c_1V_{B_\kappa}(t, s)) \geq R_{X_2}(t, s) = \exp(-V_{\hat{Y}}(t, s)).$$

and hence, due to Slepian’s inequality (see, e.g., [1, Corollary 2.4]) we find that for all $u > 0$,

$$P\left(\sup_{t \in [0, Tu^{-2/\kappa}]} X_1(t) > u\right) \leq P\left(\sup_{t \in [0, Tu^{-2/\kappa}]} X(t) > u\right) \leq P\left(\sup_{t \in [0, Tu^{-2/\kappa}]} X_2(t) > u\right).$$

Application of Proposition 2.1(i) to the inequalities above gives

$$H_{B_\kappa}^{R/c_1}(c_1^{1/\kappa}T) \leq H_{Y}^{R}(T) = H_{Y}^{R}(T^{\kappa/\alpha}) \leq H_{B_\kappa}^{R/c_2}(c_2^{1/\kappa}T),$$

where the equality follows from Lemma 4.1(ii). Note that all functions in (4.1) are increasing in $T$, and hence letting $T \to \infty$ in (4.1) completes the proof. 

4.3. Proof of Proposition 3.2 Since $\sigma_Y^2(t) = t^\alpha$, the Schwarz inequality implies that $R_Y(t, s) \leq (ts)^{\alpha/2}$ for all $t, s \geq 0$. Therefore for $t, s \geq 0$,

$$V_Y(t, s) \geq t^\alpha + s^\alpha - 2(t^\alpha s^{\alpha/2}) = t^\alpha - s^{\alpha/2} = \hat{B}_2(t, s),$$

where $\hat{B}_2(t) = B_2(t^{\alpha/2})$. Thus, by Slepian’s inequality,

$$H_Y^{R}(T) = \int R e^x P\left(\sup_{t \in [0, T]} (\sqrt{2} Y(t) - (1 + R)t^\alpha > x)\right) dx \geq \int R e^x P\left(\sup_{t \in [0, T]} (\sqrt{2} \hat{B}_2(t) - (1 + R)t^\alpha > x)\right) dx = H_{B_2}^{R}(T) = H_{B_2}^{R}(T^{\alpha/2}),$$

where the last equality follows from Lemma 4.1(ii). Letting $T \to \infty$ yields the conclusion. 

4.4. Proof of Theorem 3.4 To prove Theorem 3.4 we need some technical lemmas.

Lemma 4.4. Let $\hat{Y} \in S(\kappa, \kappa, c\hat{Y})$. For any $\epsilon \to 0^+$ there exists $\delta_\epsilon \to 0^+$ such that for any $T > 0$ and $A \geq T/\delta_\epsilon$,

$$(1 - \epsilon)c\hat{Y}|t - s|^{\kappa} \leq V_{\hat{Y}}(A + t, A + s) \leq (1 + \epsilon)c\hat{Y}|t - s|^{\kappa}$$

for all $t, s \in [0, T]$. 

Proof. Let $\epsilon \in (0, 1)$ be sufficiently small such that

\[(4.2) \quad (1 - \epsilon)c_{\hat{V}}|h|^\kappa \leq V_{\hat{V}}(1, 1 - h) \leq (1 + \epsilon)c_{\hat{V}}|h|^\kappa\]

for all $h \in [0, \delta_\epsilon]$ and $\delta_\epsilon \in (0, 1)$ (due to $S2$). Then, for any $T > 0$, $A > T/\delta_\epsilon$ and $0 \leq s \leq t \leq T$ we have $\frac{t-s}{A+t} \leq \frac{T}{A} \leq \delta_\epsilon$. Combining the fact that

\[V_{\hat{V}}(A + s, A + t) = (A + t)^\kappa V_{\hat{V}} \left(1, \frac{A + s}{A + t}\right) = (A + t)^\kappa V_{\hat{V}} \left(1, 1 - \frac{t-s}{A+t}\right)\]

with (4.2) for $h = \frac{t-s}{A+t} \leq \delta_\epsilon$, we obtain the assertion. $\blacksquare$

Lemma 4.5. Let $\hat{Y} \in S(\kappa, \kappa, c_{\hat{Y}})$ and let $c_1$, $c_2$ be as in Lemma 1.3. Consider a centered Gaussian process $\{\overline{X}(t) : t \geq 0\}$ with $R_{\overline{X}}(t, s) = \exp(-aV_{\hat{Y}}(t, s))$ with $a > 0$.

(i) Let $\{\overline{X}_i(t) : t \geq 0\}$, $i = 1, 2$, be centered stationary Gaussian processes with $R_{\overline{X}_i}(t, s) = \exp(-ac_iV_{B_\kappa}(t, s))$. Then for all $u > 0$ and any $T, A = A(u) > 0$,

\[
P \left( \sup_{t \in [0,T]} \overline{X}_1(t) > u \right) \leq P \left( \sup_{t \in [A,A+T]} \overline{X}(t) > u \right) \leq P \left( \sup_{t \in [0,T]} \overline{X}_2(t) > u \right).
\]

(ii) For any $\epsilon > 0$, let $\{\overline{X}_i(t) : t \geq 0\}$, $i = 1, 2$, be centered stationary Gaussian processes with $R_{\overline{X}_i}(t, s) = \exp(-a(1 + (-1)^i\epsilon)c_{\hat{Y}}V_{B_\kappa}(t, s))$. Then for any $\epsilon \to 0^+$ there exists $\delta_\epsilon \to 0^+$ such that for any $T > 0$ and $A = A(u) \geq T/\delta$,

\[
P \left( \sup_{t \in [0,T]} \overline{X}_1(t) > u \right) \leq P \left( \sup_{t \in [A,A+T]} \overline{X}(t) > u \right) \leq P \left( \sup_{t \in [0,T]} \overline{X}_2(t) > u \right).
\]

Proof. (i) The argument is the same as for Theorem 3.1. From Lemma 4.3 we know that for all $t, s \geq 0$,

\[V_{\overline{X}_1}(t, s) \leq V_{\overline{X}}(t, s) \leq V_{\overline{X}_2}(t, s)\]

and hence, due to Slepian’s inequality, for all $u > 0$,

\[
P \left( \sup_{t \in [A,A+T]} \overline{X}_2(t) > u \right) \leq P \left( \sup_{t \in [A,A+T]} \overline{X}(t) > u \right) \leq P \left( \sup_{t \in [A,A+T]} \overline{X}_2(t) > u \right).
\]

Due to stationarity of $\overline{X}_i(\cdot)$ we obtain the assertion.
(ii) From Lemma 4.4 for any \( \epsilon \to 0^+ \) there exists \( \delta_\epsilon \to 0^+ \) such that for any \( T > 0 \) and \( A \geq T/\delta_\epsilon \),

\[
(1 - \epsilon)c_\gamma |t - s|^{\kappa} \leq V_{\gamma}(t, s) \leq (1 + \epsilon)c_\gamma |t - s|^{\kappa}
\]

for all \( t, s \in [A, A + T] \). The same argument as in the proof of (i) completes the proof. \( \blacksquare \)

**Lemma 4.5.** Suppose that \( \lim_{u \to \infty} f(u)/u = c \) for some \( c > 0 \). Under the notation of Lemma 4.5 there exist absolute constants \( F, G > 0 \) such that

\[
P\left( \sup_{t \in [A, A+T]u^{-2/\kappa}} X(t) > f(u), \sup_{t \in [t_0, t_0+T]u^{-2/\kappa}} X(t) > f(u) \right) \leq FT^2 \exp\left(-G(t_0 - (A + T))^{\kappa}\right)\Psi(f(u))
\]

for all \( t_0 > A + T > 0, T \geq 1 \) and any \( u \geq u_0 = (2ac_2)^2(t_0 + T)^{\kappa/2} \).

**Proof.** The argument is similar to the one given in, e.g., [6 Lemma 6.2] or [12 Theorem 2.1]. Thus we only present the main steps.

Let \( u_0 = (2ac_2)^2(t_0 + T)^{\kappa/2} \) and \( \{Z_u(t_1, t_2) : (t_1, t_2) \in [A, A + T] \times [t_0, t_0 + T]\} \), where \( Z_u(t_1, t_2) = X(t_1 u^{-2/\kappa}) + X(t_2 u^{-2/\kappa}) \). Note that

\[
P\left( \sup_{t \in [A, A+T]u^{-2/\kappa}} X(t) > f(u), \sup_{t \in [t_0, t_0+T]u^{-2/\kappa}} X(t) > f(u) \right) \leq P\left( \sup_{(t_1, t_2) \in [A, A+T] \times [t_0, t_0+T]} Z_u(t_1, t_2) > 2f(u) \right).
\]

Since \( (t_0 + T)u^{-2/\kappa} \leq (2ac_2)^{-1/\kappa} \), by Lemma 4.3 for all \( t_1, t_2 \leq t_0 + T \),

\[
ac_1 u^{-2} |t_2 - t_1|^{\kappa} \leq aV_{\gamma}(t_1 u^{-2/\kappa}, t_2 u^{-2/\kappa}) \leq ac_2 u^{-2} |t_2 - t_1|^{\kappa} \leq ac_2 |(t_0 + T)u^{-2/\kappa}|^{\kappa} \leq 1/2.
\]

Hence, as \( x \leq 2(1 - e^{-x}) \leq (1 - e^{-4x}) \) for \( x \in [0, 1/2] \), we obtain

\[
V_{\gamma}(t_1 u^{-2/\kappa}, t_2 u^{-2/\kappa}) = 2\left(1 - \exp(-aV_{\gamma}(t_1 u^{-2/\kappa}, t_2 u^{-2/\kappa}))\right) \geq aV_{\gamma}(t_1 u^{-2/\kappa}, t_2 u^{-2/\kappa}) \geq ac_1 u^{-2} |t_2 - t_1|^{\kappa},
\]

\[
V_{\gamma}(t_1 u^{-2/\kappa}, t_2 u^{-2/\kappa}) \leq 2\left(1 - \exp(-ac_2 u^{-2} |t_2 - t_1|^{\kappa})\right) \leq (1 - \exp(-4ac_2 u^{-2} |t_2 - t_1|^{\kappa}))
\]

for all \( t_1, t_2 \leq t_0 + T \). Since

\[
\sigma_{Z_u}^2(t_1, t_2) = 2 + 2\exp(-aV_{\gamma}(t_1 u^{-2/\kappa}, t_2 u^{-2/\kappa})) = 4 - 2(1 - \exp(-aV_{\gamma}(t_1 u^{-2/\kappa}, t_2 u^{-2/\kappa}))),
\]
from (4.5), for any \((t_1, t_2) \in [A, A + T] \times [t_0, t_0 + T],\)

\[
2 \leq \sigma_{Z_u}^2(t_1, t_2) \leq 4 - ac_1u^{-2}(t_0 - (A + T))^\kappa.
\]

Now observe that

\[
P \left( \sup_{(t,s)\in[A,A+T]\times[t_0,t_0+T]} Z_u(t_1, t_2) > 2f(u) \right)
\leq P \left( \sup_{(t_1, t_2)\in[A,A+T]\times[t_0,t_0+T]} Z_u(t_1, t_2) > \frac{2f(u)}{\sqrt{4 - ac_1u^{-2}(t_0 - (A + T))^\kappa}} \right).
\]

Note that for any \((t_1, t_2), (s_1, s_2) \in [A, A + T] \times [t_0, t_0 + T],\) we have

\[
\text{Var}(\overline{Z_u}(t_1, t_2) - \overline{Z_u}(s_1, s_2)) \leq \frac{\text{Var}(Z_u(t_1, t_2) - Z_u(s_1, s_2))}{\sigma_{Z_u}(t_1, t_2)\sigma_{Z_u}(s_1, s_2)} \\
\leq \frac{1}{2} \mathbb{E} \left( (\overline{X}(t_1u^{-2/\kappa}) - \overline{X}(s_1u^{-2/\kappa})) + (\overline{X}(t_2u^{-2/\kappa}) - \overline{X}(s_2u^{-2/\kappa})) \right)^2 \\
\leq \overline{X}(t_1u^{-2/\kappa}, s_1u^{-2/\kappa}) + \overline{X}(t_2u^{-2/\kappa}, s_2u^{-2/\kappa}) \\
\leq (1 - \exp(-4ac_2u^{-2}|t_1 - s_1|^\kappa)) + (1 - \exp(-4ac_2u^{-2}|t_2 - s_2|^\kappa)),
\]

where the next-to-last inequality follows from \((x + y)^2 \leq 2(x^2 + y^2),\) and the last one follows from (4.6).

Denote \(u^* = \frac{2f(u)}{\sqrt{4 - ac_1u^{-2}(t_0 - (A + T))^\kappa}}\) and let \(c, \tau > 0\) be constants such that

\(c \leq f(u)/u \leq \tau\) for all \(u \geq u_0.\) Note by (4.4) that \(f(u) \leq u^* \leq \sqrt{8/7} f(u)\) for \(u \geq u_0.\) Hence, \(cu \leq u^* \leq \sqrt{8/7} \tau u\) for \(u \geq u_0.\) and therefore \(u^{-2} \leq \frac{8}{7}\tau^2 (u^*)^{-2}\) for \(u \geq u_0.\)

Consider two independent, identically distributed centered stationary Gaussian processes \(\{Z_{1,u^*}(t_1) : t_1 \geq 0\}, \{Z_{2,u^*}(t_2) : t_2 \geq 0\}\) with \(R_{Z_{1,u^*}}(t_1, s_1) = \exp(-\frac{32}{7} \alpha \sigma^2 \sigma_2(u^*)^{-2}|t_1 - s_1|^\kappa)\) and let \(Z_{u^*}(t_1, t_2) = \frac{1}{\sqrt{2}} (Z_{1,u^*}(t_1) + Z_{2,u^*}(t_2)).\) Hence, by (4.8), for any \((t_1, t_2), (s_1, s_2) \in [A, A + T] \times [t_0, t_0 + T],\)

\[
\text{Var}(\overline{Z_u}(t_1, t_2) - \overline{Z_u}(s_1, s_2)) \\
\leq (1 - \exp(-4ac_2u^{-2}|t_1 - s_1|^\kappa)) + (1 - \exp(-4ac_2u^{-2}|t_2 - s_2|^\kappa)) \\
\leq (1 - \exp(-\frac{32}{7} \alpha \sigma^2 \sigma_2(u^*)^{-2}|t_1 - s_1|^\kappa)) \\
+ (1 - \exp(-\frac{32}{7} \alpha \sigma^2 \sigma_2(u^*)^{-2}|t_2 - s_2|^\kappa)) \\
= \text{Var}(\overline{Z_{u^*}}(t_1, t_2) - \overline{Z_{u^*}}(s_1, s_2)).
\]
and due to Slepian’s inequality, we obtain

\[
\begin{align*}
(4.9) \quad P\left( \sup_{(t_1,t_2) \in [A,A+T] \times [t_0,t_0+T]} Z_u(t_1,t_2) > u^* \right) & \leq P\left( \sup_{(t_1,t_2) \in [A,A+T] \times [t_0,t_0+T]} Z_u^*(t_1,t_2) > u^* \right) \\
& = P\left( \sup_{(t_1,t_2) \in [0,T]^2} Z_u^*(t_1,t_2) > u^* \right)
\end{align*}
\]

as \( u \to \infty \), where the equality follows from stationarity of \( Z_u^*(\cdot,\cdot) \). Now

\[
\lim_{u^* \to \infty} \frac{P\left( \sup_{(t_1,t_2) \in [0,T]^2} Z_u^*(t_1,t_2) > u^* \right)}{\Psi(u^*)} = \left( \mathcal{H}_{B_{\kappa}}((16ac^2c_2/T)^{1/\kappa}T) \right)^2 \\
\leq \left( \mathcal{H}_{B_{\kappa}}(1) \right)^2 \max(1, (16ac^2c_2/7)^{2/\kappa}T^2),
\]

where the equality follows from, e.g., \([6\text{ Theorem 2.1}]\) (see also \([12\text{ Theorem 3.1}]\)) and the inequality follows from the fact that \( \mathcal{H}_{B_{\kappa}}(AT) \leq T \max(1, A) \mathcal{H}_{B_{\kappa}}(1) \) for any \( T > 1 \) and \( A > 0 \) \([26\text{ Corollary D.1}]\). Hence, there exists a constant \( F' \) (which does not depend on \( t_0, A, T \)) such that

\[
P\left( \sup_{(t_1,t_2) \in [0,T]^2} Z_u^*(t_1,t_2) > u^* \right) \leq F'T^2 \Psi(u^*)
\]

for all \( u^* \geq u_0^* = c u_0 \) (i.e. \( u \geq u_0 \)). Thus

\[
(4.10) \quad P\left( \sup_{t \in [A,A+T]u^{-2/\kappa}} X(t) > f(u), \sup_{t \in [t_0,t_0+T]u^{-2/\kappa}} X(t) > f(u) \right) \leq F'T^2 \Psi(u^*)
\]

for \( u \geq u_0 \).

Since (in view of \( \frac{1}{1-x} \geq 1 + x \) for \( x \geq 0 \))

\[
(u^*)^2 = \frac{4f^2(u)}{4 - ac_1 u^2(t_0 - (A + T))^\kappa} \geq f^2(u) + \frac{ac_1}{4} \left( \frac{f(u)}{u} \right)^2 (t_0 - (A + T))^\kappa \\
\geq f^2(u) + \frac{ac_1 c_2^2}{4} (t_0 - (A + T))^\kappa,
\]

we have

\[
(4.11) \quad \Psi(u^*) \leq \exp\left( -\frac{1}{2} \left( f^2(u) + \frac{ac_1 c_2^2}{4} (t_0 - (A + T))^\kappa \right) \right) \\
\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{f(u)}} \exp\left( -\frac{ac_1 c_2^2}{8} (t_0 - (A + T))^\kappa \right).
\]
Combination of (4.10) with (4.11) gives

\[
\mathbb{P}\left( \sup_{t \in [A,A+T]u^{-2/\kappa}} X(t) > f(u), \sup_{t \in [t_0,A+T]u^{-2/\kappa}} X(t) > f(u) \right) \\
\leq F_1 F^2 T^2 \exp\left(-\frac{ac_1 c^2}{8} (t_0 - (A+T))^\kappa\right) f(u)
\]

for any \( u \geq u_0 \) and some positive constant \( F_1 \) such that \( \frac{\exp(-\frac{1}{2} f^2(u))}{\sqrt{2\pi} f(u)} \leq F_1 \Psi(f(u)) \) for \( u > u_0 \). This completes the proof with \( F = F_1 F' \) and \( G = ac_1 c^2 / 8 \). \( \blacksquare \)

**Lemma 4.7.** With the notation of Lemma 4.5 there exist absolute constants \( F, G > 0 \) such that

\[
\mathbb{P}\left( \sup_{t \in [A,A+T]u^{-2/\kappa}} X(t) > u, \sup_{t \in [A+T,A+2T]u^{-2/\kappa}} X(t) > u \right) \\
\leq F(T^2 \exp(-G \sqrt{T^\kappa}) + \sqrt{T}) \Psi(u)
\]

for all \( A > 0, T > 1 \) and any \( u \geq u_0 = (2ac_2)^2 (A+2T)^{\kappa/2} \), i.e. \( (A+2T)u^{-2/\kappa} \leq (2ac_2)^{-1/\kappa} \).

**Proof.** Let \( u_0 = (2ac_2)^2 (A+2T)^{\kappa/2} \) and \( X_u(t) = X(t u^{-2/\kappa}) \). We have

\[
\mathbb{P}\left( \sup_{t \in [A,A+T]} X_u(t) > u, \sup_{t \in [A+T,A+2T]} X_u(t) > u \right) \\
= \mathbb{P}\left( \sup_{t \in [A,A+T]} X_u(t) > u, \sup_{t \in [A+T,A+2T]} X_u(t) > u \right) \\
\leq \mathbb{P}\left( \sup_{t \in [A+T,A+T+\sqrt{T}]} X_u(t) > u, \sup_{t \in [A+T+\sqrt{T},A+2T]} X_u(t) > u \right) \\
+ \mathbb{P}\left( \sup_{t \in [A+T,A+T+\sqrt{T}]} X_u(t) > u \right) \\
\leq F_1 T^2 \exp(-G \sqrt{T^\kappa}) \Psi(u) + \mathbb{P}\left( \sup_{t \in [A+T,A+T+\sqrt{T}]} X_u(t) > u \right),
\]

where the last inequality follows from Lemma 4.6 with \( t_0 = A+T+\sqrt{T} \). Applying Lemma 4.5(i) and Proposition 2.1 (choose \( b = 0 \)) to \( \mathbb{P}\left( \sup_{t \in [A+T,A+T+\sqrt{T}]} X_u(t) > u \right) \), we finally obtain, for sufficiently large \( u \geq u_0 \),
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\[ P \left( \sup_{t \in [A, A + T]} X_u(t) > u, \sup_{t \in [A + T, A + 2T]} X_u(t) > u \right) \]

\[ \leq F_1 T^2 \exp(-G\sqrt{T/\kappa})\Psi(u) + H_{B,\kappa}(a^{1/\kappa}T)\Psi(u)(1 + o(1)) \]

\[ \leq F_1 T^2 \exp(-G\sqrt{T/\kappa})\Psi(u) + \max(1, a^{1/\kappa})H_{B,\kappa}(1)\sqrt{T} \Psi(u)(1 + o(1)) \]

for some constant \( F > 0 \), where the next-to-last inequality follows from subadditivity of \( H_{B,\kappa} (\cdot) \) \cite[Corollary D.1]{26}.

**Proof of Theorem 3.4.** First, we prove the conclusion of the theorem for \( \hat{Y} \in S(\kappa, \kappa, c_{Y}(\kappa/\alpha)^{\kappa}) \), where \( \hat{Y}(t) = Y(t^{\kappa/\alpha}) \). Let \( c_1, c_2 \) be the constants of Lemma 4.3. Consider a centered Gaussian process \( \{ \overline{X}(t) : t \geq 0 \} \) with \( R_{\overline{X}}(t, s) = \exp(-V_{\hat{Y}}(t, s)) \) and let \( \overline{X}_u(t) = \overline{X}(tu^{-2/\kappa}) \). Let \( n \in \mathbb{N} \) and choose \( \epsilon_n \in (0, 1) \), \( \delta_{\epsilon_n} = 1/n \) so that the conclusion of Lemma 4.5(ii) holds. We find a lower bound and an upper bound separately.

**Upper bound.** Let \( T \in \mathbb{N} \) be such that \( T > n \). For any \( u > 0 \),

\[ P \left( \sup_{t \in [0, T^2]} \overline{X}_u(t) > u \right) \]

\[ \leq P \left( \sup_{t \in [0, nT]} \overline{X}_u(t) > u \right) + \sum_{k=n}^{T-1} P \left( \sup_{t \in [kT, (k+1)T]} \overline{X}_u(t) > u \right). \]

Applying Lemma 4.5 to the right side, by Proposition 2.1 we obtain

\[ H_{\hat{Y}}(T^2) \leq H_{B,\kappa}(c_2^{1/\kappa}nT) + (T - n)H_{B,\kappa}((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}T, \]

and hence

\[ \frac{H_{\hat{Y}}(T^2)}{T^2} \leq \frac{H_{B,\kappa}(c_2^{1/\kappa}nT)}{T^2} + \frac{((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}T^2}{T^2} \frac{H_{B,\kappa}(((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}T)}{((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}T}. \]

Since \( \lim_{S \to \infty} S^{-1}H_{B,\kappa}(S) = H_{B,\kappa} \), after letting \( T \to \infty \) in (4.12) we get

\[ \limsup_{T \to \infty} \frac{H_{\hat{Y}}(T)}{T} \leq ((1 + \epsilon_n)c_{\hat{Y}})^{1/\kappa}H_{B,\kappa}. \]

Since this bound holds for any \( \epsilon_n \to 0^+ \), we have

\[ \limsup_{T \to \infty} \frac{H_{\hat{Y}}(T)}{T} \leq (c_{\hat{Y}})^{1/\kappa}H_{B,\kappa}. \]
**Lower bound.** For $T \in \mathbb{N}$ such that $T > n$, with $\Delta_k = [kT, (k + 1)T]$, again from Bonferroni’s inequality we deduce that for any $u > 0$,

\begin{align}
(4.14) \quad P \left( \sup_{t \in [0, T^2]} X_u(t) > u \right) & \geq P \left( \sup_{t \in [nT, T^2]} X_u(t) > u \right) \\
& \geq \sum_{k=n}^{T-1} P \left( \sup_{t \in \Delta_k} X_u(t) > u \right) - \sum_{1 \leq k < l \leq T-1} P \left( \sup_{t \in \Delta_k} X_u(t) > u, \sup_{t \in \Delta_l} X_u(t) > u \right) \\
& \geq \sum_{k=n}^{T-1} P \left( \sup_{t \in \Delta_k} X_u(t) > u \right) - \Sigma_1 - \Sigma_2,
\end{align}

where

\begin{align*}
\Sigma_1 &= \sum_{k=1}^{T-2} P \left( \sup_{t \in \Delta_k} X_u(t) > u, \sup_{t \in \Delta_{k+1}} X_u(t) > u \right), \\
\Sigma_2 &= \sum_{1 \leq k < l \neq k+1}^{T-1} P \left( \sup_{t \in \Delta_k} X_u(t) > u, \sup_{t \in \Delta_l} X_u(t) > u \right).
\end{align*}

By Lemma 4.5(ii) and Proposition 2.1 as $u \to \infty$,

\begin{align}
(4.15) \quad \sum_{k=n}^{T-1} P \left( \sup_{t \in \Delta_k} X_u(t) > u \right) & \geq (T - n) \mathcal{H}_{B_\kappa} \left( \left((1 - \epsilon_n)c_{\hat{Y}}\right)^{1/\kappa}T \right) \Psi(u) (1 + o(1)).
\end{align}

From Lemma 4.7 for sufficiently large $u$, we have

\begin{align}
(4.16) \quad \Sigma_1 & \leq TF_1 \left( T^2 \exp(-G_1 \sqrt{T^\kappa}) + \sqrt{T} \right) \Psi(u)
\end{align}

From Lemma 4.6 for sufficiently large $u$,

\begin{align}
(4.17) \quad \Sigma_2 & \leq T^2 F_2 T^2 \exp(-G_2 T^\kappa) \Psi(u).
\end{align}

Inserting (4.15)–(4.17) in (4.14) (and using Proposition 2.1), we obtain

\begin{align*}
\frac{\mathcal{H}_{\hat{Y}}(T^2)}{T^2} & \geq \frac{(T - n) \mathcal{H}_{B_\kappa} \left( \left((1 - \epsilon_n)c_{\hat{Y}}\right)^{1/\kappa}T \right)}{T^2} \\
& \quad - \frac{F_1 (T^3 \exp(-G_1 \sqrt{T^\kappa}) + T^{3/2}) + F_2 T^4 \exp(-G_2 T^\kappa)}{T^2}.
\end{align*}

Letting $T \to \infty$ and then $\epsilon_n \to 0^+$ yields

\begin{align}
(4.18) \quad \liminf_{T \to \infty} \frac{\mathcal{H}_{\hat{Y}}(T)}{T} \geq (c_{\hat{Y}})^{1/\kappa} \mathcal{H}_{B_\kappa}.
\end{align}

From the upper bound (4.13) and the lower bound (4.18) we conclude that

\begin{align*}
\lim_{T \to \infty} \frac{\mathcal{H}_{\hat{Y}}(T)}{T} = (c_{\hat{Y}})^{1/\kappa} \mathcal{H}_{B_\kappa}.
\end{align*}
For $Y \in S(\alpha, \kappa, c_Y)$, $c_Y = c_Y (\kappa/\alpha)^\kappa$ and from Lemma 4.1(ii) it follows that
\[
\frac{\kappa}{\alpha} (c_Y)^{1/\kappa} H_{B_\kappa} = \left( c_Y \right)^{1/\kappa} H_{B_\kappa} = \lim_{T \to \infty} \frac{H_{Y}(T)}{T} = \lim_{T \to \infty} \frac{H_{Y}(T^{\kappa/\alpha})}{T} = \lim_{T \to \infty} \frac{H_{Y}(T)}{T^{\alpha/\kappa}}.
\]

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Krzysztof Dębicki, Kamil Tabiś
Mathematical Institute
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: debicki@math.uni.wroc.pl
kamil.tabis@math.uni.wroc.pl

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