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PICKANDS–PITERBARG CONSTANTS FOR SELF-SIMILAR GAUSSIAN PROCESSES

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Abstract. For a centered self-similar Gaussian process $\{Y(t) : t \in [0, \infty)\}$ and $R \geqslant 0$ we analyze the asymptotic behavior of

$$\mathcal{H}_Y^R(T) = \mathbf{E} \exp\Big(\sup_{t \in [0,T]} \big(\sqrt{2}\, Y(t) - (1+R)\sigma_Y^2(t) \big) \Big)$$

as $T \to \infty$. We prove that $\mathcal{H}_Y^R = \lim_{T \to \infty} \mathcal{H}_Y^R(T) \in (0, \infty)$ for $R > 0$ and

$$\mathcal{H}_Y = \lim_{T \to \infty} \frac{\mathcal{H}_Y^0(T)}{T^\gamma} \in (0, \infty)$$

for suitably chosen $\gamma > 0$. Additionally, we find bounds for $\mathcal{H}_Y^R$, $R > 0$, and a surprising relation between $\mathcal{H}_Y$ and the classical Pickands constants.

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1. INTRODUCTION

For a centered Gaussian process $\{Y(t) : t \in [0, \infty)\}$ with a.s. continuous sample paths, $\mathbf{Var}(Y(t)) = \sigma_Y^2(t)$ and $Y(0) = 0$ a.s., let

$$\mathcal{H}_Y^R(T) = \mathbf{E} \exp\Big(\sup_{t \in [0,T]} \big(\sqrt{2}\, Y(t) - (1+R)\sigma_Y^2(t) \big) \Big), \tag{1.1}$$

where $R \geqslant 0$ and let $\mathcal{H}_Y(T) := \mathcal{H}_Y^0(T)$.

The functionals $\mathcal{H}_Y^R(T), \mathcal{H}_Y(T)$ play an important role in many areas of probability theory. For example, consider a fractional Brownian motion $\{B_\kappa(t) : t \in [0, \infty)\}$ with Hurst parameter $\kappa/2 \in (0, 1]$, i.e. a centered Gaussian process with stationary increments, continuous sample paths a.s. and variance function $\mathbf{Var}(B_\kappa(t)) = t^\kappa$. Then, for $\kappa \in (0, 2]$, the *Pickands constants* $\mathcal{H}_{B_\kappa}$ defined as

$$\mathcal{H}_{B_\kappa} = \lim_{T\to\infty} \frac{\mathcal{H}_{B_\kappa}(T)}{T}, \tag{1.2}$$

and the *Piterbarg constants* $\mathcal{H}^R_{B_\kappa}$, for $R > 0$, defined as

$$\mathcal{H}^R_{B_\kappa} = \lim_{T\to\infty} \mathcal{H}^R_{B_\kappa}(T) \tag{1.3}$$

play a key role in the extreme value theory of Gaussian processes; see, e.g., [25], [26], [27] or more recent contributions [18], [24]. In [6] it was observed that the notion of Pickands and Piterbarg constants can be extended to *generalized* Pickands and Piterbarg constants, defined as

$$\mathcal{H}_\eta = \lim_{T\to\infty} \frac{\mathcal{H}_\eta(T)}{T} \quad \text{and} \quad \mathcal{H}^R_\eta = \lim_{T\to\infty} \mathcal{H}^R_\eta(T)$$

respectively, where $R > 0$ and $\{\eta(t) : t \in (0,\infty)\}$ is a centered Gaussian process with stationary increments. We refer to [2], [3], [5], [12], [14], [16] for properties and other representations of $\mathcal{H}_{B_\kappa}$, $\mathcal{H}^R_{B_\kappa}$ and generalized Pickands–Piterbarg constants, and to [10], [11] for multidimensional analogs of Pickands–Piterbarg constants.

Recently (see e.g. [12]), it was found that for general Gaussian processes Y (satisfying some regularity conditions) the functionals (1.1) appear in the formulas for exact asymptotics of suprema of some Gaussian processes (see Proposition 2.1).

The interest in (1.1) also stems from an important contribution [16] which established a direct connection between Pickands constants and max-stationary stable processes (see also [7], [8], [9]).

The constants $\mathcal{H}_Y(T)$ also appear in the context of convex geometry where they are known as *Wills functionals* (see [28]).

In this contribution we analyze the properties of $\mathcal{H}^R_Y(T)$ and $\mathcal{H}_Y(T)$ for a class of general self-similar Gaussian processes Y with non-stationary increments. In particular, we find analogs of limits (1.2), (1.3) and give some bounds for them. Surprisingly, it appears that, up to some explicitly given constant, $\mathcal{H}_Y$ is equal to the classical $\mathcal{H}_{B_\kappa}$ for some appropriately chosen κ.

2. NOTATION AND PRELIMINARY RESULTS

Let $\{Y(t) : t \geqslant 0\}$ be a centered Gaussian process with a.s. continuous sample paths and let

$$V_Y(s,t) := \mathbf{Var}(Y(s) - Y(t)), \quad R_Y(s,t) := \mathbf{Cov}(Y(s), Y(t)).$$

We say that a stochastic process $Y(\cdot)$ is *self-similar* with index $H > 0$ if for all $a > 0$,

$$\{Y(at) : t \geqslant 0\} \stackrel{\mathcal{D}}{=} \{a^H Y(t) : t \geqslant 0\}. \tag{2.1}$$

A straightforward consequence of (2.1) is that for self-similar Gaussian processes, $\sigma_Y^2(t) = \sigma_Y^2(1)t^{2H}$ for $t \geqslant 0$.

We write $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$ if

S1. $Y(\cdot)$ is self-similar with index $\alpha/2 > 0$ and $\sigma_Y^2(1) = 1$;

S2. there exist $\kappa \in (0, 2]$ and $c_Y > 0$ such that

$$\mathbf{Var}(Y(1) - Y(1-h)) = c_Y|h|^\kappa + o(|h|^\kappa) \quad \text{as } h \to 0.$$

It is well known (see Lamperti [21]) that $\{Y(t) : t \geqslant 0\}$ is a self-similar Gaussian process with index $\alpha/2$ if and only if its Lamperti transform $X(t) = e^{-(\alpha/2)t}Y(e^t)$ is a stationary Gaussian process. Thus, there is a unique correspondence between self-similar Gaussian processes and stationary Gaussian processes. In fact condition **S2** relates to regularity of the covariance function of the stationary counterpart of Y. More precisely, let $\{X(t) : t \in \mathbb{R}\}$ be a stationary Gaussian process such that $R_X(t, 0) = 1 - a|t|^\kappa + o(|t|^\kappa)$ as $t \to 0$ with $\kappa \in (0, 2], a > 0$. Then one can check that the self-similar process $Y(t) := t^{\alpha/2}X(\log t)$ for $\alpha \in (0, 2]$ is $\mathbf{S}(\alpha, \kappa, c_Y)$ with

$$c_Y = \begin{cases} 2a & \text{for } \kappa < 2, \\ \alpha^2/4 + 2a & \text{for } \kappa = 2. \end{cases}$$

Below we specify some important classes of self-similar Gaussian processes that satisfy **S1–S2**.

- ◇ *Fractional Brownian motion* B_α is in $\mathbf{S}(\alpha, \alpha, 1)$ with $\alpha/2 \in (0, 1]$.

- ◇ *Bifractional Brownian motion* $\{Y^{(1)}(t) : t \geqslant 0\}$ with parameters $\alpha \in (0, 2)$ and $K \in (0, 1]$ is a centered Gaussian process with covariance function

$$R_{Y^{(1)}}(t, s) = \frac{1}{2^K}((t^\alpha + s^\alpha)^K - |t - s|^{\alpha K})$$

  (see e.g. [19], [22]). We have $Y^{(1)} \in \mathbf{S}(\alpha K, \alpha K, 2^{1-K})$.

- ◇ *Sub-fractional Brownian motion* $\{Y^{(2)}(t) : t \geqslant 0\}$ with parameter $\alpha \in (0, 2)$ is a centered Gaussian process with covariance function

$$R_{Y^{(2)}}(t, s) = \frac{1}{2 - 2^{\alpha - 1}}\left(t^\alpha + s^\alpha - \frac{(t + s)^\alpha + |t - s|^\alpha}{2}\right)$$

  (see [4], [17]). Then $Y^{(2)} \in \mathbf{S}(\alpha, \alpha, (2 - 2^{\alpha-1})^{-1})$.

◇ *k-fold integrated fractional Brownian motion* $\{Y^{(3),k}(t) : t \geqslant 0\}$ with parameters $k \in \mathbb{N} := \{1, 2, \dots\}$ and $\alpha \in (0, 2]$ is a Gaussian process defined as

$$Y^{(3),1}(t) = \sqrt{\alpha + 2} \int_0^t B_\alpha(s)\, ds,$$

$$Y^{(3),k}(t) = \sqrt{\frac{k(\alpha + 2k)(\alpha + k - 1)}{\alpha + 2k - 2}} \int_0^t Y^{(3),k-1}(s)\, ds \quad \text{for } k \geqslant 2.$$

Then $Y^{(3),k} \in \mathbf{S}\big(\alpha + 2k, 2, \frac{k(\alpha+2k)(\alpha+k-1)}{\alpha+2k-2}\big)$ for $k \geqslant 1$.

◇ *Time-average of fractional Brownian motion* $\{Y^{(4)}(t) : t \geqslant 0\}$ with parameter $\alpha \in (0, 2]$ is a Gaussian process defined as

$$Y^{(4)}(t) = \sqrt{\alpha + 2}\, \frac{1}{t} \int_0^t B_\alpha(s)\, ds.$$

Its covariance function is

$$R_{Y^{(4)}}(t, s) = \frac{(\alpha + 2)(s^{\alpha+1} t + s t^{\alpha+1}) + |t - s|^{\alpha+2} - t^{\alpha+2} - s^{\alpha+2}}{2(\alpha + 1)ts}$$

and we have $Y^{(4)} \in \mathbf{S}(\alpha, 2, 1)$.

◇ *Dual fractional Brownian motion* $\{Y^{(5)}(t) : t \geqslant 0\}$ with parameter $\alpha \in (0, 2]$ is a centered Gaussian process defined as

$$Y^{(5)}(t) = t^{\alpha+1} \sqrt{\frac{2}{\Gamma(\alpha + 1)}} \int_0^\infty B_\alpha(s) e^{-st}\, ds$$

(see [23]). We have

$$R_{Y^{(5)}}(t, s) = \frac{t^\alpha s + s^\alpha t}{t + s}$$

and $Y^{(5)} \in \mathbf{S}(\alpha, 2, \alpha/2)$.

In the rest of the paper, $\overline{X}(s) := X(s)/\sigma_X(s)$, and $\Psi(\cdot)$ denotes the tail distribution function of the standard normal random variable.

The following proposition plays a key role in the proofs of our main results, confirming also that the functionals $\mathcal{H}_Y(\cdot)$ and $\mathcal{H}_Y^R(\cdot)$ for $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$ appear in the asymptotics of extremes of Gaussian processes.

PROPOSITION 2.1. *Let $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$ and let $\{X(t) : t \geqslant 0\}$ be a centered Gaussian process with $R_{\overline{X}}(t, s) = \exp(-aV_Y(t, s))$ for $a > 0$ and $\sigma_X(t) = \frac{1}{1+bt^\beta}$ for $b \geqslant 0$, $\beta > 0$.*

(i) *If $\alpha = \beta$, then as $u \to \infty$,*

$$\mathbf{P}\Big(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t) > u\Big) = \mathcal{H}_Y^{b/a}(a^{1/\alpha}T)\Psi(u)(1+o(1)).$$

(ii) *If $\alpha < \beta$, then as $u \to \infty$,*

$$\mathbf{P}\Big(\sup_{t\in[0,Tu^{-2/\alpha}]} X(t) > u\Big) = \mathcal{H}_Y(a^{1/\alpha}T)\Psi(u)(1+o(1)).$$

The proof of Proposition 2.1 is given in Section 4.1.

3. PICKANDS–PITERBARG CONSTANTS FOR SELF-SIMILAR GAUSSIAN PROCESSES

The aim of this section is to find analogs of Pickands and Piterbarg constants for self-similar Gaussian processes $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$.

3.1. Piterbarg constants. For $R > 0$ and $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$ let us introduce an analog of the Piterbarg constant $\mathcal{H}_{B_\kappa}^R$ as follows:

$$\mathcal{H}_Y^R := \lim_{T\to\infty} \mathcal{H}_Y^R(T) = \lim_{T\to\infty} \mathbf{E}\exp\Big(\sup_{t\in[0,T]} \big(\sqrt{2}\,Y(t) - (1+R)t^\alpha\big)\Big).$$

In the next theorem we prove that $\mathcal{H}_Y^R$ is well-defined and we compare it with the classical Piterbarg constants.

THEOREM 3.1. *Let $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$. Then, for any $R > 0$,*

$$\mathcal{H}_Y^R \in (0, \infty).$$

Furthermore

$$\mathcal{H}_{B_\kappa}^{R/c_1} \leqslant \mathcal{H}_Y^R \leqslant \mathcal{H}_{B_\kappa}^{R/c_2},$$

where

$$c_1 = \inf_{x\in[0,1)} \frac{V_Y(1, x^{\kappa/\alpha})}{|1-x|^\kappa} \quad \textit{and} \quad c_2 = \sup_{x\in[0,1)} \frac{V_Y(1, x^{\kappa/\alpha})}{|1-x|^\kappa}.$$

The proof of Theorem 3.1 is given in Section 4.2.

PROPOSITION 3.2. *Let $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$. Then*

$$\mathcal{H}_Y^R \geqslant \frac{1}{2}\bigg(1 + \sqrt{1 + \frac{1}{R}}\bigg).$$

The proof of Proposition 3.2 is postponed to Section 4.3.

The following corollary follows from Theorem 3.1 combined with the fact that $\mathcal{H}_{B_1}^R = 1 + 1/R$ (see, e.g., [13]) and $\mathcal{H}_{B_2}^R = \frac{1}{2}(1 + \sqrt{1 + 1/R})$ (see, e.g., [20]).

COROLLARY 3.3. *Let* $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$.

(i) *If* $\kappa = 1$, *then*

$$1 + \frac{1}{R}\left(\inf_{x\in[0,1)} \frac{V_Y(1, x^{1/\alpha})}{|1-x|}\right) \leqslant \mathcal{H}_Y^R \leqslant 1 + \frac{1}{R}\left(\sup_{x\in[0,1)} \frac{V_Y(1, x^{1/\alpha})}{|1-x|}\right).$$

(ii) *If* $\kappa = 2$, *then*

$$\frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{R}}\right) \leqslant \mathcal{H}_Y^R \leqslant \frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{R}\left(\sup_{x\in[0,1)} \frac{V_Y(1, x^{2/\alpha})}{|1-x|^2}\right)}\right).$$

In the following example we specify Corollary 3.3 for some particular self-similar processes introduced in Section 2.

EXAMPLE 3.1. The following bounds hold:

◇ k-fold integrated fractional Brownian motion $Y^{(3),k}$:

$$\frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{R}}\right) \leqslant \mathcal{H}_{Y^{(3),k}}^R \leqslant \frac{1}{2}\left(1 + \sqrt{1 + \frac{4k(\alpha + k - 1)}{R(\alpha + 2k)(\alpha + 2k - 2)}}\right).$$

◇ Time-average of fractional Brownian motion $Y^{(4)}$ with parameter $\alpha \in (0, 2]$:

$$\frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{R}}\right) \leqslant \mathcal{H}_{Y^{(4)}}^R \leqslant \frac{1}{2}\left(1 + \sqrt{1 + \frac{4}{R(\alpha + 2)}}\right).$$

The above bounds improve the results obtained in [15] for the constants

$$\begin{aligned}
\mathcal{F}_\alpha &= \lim_{T\to\infty} \mathbf{E}\exp\left(\sup_{t\in(0,T]} \frac{1}{t}\int_0^t (\sqrt{2}\, B_\alpha(s) - s^\alpha)\, ds\right) \\
&= \lim_{T\to\infty} \mathbf{E}\exp\left(\sup_{t\in(0,T]} \left(Y^{(4)}(t) - \frac{\alpha+2}{\alpha+1} t^\alpha\right)\right) = \mathcal{H}_{Y^{(4)}}^{1/(\alpha+1)},
\end{aligned}$$

leading to

$$\frac{1}{2}(1 + \sqrt{2 + \alpha}) \leqslant \mathcal{F}_\alpha \leqslant \frac{1}{2}(1 + \sqrt{1 + 4(\alpha + 1)/\alpha^2}),$$

while in [15] it was proved that $\mathcal{F}_\alpha \leqslant 2 + \alpha$ for $\alpha \in [1, 2)$.

◇ Dual fractional Brownian motion $Y^{(5)}$ with parameter $\alpha \in (0, 2]$:

$$\frac{1}{2}\left(1 + \sqrt{1 + \frac{1}{R}}\right) \leqslant \mathcal{H}_{Y^{(5)}}^R \leqslant \frac{1}{2}\left(1 + \sqrt{1 + \frac{2}{R\alpha}}\right).$$

3.2. Pickands constants. In this section we focus on an analog of Pickands constants for $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$. Let

$$\mathcal{H}_Y := \lim_{T\to\infty} \frac{\mathcal{H}_Y(T)}{T^{\alpha/\kappa}} = \lim_{T\to\infty} \frac{\mathbf{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}\,Y(t) - t^\alpha))}{T^{\alpha/\kappa}}.$$

We observe that for $Y(t) = B_\kappa(t)$ the above definition agrees with the notion of the classical Pickands constant $\mathcal{H}_{B_\kappa}$, since $\alpha = \kappa$ in this case.

In the following theorem we show that $\mathcal{H}_Y$ is well-defined and find a surprising relation between $\mathcal{H}_Y$ and $\mathcal{H}_{B_\kappa}$.

THEOREM 3.4. *Let $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$. Then $\mathcal{H}_Y \in (0, \infty)$ and*

$$\mathcal{H}_Y = \frac{\kappa}{\alpha}(c_Y)^{1/\kappa}\mathcal{H}_{B_\kappa}.$$

A complete proof of Theorem 3.4 is presented in Section 4.4.

The following corollary is an immediate consequence of Theorem 3.4 and the fact that $\mathcal{H}_{B_1} = 1$ and $\mathcal{H}_{B_2} = 1/\sqrt{\pi}$.

COROLLARY 3.5. *Let $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$.*

(i) *If $\kappa = 1$, then*

$$\mathcal{H}_Y = \lim_{T\to\infty} \frac{\mathbf{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}\,Y(t) - t^\alpha))}{T^{\alpha}} = \frac{c_Y}{\alpha}.$$

(ii) *If $\kappa = 2$, then*

$$\mathcal{H}_Y = \lim_{T\to\infty} \frac{\mathbf{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}\,Y(t) - t^\alpha))}{T^{\alpha/2}} = \frac{2}{\alpha}\sqrt{\frac{c_Y}{\pi}}.$$

In the following example we specify the findings of this section for self-similar Gaussian processes introduced in Section 2.

EXAMPLE 3.2. The following equalities hold:

◇ Bifractional Brownian motion with parameters $\alpha \in (0, 2)$ and $K \in (0, 1]$:

$$\mathcal{H}_{Y^{(1)}} = \lim_{T\to\infty} \frac{\mathbf{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}\,Y^{(1)}(t) - t^{\alpha K}))}{T} = 2^{\frac{1-K}{\alpha K}}\mathcal{H}_{B_{\alpha K}}.$$

◇ Sub-fractional Brownian motion with parameter $\alpha \in (0, 2)$:

$$\mathcal{H}_{Y^{(2)}} = \lim_{T\to\infty} \frac{\mathbf{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}\,Y^{(2)}(t) - t^{\alpha}))}{T} = (2 - 2^{\alpha-1})^{-1/\alpha}\mathcal{H}_{B_\alpha}.$$

◇ k-fold integrated fractional Brownian motion with parameters $k \in \mathbb{N}$ and $\alpha \in (0, 2]$:

$$\begin{aligned}\mathcal{H}_{Y^{(3),k}} &= \lim_{T\to\infty} \frac{\mathbf{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}\,Y^{(3),k}(t) - t^{k+\alpha/2}))}{T^{k+\alpha/2}} \\ &= \sqrt{\frac{4k(\alpha+k-1)}{\pi(\alpha+2k)(\alpha+2k-2)}}.\end{aligned}$$

◇ Time-average of fractional Brownian motion with parameter $\alpha \in (0, 2]$:

$$\mathcal{H}_{Y^{(4)}} = \lim_{T\to\infty} \frac{\mathbf{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}\,Y^{(4)}(t) - t^{\alpha}))}{T^{\alpha/2}} = \frac{2}{\sqrt{\pi}\,\alpha}.$$

◇ Dual fractional Brownian motion with parameter $\alpha \in (0, 2]$:

$$\mathcal{H}_{Y^{(5)}} = \lim_{T\to\infty} \frac{\mathbf{E}\exp(\sup_{t\in[0,T]}(\sqrt{2}\,Y^{(5)}(t) - t^{\alpha}))}{T^{\alpha/2}} = \sqrt{\frac{2}{\pi\alpha}}.$$

4. PROOFS

In the rest of the paper we use the notation $v_Y(t) := V_Y(1, t)$. We begin with the following lemma, skipping its straightforward proof.

LEMMA 4.1. *Let $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$ and $\hat{Y}_{\alpha_1} = Y(t^{\alpha_1})$ for some $\alpha_1 > 0$. Then, for any $R \geqslant 0$ and $T > 0$,*

(i) $\mathcal{H}^R_{cY}(T) = \mathcal{H}^R_Y(c^{2/\alpha}T)$ *for any* $c > 0$*;*

(ii) $\mathcal{H}^R_{\hat{Y}_{\alpha_1}}(T) = \mathcal{H}^R_Y(T^{\alpha_1})$.

4.1. Proof of Proposition 2.1. In the next lemma we present a useful bound on $V_Y(\cdot, \cdot)$ for $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$.

LEMMA 4.2. *Let $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$. Then there exists a positive constant C such that for $\gamma = \min(\alpha, \kappa)$, $T > 0$ and all $t, s \in [0, T]$,*

$$V_Y(t, s) \leqslant CT^{\alpha-\gamma}|t - s|^{\gamma}.$$

Proof. For $t = s$ the conclusion is obvious. Suppose that $0 \leqslant s < t \leqslant T$ and let $\epsilon \in (0, 1)$ be such that for $\delta \in (0, 1)$,

$$(1-\epsilon)c_Y|1-x|^{\kappa} \leqslant V_Y(1, x) \leqslant (1+\epsilon)c_Y|1-x|^{\kappa}$$

for all $x \in [\delta, 1]$ (due to **S2**). For $s/t \geqslant \delta$ we have

$$
\begin{aligned}
V_Y(t,s) &= t^\alpha V_Y(1, s/t) \leqslant t^\alpha (1+\epsilon) c_Y |1 - s/t|^\kappa \\
&\leqslant t^\alpha (1+\epsilon) c_Y |1 - s/t|^{\min(\alpha,\kappa)} \\
&= t^{\alpha-\gamma}(1+\epsilon) c_Y |t-s|^\gamma \leqslant T^{\alpha-\gamma}(1+\epsilon) c_Y |t-s|^\gamma.
\end{aligned}
$$

For $s/t \leqslant \delta$ we have $|1-\delta|^\gamma \leqslant |1-s/t|^\gamma$. Hence, $t^\gamma |1-\delta|^\gamma \leqslant |t-s|^\gamma$. Then

$$
\begin{aligned}
V_Y(t,s) &= t^\alpha V_Y(1, s/t) \\
&\leqslant t^\gamma t^{\alpha-\gamma} \frac{|1-\delta|^\gamma}{|1-\delta|^\gamma} \max_{x\in[0,\delta]} V_Y(1,x) \leqslant T^{\alpha-\gamma} \frac{\max_{x\in[0,\delta]} V_Y(1,x)}{|1-\delta|^\gamma} |t-s|^\gamma.
\end{aligned}
$$

Hence the proof is completed with $C = \max\big((1+\epsilon)c_Y, \frac{\max_{x\in[0,\delta]} V_Y(1,x)}{|1-\delta|^\gamma}\big)$. ∎

Proof of Proposition 2.1. Since for any Gaussian process $Y(\cdot)$, the variogram function $V_Y(\cdot,\cdot)$ is negative definite, by the Schoenberg theorem the function $\exp(-V_Y(\cdot,\cdot))$ is positive definite. Thus there exists a Gaussian process $\{X(t) : t \geqslant 0\}$ with $R_X(t,s) = \exp(-V_Y(t,s))$.

The rest of the proof follows straightforwardly from [12, Theorem 2.1] and Lemma 4.2 applied to $X_u(t) = X(tu^{-2/\alpha})$. ∎

4.2. Proof of Theorem 3.1

LEMMA 4.3. *Let $Y \in \mathbf{S}(\alpha,\kappa,c_Y)$ and define $\hat{Y}(t) = Y(t^{\kappa/\alpha})$. Then $\hat{Y} \in \mathbf{S}(\kappa,\kappa,c_Y(\kappa/\alpha)^\kappa)$ and there exist finite and positive constants*

$$
\begin{aligned}
c_1 &= \inf_{x\in[0,1)} \frac{V_Y(1, x^{\kappa/\alpha})}{|1-x|^\kappa} = \inf_{x\in[0,1)} \frac{V_{\hat{Y}}(1,x)}{|1-x|^\kappa}, \\
c_2 &= \sup_{x\in[0,1)} \frac{V_Y(1, x^{\kappa/\alpha})}{|1-x|^\kappa} = \sup_{x\in[0,1)} \frac{V_{\hat{Y}}(1,x)}{|1-x|^\kappa}.
\end{aligned}
$$

Moreover for all $t, s \geqslant 0$,

$$
c_1|t-s|^\kappa \leqslant V_Y(t^{\kappa/\alpha}, s^{\kappa/\alpha}) = V_{\hat{Y}}(t,s) \leqslant c_2|t-s|^\kappa.
$$

Proof. Observe that $\hat{Y} \in \mathbf{S}(\kappa,\kappa,c_Y(\kappa/\alpha)^\kappa)$ with $V_{\hat{Y}}(t,s) = V_Y(t^{\kappa/\alpha}, s^{\kappa/\alpha})$. Consider the function $f(x) = \frac{V_{\hat{Y}}(1,x)}{|1-x|^\kappa}$ for $x \in [0,1)$. Due to **S2**, $\lim_{x\to 1^-} f(x) = c_{\hat{Y}} > 0$, $f(0) = 1$ and $f(x) = 0$ only for $x = 1$. Hence, $c_1, c_2 > 0$ exist.

Moreover, for all $t \geqslant s \geqslant 0$,

$$
c_1|t-s|^\kappa = c_1 t^\kappa |1-s/t|^\kappa \leqslant t^\kappa V_{\hat{Y}}(1, s/t) \leqslant c_2 t^\kappa |1-s/t|^\kappa = c_2|t-s|^\kappa.
$$

This completes the proof. ∎

Proof of Theorem 3.1. Let $R,T>0$. Define $\hat{Y}(t)=Y(t^{\kappa/\alpha})$. Then $\hat{Y}\in\mathbf{S}(\kappa,\kappa,c_Y(\kappa/\alpha)^\kappa)$ with $V_{\hat{Y}}(1,x)=V_Y(1,x^{\kappa/\alpha})$ and $c_1,c_2>0$ exist. Let $\{X(t):t\geqslant 0\}$, $\{X_1(t):t\geqslant 0\}$, $\{X_2(t):t\geqslant 0\}$ be centered Gaussian processes with $R_{\overline{X}}(t,s)=\exp(-V_{\hat{Y}}(t,s))$, $R_{\overline{X}_i}(t,s)=\exp(-c_iV_{B_\kappa}(t,s))$ and $\sigma_X(t)=\sigma_{X_i}(t)=\frac{1}{1+Rt^\kappa}$. Then, for all $t,s\geqslant 0$,

$$\begin{aligned}R_{\overline{X}_1}(t,s)&=\exp(-c_1V_{B_\kappa}(t,s))\geqslant R_{\overline{X}}(t,s)=\exp(-V_{\hat{Y}}(t,s))\\&\geqslant\exp(-c_2V_{B_\kappa}(t,s))=R_{\overline{X}_2}(t,s)\end{aligned}$$

and hence, due to Slepian's inequality (see, e.g., [1, Corollary 2.4]) we find that for all $u>0$,

$$\begin{aligned}\mathbf{P}\Big(\sup_{t\in[0,Tu^{-2/\kappa}]}X_1(t)>u\Big)&\leqslant\mathbf{P}\Big(\sup_{t\in[0,Tu^{-2/\kappa}]}X(t)>u\Big)\\&\leqslant\mathbf{P}\Big(\sup_{t\in[0,Tu^{-2/\kappa}]}X_2(t)>u\Big).\end{aligned}$$

Application of Proposition 2.1(i) to the inequalities above gives

$$\mathcal{H}_{B_\kappa}^{R/c_1}(c_1^{1/\kappa}T)\leqslant\mathcal{H}_{\hat{Y}}^{R}(T)=\mathcal{H}_{Y}^{R}(T^{\kappa/\alpha})\leqslant\mathcal{H}_{B_\kappa}^{R/c_2}(c_2^{1/\kappa}T),\tag{4.1}$$

where the equality follows from Lemma 4.1(ii). Note that all functions in (4.1) are increasing in T, and hence letting $T\to\infty$ in (4.1) completes the proof. ■

4.3. Proof of Proposition 3.2. Since $\sigma_Y^2(t)=t^\alpha$, the Schwarz inequality implies that $R_Y(t,s)\leqslant(ts)^{\alpha/2}$ for all $t,s\geqslant 0$. Therefore for $t,s\geqslant 0$,

$$V_Y(t,s)\geqslant t^\alpha+s^\alpha-2(ts)^{\alpha/2}=|t^{\alpha/2}-s^{\alpha/2}|^2=V_{\hat{B}_2}(t,s),$$

where $\hat{B}_2(t)=B_2(t^{\alpha/2})$. Thus, by Slepian's inequality,

$$\begin{aligned}\mathcal{H}_Y^R(T)&=\int_{\mathbb{R}}e^x\mathbf{P}\Big(\sup_{t\in[0,T]}(\sqrt{2}\,Y(t)-(1+R)t^\alpha>x)\Big)\,dx\\&\geqslant\int_{\mathbb{R}}e^x\mathbf{P}\Big(\sup_{t\in[0,T]}(\sqrt{2}\,\hat{B}_2(t)-(1+R)t^\alpha>x)\Big)\,dx\\&=\mathcal{H}_{\hat{B}_2}^R(T)=\mathcal{H}_{B_2}^R(T^{\alpha/2}),\end{aligned}$$

where the last equality follows from Lemma 4.1(ii). Letting $T\to\infty$ yields the conclusion. ■

4.4. Proof of Theorem 3.4. To prove Theorem 3.4 we need some technical lemmas.

LEMMA 4.4. *Let $\hat{Y}\in\mathbf{S}(\kappa,\kappa,c_{\hat{Y}})$. For any $\epsilon\to 0^+$ there exists $\delta_\epsilon\to 0^+$ such that for any $T>0$ and $A\geqslant T/\delta_\epsilon$,*

$$(1-\epsilon)c_{\hat{Y}}|t-s|^\kappa\leqslant V_{\hat{Y}}(A+t,A+s)\leqslant(1+\epsilon)c_{\hat{Y}}|t-s|^\kappa$$

for all $t,s\in[0,T]$.

Proof. Let $\epsilon \in (0, 1)$ be sufficiently small such that

$$(1-\epsilon)c_{\hat{Y}}|h|^{\kappa} \leqslant V_{\hat{Y}}(1, 1-h) \leqslant (1+\epsilon)c_{\hat{Y}}|h|^{\kappa} \tag{4.2}$$

for all $h \in [0, \delta_\epsilon]$ and $\delta_\epsilon \in (0, 1)$ (due to **S2**). Then, for any $T > 0$, $A \geqslant T/\delta_\epsilon$ and $0 \leqslant s \leqslant t \leqslant T$ we have $\frac{t-s}{A+t} \leqslant \frac{T}{A} \leqslant \delta_\epsilon$. Combining the fact that

$$V_{\hat{Y}}(A+s, A+t) = (A+t)^{\kappa} V_{\hat{Y}}\left(1, \frac{A+s}{A+t}\right) = (A+t)^{\kappa} V_{\hat{Y}}\left(1, 1-\frac{t-s}{A+t}\right)$$

with (4.2) for $h = \frac{t-s}{A+t} \leqslant \delta_\epsilon$, we obtain the assertion. ■

LEMMA 4.5. *Let $\hat{Y} \in \mathbf{S}(\kappa, \kappa, c_{\hat{Y}})$ and let c_1, c_2 be as in Lemma 4.3. Consider a centered Gaussian process $\{\overline{X}(t) : t \geqslant 0\}$ with $R_{\overline{X}}(t, s) = \exp(-aV_{\hat{Y}}(t, s))$ with $a > 0$.*

(i) *Let $\{\overline{X}_i(t) : t \geqslant 0\}$, $i = 1, 2$, be centered stationary Gaussian processes with $R_{\overline{X}_i}(t, s) = \exp(-ac_iV_{B_\kappa}(t, s))$. Then for all $u > 0$ and any $T, A = A(u) > 0$,*

$$\mathbf{P}\Big(\sup_{t\in[0,T]u^{-2/\kappa}} \overline{X}_1(t) > u\Big) \leqslant \mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}(t) > u\Big) \leqslant \mathbf{P}\Big(\sup_{t\in[0,T]u^{-2/\kappa}} \overline{X}_2(t) > u\Big).$$

(ii) *For any $\epsilon > 0$, let $\{\overline{X}_i(t) : t \geqslant 0\}$, $i = 1, 2$, be centered stationary Gaussian processes with $R_{\overline{X}_i}(t, s) = \exp(-a(1 + (-1)^i\epsilon)c_{\hat{Y}}V_{B_\kappa}(t, s))$. Then for any $\epsilon \to 0^+$ there exists $\delta_\epsilon \to 0^+$ such that for any $T > 0$ and $A = A(u) \geqslant T/\delta$,*

$$\mathbf{P}\Big(\sup_{t\in[0,T]u^{-2/\kappa}} \overline{X}_1(t) > u\Big) \leqslant \mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}(t) > u\Big) \leqslant \mathbf{P}\Big(\sup_{t\in[0,T]u^{-2/\kappa}} \overline{X}_2(t) > u\Big).$$

Proof. (i) The argument is the same as for Theorem 3.1. From Lemma 4.3 we know that for all $t, s \geqslant 0$,

$$V_{\overline{X}_1}(t, s) \leqslant V_{\overline{X}}(t, s) \leqslant V_{\overline{X}_2}(t, s)$$

and hence, due to Slepian's inequality, for all $u > 0$,

$$\mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}_2(t) > u\Big) \leqslant \mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}(t) > u\Big) \leqslant \mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}_2(t) > u\Big).$$

Due to stationarity of $\overline{X}_i(\cdot)$ we obtain the assertion.

(ii) From Lemma 4.4, for any $\epsilon \to 0^+$ there exists $\delta_\epsilon \to 0^+$ such that for any $T > 0$ and $A \geqslant T/\delta_\epsilon$,

$$(1-\epsilon)c_{\hat{Y}}|t-s|^\kappa \leqslant V_{\hat{Y}}(t,s) \leqslant (1+\epsilon)c_{\hat{Y}}|t-s|^\kappa$$

for all $t, s \in [A, A+T]$. The same argument as in the proof of (i) completes the proof. ■

LEMMA 4.6. *Suppose that* $\lim_{u\to\infty} f(u)/u = c$ *for some* $c > 0$*. Under the notation of Lemma* 4.5*, there exist absolute constants* $F, G > 0$ *such that*

$$\mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}(t) > f(u), \sup_{t\in[t_0,t_0+T]u^{-2/\kappa}} \overline{X}(t) > f(u)\Big) \leqslant FT^2 \exp(-G(t_0-(A+T))^\kappa)\Psi(f(u))$$

for all $t_0 > A+T > 0$*,* $T \geqslant 1$ *and any* $u \geqslant u_0 = (2ac_2)^2(t_0+T)^{\kappa/2}$*.*

Proof. The argument is similar to the one given in, e.g., [6, Lemma 6.2] or [12, Theorem 2.1]. Thus we only present the main steps.

Let $u_0 = (2ac_2)^2(t_0+T)^{\kappa/2}$ and $\{Z_u(t_1,t_2) : (t_1,t_2) \in [A, A+T] \times [t_0, t_0+T]\}$, where $Z_u(t_1,t_2) = \overline{X}(t_1u^{-2/\kappa}) + \overline{X}(t_2u^{-2/\kappa})$. Note that

$$\mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}(t) > f(u), \sup_{t\in[t_0,t_0+T]u^{-2/\kappa}} \overline{X}(t) > f(u)\Big) \leqslant \mathbf{P}\Big(\sup_{(t_1,t_2)\in[A,A+T]\times[t_0,t_0+T]} Z_u(t_1,t_2) > 2f(u)\Big). \tag{4.3}$$

Since $(t_0+T)u^{-2/\kappa} \leqslant (2ac_2)^{-1/\kappa}$, by Lemma 4.3 for all $t_1, t_2 \leqslant t_0+T$,

$$\begin{aligned} ac_1u^{-2}|t_2-t_1|^\kappa &\leqslant aV_{\hat{Y}}(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}) \\ &\leqslant ac_2u^{-2}|t_2-t_1|^\kappa \leqslant ac_2|(t_0+T)u^{-2/\kappa}|^\kappa \leqslant 1/2. \end{aligned} \tag{4.4}$$

Hence, as $x \leqslant 2(1-e^{-x}) \leqslant (1-e^{-4x})$ for $x \in [0, 1/2]$, we obtain

$$\begin{aligned} V_{\overline{X}}(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}) &= 2\big(1-\exp(-aV_{\hat{Y}}(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}))\big) \\ &\geqslant aV_{\hat{Y}}(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}) \geqslant ac_1u^{-2}|t_2-t_1|^\kappa, \end{aligned} \tag{4.5}$$

$$\begin{aligned} V_{\overline{X}}(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}) &\leqslant 2(1-\exp(-ac_2u^{-2}|t_2-t_1|^\kappa)) \\ &\leqslant (1-\exp(-4ac_2u^{-2}|t_2-t_1|^\kappa)) \end{aligned} \tag{4.6}$$

for all $t_1, t_2 \leqslant t_0+T$. Since

$$\begin{aligned} \sigma^2_{Z_u}(t_1,t_2) &= 2 + 2\exp(-aV_{\hat{Y}}(t_1u^{-2/\kappa}, t_2u^{-2/\kappa})) \\ &= 4 - 2\big(1-\exp(-aV_{\hat{Y}}(t_1u^{-2/\kappa}, t_2u^{-2/\kappa}))\big), \end{aligned}$$

from (4.5), for any $(t_1, t_2) \in [A, A+T] \times [t_0, t_0+T]$,

$$2 \leqslant \sigma^2_{Z_u}(t_1, t_2) \leqslant 4 - ac_1 u^{-2}(t_0 - (A+T))^\kappa. \tag{4.7}$$

Now observe that

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{(t,s)\in[A,A+T]\times[t_0,t_0+T]} Z_u(t_1,t_2) > 2f(u)\Big)\\
&\quad\leqslant \mathbf{P}\bigg(\sup_{(t_1,t_2)\in[A,A+T]\times[t_0,t_0+T]} \overline{Z}_u(t_1,t_2) > \frac{2f(u)}{\sqrt{4-ac_1u^{-2}(t_0-(A+T))^\kappa}}\bigg).
\end{aligned}$$

Note that for any $(t_1, t_2), (s_1, s_2) \in [A, A+T] \times [t_0, t_0+T]$, we have

$$\begin{aligned}
\mathbf{Var}(\overline{Z}_u(t_1,t_2) - \overline{Z}_u(s_1,s_2)) &\leqslant \frac{\mathbf{Var}(Z_u(t_1,t_2) - Z_u(s_1,s_2))}{\sigma_{Z_u}(t_1,t_2)\sigma_{Z_u}(s_1,s_2)}\\
&\leqslant \frac{1}{2}\,\mathbf{E}\big((\overline{X}(t_1u^{-2/\kappa}) - \overline{X}(s_1u^{-2/\kappa})) + (\overline{X}(t_2u^{-2/\kappa}) - \overline{X}(s_2u^{-2/\kappa}))\big)^2\\
&\leqslant V_{\overline{X}}(t_1u^{-2/\kappa}, s_1u^{-2/\kappa}) + V_{\overline{X}}(t_2u^{-2/\kappa}, s_2u^{-2/\kappa})\\
&\leqslant (1-\exp(-4ac_2u^{-2}|t_1-s_1|^\kappa)) + (1-\exp(-4ac_2u^{-2}|t_2-s_2|^\kappa)),
\end{aligned} \tag{4.8}$$

where the next-to-last inequality follows from $(x+y)^2 \leqslant 2(x^2+y^2)$, and the last one follows from (4.6).

Denote $u^* = \frac{2f(u)}{\sqrt{4-ac_1u^{-2}(t_0-(A+T))^\kappa}}$ and let $\underline{c}, \overline{c} > 0$ be constants such that $\underline{c} \leqslant f(u)/u \leqslant \overline{c}$ for all $u \geqslant u_0$. Note (by (4.4)) that $f(u) \leqslant u^* \leqslant \sqrt{8/7}\, f(u)$ for $u \geqslant u_0$. Hence, $\underline{c}u \leqslant u^* \leqslant \sqrt{8/7}\,\overline{c}u$ for $u \geqslant u_0$, and therefore $u^{-2} \leqslant \frac{8}{7}\overline{c}^2(u^*)^{-2}$ for $u \geqslant u_0$.

Consider two independent, identically distributed centered stationary Gaussian processes $\{Z_{1,u^*}(t_1) : t_1 \geqslant 0\}$, $\{Z_{2,u^*}(t_2) : t_2 \geqslant 0\}$ with $R_{Z_{1,u^*}}(t_1, s_1) = \exp\big(-\frac{32}{7}a\overline{c}^2c_2(u^*)^{-2}|t_1-s_1|^\kappa\big)$ and let $\underline{Z}_{u^*}(t_1,t_2) = \frac{1}{\sqrt{2}}(Z_{1,u^*}(t_1) + Z_{2,u^*}(t_2))$. Hence, by (4.8), for any $(t_1, t_2), (s_1, s_2) \in [A, A+T] \times [t_0, t_0+T]$,

$$\begin{aligned}
&\mathbf{Var}(\overline{Z}_u(t_1,t_2) - \overline{Z}_u(s_1,s_2))\\
&\quad\leqslant (1-\exp(-4ac_2u^{-2}|t_1-s_1|^\kappa)) + (1-\exp(-4ac_2u^{-2}|t_2-s_2|^\kappa))\\
&\quad\leqslant \big(1-\exp\big(-\tfrac{32}{7}a\overline{c}^2c_2(u^*)^{-2}|t_1-s_1|^\kappa\big)\big)\\
&\qquad+ \big(1-\exp\big(-\tfrac{32}{7}a\overline{c}^2c_2(u^*)^{-2}|t_2-s_2|^\kappa\big)\big)\\
&\quad= \mathbf{Var}(\underline{Z}_{u^*}(t_1,t_2) - \underline{Z}_{u^*}(s_1,s_2)),
\end{aligned}$$

and due to Slepian's inequality, we obtain

$$\text{(4.9)}\qquad \begin{aligned}\mathbf{P}\Big(\sup_{(t_1,t_2)\in[A,A+T]\times[t_0,t_0+T]} \overline{Z}_u(t_1,t_2) > u^*\Big) &\leqslant \mathbf{P}\Big(\sup_{(t_1,t_2)\in[A,A+T]\times[t_0,t_0+T]} \underline{Z}_{u^*}(t_1,t_2) > u^*\Big)\\ &= \mathbf{P}\Big(\sup_{(t_1,t_2)\in[0,T]^2} \underline{Z}_{u^*}(t_1,t_2) > u^*\Big)\end{aligned}$$

as $u\to\infty$, where the equality follows from stationarity of $\underline{Z}_{u^*}(\cdot,\cdot)$. Now

$$\begin{aligned}\lim_{u^*\to\infty} \frac{\mathbf{P}(\sup_{(t_1,t_2)\in[0,T]^2} \underline{Z}_{u^*}(t_1,t_2) > u^*)}{\Psi(u^*)} &= \big(\mathcal{H}_{B_\kappa}((16a\overline{c}^2 c_2/7)^{1/\kappa}T)\big)^2\\ &\leqslant (\mathcal{H}_{B_\kappa}(1))^2 \max(1,(16a\overline{c}^2c_2/7)^{2/\kappa})T^2,\end{aligned}$$

where the equality follows from, e.g., [6, Theorem 2.1] (see also [12, Theorem 3.1]) and the inequality follows from the fact that $\mathcal{H}_{B_\kappa}(AT) \leqslant T\max(1,A)\mathcal{H}_{B_\kappa}(1)$ for any $T>1$ and $A>0$ [26, Corollary D.1]. Hence, there exists a constant F' (which does not depend on t_0, A, T) such that

$$\mathbf{P}\Big(\sup_{(t_1,t_2)\in[0,T]^2} \underline{Z}_{u^*}(t_1,t_2) > u^*\Big) \leqslant F'T^2\Psi(u^*)$$

for all $u^*\geqslant u_0^* = \underline{c}u_0$ (i.e. $u\geqslant u_0$). Thus

$$\text{(4.10)}\qquad \mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}(t) > f(u),\ \sup_{t\in[t_0,t_0+T]u^{-2/\kappa}} \overline{X}(t) > f(u)\Big) \leqslant F'T^2\Psi(u^*)$$

for $u\geqslant u_0$.

Since (in view of $\frac{1}{1-x}\geqslant 1+x$ for $x\geqslant 0$)

$$\begin{aligned}(u^*)^2 = \frac{4f^2(u)}{4-ac_1u^{-2}(t_0-(A+T))^\kappa} &\geqslant f^2(u) + \frac{ac_1}{4}\Big(\frac{f(u)}{u}\Big)^2(t_0-(A+T))^\kappa\\ &\geqslant f^2(u) + \frac{ac_1\underline{c}^2}{4}(t_0-(A+T))^\kappa,\end{aligned}$$

we have

$$\text{(4.11)}\qquad \begin{aligned}\Psi(u^*) &\leqslant \frac{\exp\big(-\frac12\big(f^2(u)+\frac{ac_1\underline{c}^2}{4}(t_0-(A+T))^\kappa\big)\big)}{\sqrt{2\pi}\sqrt{f^2(u)+\frac{ac_1\underline{c}^2}{4}(t_0-(A+T))^\kappa}}\\ &\leqslant \frac{\exp\big(-\frac12 f^2(u)\big)}{\sqrt{2\pi}\,f(u)}\exp\Big(-\frac{ac_1\underline{c}^2}{8}(t_0-(A+T))^\kappa\Big).\end{aligned}$$

Combination of (4.10) with (4.11) gives

$$\mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}(t) > f(u), \sup_{t\in[t_0,t_0+T]u^{-2/\kappa}} \overline{X}(t) > f(u)\Big) \leqslant F_1 F' T^2 \exp\Big(-\frac{ac_1\underline{c}^2}{8}(t_0-(A+T))^\kappa\Big)\Psi(f(u))$$

for any $u \geqslant u_0$ and some positive constant F_1 such that $\frac{\exp(-\frac{1}{2}f^2(u))}{\sqrt{2\pi}\,f(u)} \leqslant F_1\Psi(f(u))$ for $u > u_0$. This completes the proof with $F = F_1F'$ and $G = ac_1\underline{c}^2/8$. ∎

LEMMA 4.7. *With the notation of Lemma* 4.5*, there exist absolute constants* $F, G > 0$ *such that*

$$\mathbf{P}\Big(\sup_{t\in[A,A+T]u^{-2/\kappa}} \overline{X}(t) > u, \sup_{t\in[A+T,A+2T]u^{-2/\kappa}} \overline{X}(t) > u\Big) \leqslant F\big(T^2\exp(-G\sqrt{T^\kappa}) + \sqrt{T}\big)\Psi(u)$$

for all $A > 0$, $T > 1$ *and any* $u \geqslant u_0 = (2ac_2)^2(A+2T)^{\kappa/2}$, *i.e.* $(A+2T)u^{-2/\kappa} \leqslant (2ac_2)^{-1/\kappa}$.

Proof. Let $u_0 = (2ac_2)^2(A+2T)^{\kappa/2}$ and $\overline{X}_u(t) = \overline{X}(tu^{-2/\kappa})$. We have

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{t\in[A,A+T]} \overline{X}_u(t) > u, \sup_{t\in[A+T,A+2T]} \overline{X}_u(t) > u\Big)\\
&= \mathbf{P}\Big(\sup_{t\in[A,A+T]} \overline{X}_u(t) > u,\\
&\qquad \sup_{t\in[A+T,A+T+\sqrt{T}]} \overline{X}_u(t) > u \vee \sup_{t\in[A+T+\sqrt{T},A+2T]} \overline{X}_u(t) > u\Big)\\
&\leqslant \mathbf{P}\Big(\sup_{t\in[A,A+T]} \overline{X}_u(t) > u, \sup_{t\in[A+T+\sqrt{T},A+2T+\sqrt{T}]} \overline{X}_u(t) > u\Big)\\
&\quad + \mathbf{P}\Big(\sup_{t\in[A+T,A+T+\sqrt{T}]} \overline{X}_u(t) > u\Big)\\
&\leqslant F_1T^2\exp(-G\sqrt{T^\kappa})\Psi(u) + \mathbf{P}\Big(\sup_{t\in[A+T,A+T+\sqrt{T}]} \overline{X}_u(t) > u\Big),
\end{aligned}$$

where the last inequality follows from Lemma 4.6 with $t_0 = A+T+\sqrt{T}$. Applying Lemma 4.5(i) and Proposition 2.1 (choose $b = 0$) to $\mathbf{P}(\sup_{t\in[A+T,A+T+\sqrt{T}]} \overline{X}_u(t) > u)$, we finally obtain, for sufficiently large $u \geqslant u_0$,

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{t\in[A,A+T]}\overline{X}_u(t)>u,\ \sup_{t\in[A+T,A+2T]}\overline{X}_u(t)>u\Big)\\
&\quad\leqslant F_1T^2\exp(-G\sqrt{T^\kappa})\Psi(u)+\mathcal{H}_{B_\kappa}(a^{1/\kappa}\sqrt{T})\Psi(u)(1+o(1))\\
&\quad\leqslant F_1T^2\exp(-G\sqrt{T^\kappa})\Psi(u)+\max(1,a^{1/\kappa})\mathcal{H}_{B_\kappa}(1)\sqrt{T}\,\Psi(u)(1+o(1))\\
&\quad\leqslant F\big(T^2\exp(-G\sqrt{T^\kappa})+\sqrt{T}\big)\Psi(u)
\end{aligned}$$

for some constant $F>0$, where the next-to-last inequality follows from subadditivity of $\mathcal{H}_{B_\kappa}(\cdot)$ [26, Corollary D.1]. ■

Proof of Theorem 3.4. First, we prove the conclusion of the theorem for $\hat{Y}\in\mathbf{S}(\kappa,\kappa,c_Y(\kappa/\alpha)^\kappa)$, where $\hat{Y}(t)=Y(t^{\kappa/\alpha})$. Let c_1,c_2 be the constants of Lemma 4.3. Consider a centered Gaussian process $\{\overline{X}(t):t\geqslant 0\}$ with $R_{\overline{X}}(t,s)=\exp(-V_{\hat{Y}}(t,s))$ and let $\overline{X}_u(t)=\overline{X}(tu^{-2/\kappa})$. Let $n\in\mathbb{N}$ and choose $\epsilon_n\in(0,1)$, $\delta_{\epsilon_n}=1/n$ so that the conclusion of Lemma 4.5(ii) holds. We find a lower bound and an upper bound separately.

Upper bound. Let $T\in\mathbb{N}$ be such that $T>n$. For any $u>0$,

$$\begin{aligned}
&\mathbf{P}\Big(\sup_{t\in[0,T^2]}\overline{X}_u(t)>u\Big)\\
&\qquad\leqslant\mathbf{P}\Big(\sup_{t\in[0,nT]}\overline{X}_u(t)>u\Big)+\sum_{k=n}^{T-1}\mathbf{P}\Big(\sup_{t\in[kT,(k+1)T]}\overline{X}_u(t)>u\Big).
\end{aligned}$$

Applying Lemma 4.5 to the right side, by Proposition 2.1 we obtain

$$\mathcal{H}_{\hat{Y}}(T^2)\leqslant\mathcal{H}_{B_\kappa}(c_2^{1/\kappa}nT)+(T-n)\mathcal{H}_{B_\kappa}\big(((1+\epsilon_n)c_{\hat{Y}})^{1/\kappa}T\big),$$

and hence

$$\frac{\mathcal{H}_{\hat{Y}}(T^2)}{T^2}\leqslant\frac{\mathcal{H}_{B_\kappa}(c_2^{1/\kappa}nT)}{T^2}+\frac{((1+\epsilon_n)c_{\hat{Y}})^{1/\kappa}T^2}{T^2}\,\frac{\mathcal{H}_{B_\kappa}(((1+\epsilon_n)c_{\hat{Y}})^{1/\kappa}T)}{((1+\epsilon_n)c_{\hat{Y}})^{1/\kappa}T}.\tag{4.12}$$

Since $\lim_{S\to\infty}S^{-1}\mathcal{H}_{B_\kappa}(S)=\mathcal{H}_{B_\kappa}$, after letting $T\to\infty$ in (4.12) we get

$$\limsup_{T\to\infty}\frac{\mathcal{H}_{\hat{Y}}(T)}{T}\leqslant((1+\epsilon_n)c_{\hat{Y}})^{1/\kappa}\mathcal{H}_{B_\kappa}.$$

Since this bound holds for any $\epsilon_n\to 0^+$, we have

$$\limsup_{T\to\infty}\frac{\mathcal{H}_{\hat{Y}}(T)}{T}\leqslant(c_{\hat{Y}})^{1/\kappa}\mathcal{H}_{B_\kappa}.\tag{4.13}$$

Lower bound. For $T \in \mathbb{N}$ such that $T > n$, with $\Delta_k = [kT, (k+1)T]$, again from Bonferroni's inequality we deduce that for any $u > 0$,

$$\begin{aligned}
(4.14)\qquad & \mathbf{P}\Big(\sup_{t\in[0,T^2]} \overline{X}_u(t) > u\Big) \geqslant \mathbf{P}\Big(\sup_{t\in[nT,T^2]} \overline{X}_u(t) > u\Big) \\
& \geqslant \sum_{k=n}^{T-1} \mathbf{P}\Big(\sup_{t\in\Delta_k} \overline{X}_u(t) > u\Big) - \sum_{1\leqslant k<l\leqslant T-1} \mathbf{P}\Big(\sup_{t\in\Delta_k} \overline{X}_u(t) > u, \sup_{t\in\Delta_l} \overline{X}_u(t) > u\Big) \\
& \geqslant \sum_{k=n}^{T-1} \mathbf{P}\Big(\sup_{t\in\Delta_k} \overline{X}_u(t) > u\Big) - \Sigma_1 - \Sigma_2,
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1 &= \sum_{k=1}^{T-2} \mathbf{P}\Big(\sup_{t\in\Delta_k} \overline{X}_u(t) > u, \sup_{t\in\Delta_{k+1}} \overline{X}_u(t) > u\Big), \\
\Sigma_2 &= \sum_{1\leqslant k<l\neq k+1}^{T-1} \mathbf{P}\Big(\sup_{t\in\Delta_k} \overline{X}_u(t) > u, \sup_{t\in\Delta_l} \overline{X}_u(t) > u\Big).
\end{aligned}$$

By Lemma 4.5(ii) and Proposition 2.1, as $u \to \infty$,

(4.15)
$$\sum_{k=n}^{T-1} \mathbf{P}\Big(\sup_{t\in\Delta_k} \overline{X}_u(t) > u\Big) \geqslant (T-n)\mathcal{H}_{B_\kappa}\big(((1-\epsilon_n)c_{\hat{Y}})^{1/\kappa} T\big)\Psi(u)(1+o(1)).$$

From Lemma 4.7, for sufficiently large u, we have

$$\Sigma_1 \leqslant TF_1\big(T^2\exp(-G_1\sqrt{T^\kappa}) + \sqrt{T}\big)\Psi(u). \tag{4.16}$$

From Lemma 4.6, for sufficiently large u,

$$\Sigma_2 \leqslant T^2 F_2 T^2 \exp(-G_2 T^\kappa)\Psi(u). \tag{4.17}$$

Inserting (4.15)–(4.17) in (4.14) (and using Proposition 2.1), we obtain

$$\begin{aligned}
\frac{\mathcal{H}_{\hat{Y}}(T^2)}{T^2} \geqslant{} & \frac{(T-n)\mathcal{H}_{B_\kappa}(((1-\epsilon_n)c_{\hat{Y}})^{1/\kappa}T)}{T^2} \\
& - \frac{F_1(T^3\exp(-G_1\sqrt{T^\kappa}) + T^{3/2}) + F_2T^4\exp(-G_2T^\kappa)}{T^2}.
\end{aligned}$$

Letting $T \to \infty$ and then $\epsilon_n \to 0^+$ yields

$$\liminf_{T\to\infty} \frac{\mathcal{H}_{\hat{Y}}(T)}{T} \geqslant (c_{\hat{Y}})^{1/\kappa}\mathcal{H}_{B_\kappa}. \tag{4.18}$$

From the upper bound (4.13) and the lower bound (4.18) we conclude that

$$\lim_{T\to\infty} \frac{\mathcal{H}_{\hat{Y}}(T)}{T} = (c_{\hat{Y}})^{1/\kappa}\mathcal{H}_{B_\kappa}.$$

For $Y \in \mathbf{S}(\alpha, \kappa, c_Y)$, $c_{\hat{Y}} = c_Y(\kappa/\alpha)^\kappa$ and from Lemma 4.1(ii) it follows that

$$\frac{\kappa}{\alpha}(c_Y)^{1/\kappa}\mathcal{H}_{B_\kappa} = (c_{\hat{Y}})^{1/\kappa}\mathcal{H}_{B_\kappa} = \lim_{T\to\infty}\frac{\mathcal{H}_{\hat{Y}}(T)}{T}$$
$$= \lim_{T\to\infty}\frac{\mathcal{H}_Y(T^{\kappa/\alpha})}{T} = \lim_{T\to\infty}\frac{\mathcal{H}_Y(T)}{T^{\alpha/\kappa}}. \blacksquare$$

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REFERENCES

[1] R. Adler, *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes*, Inst. Math. Statist., Hayward, CA, 1990.

[2] J. M. P. Albin and H. Choi, *A new proof of an old result by Pickands*, Electron. Comm. Probab. 15 (2010), 339–345.

[3] L. Bai, K. Dębicki, E. Hashorva, and L. Luo, *On generalised Piterbarg constants*, Methodology Computing Appl. Probab. 20 (2018), 137–164.

[4] T. Bojdecki, L. G. Gorostiza, and A. Talarczyk, *Sub-fractional Brownian motion and its relation to occupation times*, Statist. Probab. Lett. 69 (2004), 405–419.

[5] K. Burnecki and Z. Michna, *Simulation of Pickands constants*, Probab. Math. Statist. 22 (2002), 193–199.

[6] K. Dębicki, *Ruin probability for Gaussian integrated processes*, Stoch. Process. Appl. 98 (2002), 151–174.

[7] K. Dębicki, S. Engelke, and E. Hashorva, *Generalized Pickands constants and stationary max-stable processes*, Extremes 20 (2017), 493–517.

[8] K. Dębicki and E. Hashorva, *On extremal index of max-stable processes*, Probab. Math. Statist. 27 (2017), 299–317.

[9] K. Dębicki and E. Hashorva, *Approximation of supremum of max-stable stationary processes & Pickands constants*, J. Appl. Probab. 33 (2020), 444–464.

[10] K. Dębicki, E. Hashorva, L. Ji, and T. Rolski, *Extremal behavior of hitting a cone by correlated Brownian motion with drift*, Stoch. Process. Appl. 128 (2018), 4171–4206.

[11] K. Dębicki, E. Hashorva, L. Ji, and K. Tabiś, *Extremes of vector-valued Gaussian processes: Exact asymptotics*, Stoch. Process. Appl. 125 (2015), 4039–4065.

[12] K. Dębicki, E. Hashorva, and P. Liu, *Uniform tail approximation of homogeneous functionals of Gaussian fields*, Adv. Appl. Probab. 49 (2017), 1037–1066.

[13] K. Dębicki and M. Mandjes, *Exact overflow asymptotics for queues with many Gaussian inputs*, J. Appl. Probab. 40 (2003), 704–720.

[14] K. Dębicki, Z. Michna, and T. Rolski, *Simulation of the asymptotic constant in some fluid models*, Stoch. Models 19 (2003), 407–423.

[15] K. Dębicki and K. Tabiś, *Extremes of the time-average of stationary Gaussian processes*, Stoch. Process. Appl. 121 (2011), 2049–2063.

[16] A. B. Dieker and T. Mikosch, *Exact simulation of Brown–Resnick random fields at a finite number of locations*, Extremes 18 (2015), 301–314.

[17] K. Dzhaparidze and H. Van Zanten, *A series expansion of fractional Brownian motion*, Probab. Theory Related Fields 130 (2004), 39–55.

[18] E. Hashorva, S. Kobelkov, and V. I. Piterbarg, *On maximum of Gaussian process with unique maximum point of its variance*, arXiv:1901.09753 (2019).

[19] Ch. Houdré and J. Villa, *An example of infinite dimensional quasi-helix*, in: Stochastic Models (Mexico City, 2002), Contemp. Math. 336, Amer. Math. Soc., 2003, 195–202.

[20] D. G. Konstant and V. I. Piterbarg, *Extreme values of the cyclostationary Gaussian random process*, J. Appl. Probab. 30 (1993), 82–97.
[21] J. Lamperti, *Semi-stable stochastic processes*, Trans. Amer. Math. Soc. 104 (1962), 62–78.
[22] P. Lei and D. Nualart, *A decomposition of the bifractional Brownian motion and some applications*, Statist. Probab. Lett. 79 (2009), 619–624.
[23] W. Li and Q. Shao, *Lower tail probabilities for Gaussian processes*, Ann. Probab. 32 (2004), 216–242.
[24] Z. Michna, *Remarks on Pickands' theorem*, Probab. Math. Statist. 37 (2017), 373–393.
[25] J. Pickands, *Asymptotic properties of the maximum in a stationary Gaussian process*, Trans. Amer. Math. Soc. 145 (1969), 75–86.
[26] V. I. Piterbarg, *Asymptotic Methods in the Theory of Gaussian Processes and Fields*, Transl. Math. Monogr. 148, Amer. Math. Soc., 2012.
[27] V. I. Piterbarg and V. P. Prisiazhniuk, *Asymptotic analysis of the probability of large excursions for a nonstationary Gaussian process*, Teor. Veroyatnost. Mat. Statist. 18 (1978), 121–134 (in Russian).
[28] R. A. Vitale, *The Wills functional and Gaussian processes*, Ann. Probab. 24 (1996), 2172–2178.

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