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ENERGY OF TAUT STRINGS ACCOMPANYING RANDOM WALK

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Abstract. We consider the kinetic energy of the taut strings accompanying trajectories of a Wiener process and a random walk. Under certain assumptions on the band width, it is shown that the energy of a taut string accompanying a random walk within a band satisfies the same strong law of large numbers as proved earlier for a Wiener process and a fixed band width. New results for Wiener processes are also obtained.

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1. INTRODUCTION

Consider a time interval $[0, T]$ and some continuous functional boundaries $g_1(t) \leqslant g_2(t)$, $0 \leqslant t \leqslant T$. A *taut string* is a function h_* that has the remarkable property of universal optimization: for every convex function $\varphi(\cdot)$ it minimizes the functional

$$\int_0^T \varphi(h'(s))\, ds$$

over all absolutely continuous functions h having the same values at 0 and T and satisfying $g_1(t) \leqslant h(t) \leqslant g_2(t)$, $0 \leqslant t \leqslant T$. In particular, such functionals as the *kinetic energy* $\int_0^T h'(t)^2\, dt$ and the graph length $\int_0^T \sqrt{1 + h'(t)^2}\, dt$ are minimized by taut strings [10, Theorem 5.2, Remark 5.2], [9, Theorems 4.1, 5.1, 5.2], [19, Theorem 4.35, p. 141].

If the values at 0 and/or T are not fixed, the universality is maintained for the smaller set of *even* convex functions.

Recall that for every function φ a solution of this problem exists, while for strictly convex functions φ the solution is unique; however, if φ is not strictly

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convex, the uniqueness may break down. For example, for $\varphi(x) = |x|$ there exist curious alternative solutions called "lazy functions" [13]. In this article we deal only with the kinetic energy, therefore uniqueness is not a problem.

Taut strings appeared in the literature for the first time in G. Dantzig's article [1] in connection with some problems of optimal control. For further developments and applications of taut strings in optimal planning, discrete optimization, statistics, image processing, and information transmission, we refer to [2], [3], [9], [15], [16], [19], [20].

In [12], M. Lifshits and E. Setterqvist studied taut strings accompanying a Wiener process. This work opened the way to further studies of taut strings related to other random processes, other types of energy, and to other distances between the string and the approximated process [4], [5], [6], [18].

In this work, we will consider taut strings accompanying a *random walk* and extend to this case the results of [12], which we briefly recall now. Let us consider taut strings running through a band of constant width around a sample path of a Wiener process W, i.e. for some $r > 0$ we define the functional boundaries as follows: $g_1(t) := W(t) - r$, $g_2(t) := W(t) + r$ (see Fig. 1). The results of [12] show that, as $T \to \infty$, the string expends an asymptotically constant amount of the kinetic energy $r^{-2}\mathcal{C}^2$ per unit of time. More precise statements are cited below in Theorems 2.1 and 2.2.

In this article we establish an analogous result for the energy of taut strings going through bands of different widths, i.e., we consider a system of problems with boundaries $g_1(t) := W(t) - r_T$, $g_2(t) := W(t) + r_T$, $0 \leqslant t \leqslant T$. The precise statement is given in Theorem 2.3.

Then the result obtained is transferred to the case of random walk. Notice that the assumptions imposed on the function $T \mapsto r_T$ depend on the moment properties of the walk's step. The precise statement is given in Theorem 2.4.

All proofs are collected in Section 4.

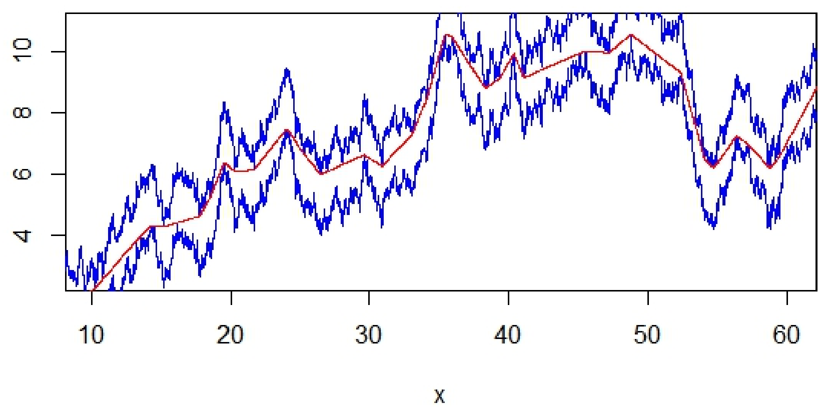

FIGURE 1. The taut string accompanying a Wiener process

2. ENERGY OF ACCOMPANYING TAUT STRINGS

2.1. Wiener process. Throughout the article we consider the uniform norms

$$\|h\|_T := \sup_{0\leqslant t\leqslant T} |h(t)|, \qquad h \in C[0,T],$$

and the Sobolev norms

$$|h|_T^2 := \int_0^T h'(t)^2\,dt, \qquad h \in AC[0,T],$$

where $AC[0,T]$ denotes the space of absolutely continuous functions defined on the time interval $[0,T]$. We call $|h|_T^2$ the *kinetic energy* or simply the energy of the function h.

Let W be a Wiener process. We are interested in the following approximation characteristics:

$$I_W(T,r) := \inf\{|h|_T : h \in AC[0,T],\ \|h-W\|_T \leqslant r,\ h(0)=0\},$$
$$I_W^0(T,r) := \inf\{|h|_T : h \in AC[0,T],\ \|h-W\|_T \leqslant r,\ h(0)=0,\ h(T)=W(T)\}.$$

The unique functions at which these infima are attained are called the *taut string* and the *taut string with fixed end*, respectively.

The main results obtained in [12] are as follows.

THEOREM 2.1. *There exists a constant $\mathcal{C} \in (0,\infty)$ such that, as $r_T/T^{1/2} \to 0$, we have*

$$\frac{r_T}{T^{1/2}} I_W(T,r_T) \xrightarrow{L_q} \mathcal{C}, \qquad \frac{r_T}{T^{1/2}} I_W^0(T,r_T) \xrightarrow{L_q} \mathcal{C}$$

for every $q > 0$.

THEOREM 2.2. *For every fixed $r > 0$, as $T \to \infty$, we have*

$$\frac{r}{T^{1/2}} I_W(T,r) \to \mathcal{C} \quad a.s., \qquad \frac{r}{T^{1/2}} I_W^0(T,r) \to \mathcal{C} \quad a.s.$$

The constant $\mathcal{C}^2$ is the energy per unit time that must be expended by an absolutely continuous trajectory if it is bound to stay within unit distance of a non-differentiable sample path of the Wiener process W. The precise value of $\mathcal{C}$ remains unknown, while computer simulation yields the approximate value $\mathcal{C} \approx 0.63$; moreover, in [12] the following theoretical bounds for $\mathcal{C}$ are obtained: $0.381 \leqslant \mathcal{C} \leqslant \pi/2$. An alternative theoretical approach to $\mathcal{C}$ is given in [18].

In the present work we need an intermediate result between Theorems 2.1 and 2.2: the band width will be varying as in Theorem 2.1 but the convergence with probability one will be obtained, as in Theorem 2.2. The range of admissible variation of the band width turns out to be slightly more narrow than in Theorem 2.1.

THEOREM 2.3. *Let W be a Wiener process. Assume that a band width r_T is non-decreasing but*

$$\frac{r_T\,(\ln\ln T)^{1/2}}{T^{1/2}} \to 0 \quad \textit{as } T\to\infty, \tag{2.1}$$

the left hand side being non-increasing. Then

$$\lim_{T\to\infty} \frac{r_T}{T^{1/2}} I_W(T,r_T) = \mathcal{C} \quad \textit{a.s.}, \tag{2.2}$$

$$\lim_{T\to\infty} \frac{r_T}{T^{1/2}} I^0_W(T,r_T) = \mathcal{C} \quad \textit{a.s.} \tag{2.3}$$

where $\mathcal{C}\in(0,\infty)$ is the constant appearing in Theorem 2.1.

REMARK 2.1. The claim of the theorem obviously remains true if we replace the band width r_T by $\rho_T = r_T(1+o(1))$. This observation essentially enables one to drop the monotonicity assumptions.

REMARK 2.2. It follows from the proof of the theorem that the claim (2.3) remains valid if the condition of arriving at the end point $W(T)$ is replaced by arrival at some given point $W(T)+x_T$ with $x_T = o(r_T)$.

REMARK 2.3. In Theorem 2.3 a system of bands of constant widths is considered. However, one can derive from it information about the energy of taut strings running through a band of varying width. Namely, suppose $T\mapsto r_T$ satisfies the assumptions of Theorem 2.3. Let

$$I_W(T,r.) := \inf\{|h|_T : h\in AC[0,T],\ h(0)=0,\ |h(t)-W(t)|\leqslant r_t,\ 0\leqslant t\leqslant T\}.$$

Then, as $T\to\infty$,

$$I_W(T,r.)^2 \sim \mathcal{C}^2 \int_0^T \frac{dt}{r_t^2} \quad \text{a.s.}$$

The proof requires a considerable amount of computation and will be provided in a subsequent publication.

Assumption (2.1) may look strange at first glance but the following proposition shows that the iterated logarithm is essential.

PROPOSITION 2.1. *Let $M>0$ and $r_T = M(T/\ln\ln T)^{1/2}$. Then*

$$\limsup_{T\to\infty} \frac{r_T}{T^{1/2}} I_W(T,r_T) \geqslant \sqrt{2}\,M \quad \textit{a.s.}$$

2.2. Random walk. Let $X_1, X_2, \dots$ be a sequence of i.i.d. real random variables. Define their partial sums by $S_0 := 0$ and

$$S_k := \sum_{j=1}^k X_j, \qquad k\geqslant 1.$$

Define on $[0, \infty)$ a random broken line $S(\cdot)$ as

$$S(t) := \begin{cases} S_k, & t = k,\ k = 0, 1, \dots, \\ (k+1-t)S_k + (t-k)S_{k+1}, & t \in (k, k+1),\ k = 0, 1, \dots. \end{cases}$$

We will consider the taut strings running through the band of width r around the broken line S. We introduce the approximation characteristics for S similar to those introduced previously for the Wiener process:

$$I_S(T, r) := \inf\{|h|_T : h \in AC[0, T],\ \|h - S\|_T \leqslant r,\ h(0) = 0\},$$
$$I_S^0(T, r) := \inf\{|h|_T : h \in AC[0, T],\ \|h - S\|_T \leqslant r,\ h(0) = 0,\ h(T) = S(T)\}.$$

In the following, just for simplicity, we only consider $T \in \mathbb{N}$. Then the following result on the energy of the taut string accompanying the random broken line S (random walk) is true.

THEOREM 2.4. *Let S be the random broken line based as above on the partial sums of i.i.d. random variables X_j having zero expectation and unit variance. Suppose each X_j has finite moment of order $p > 2$ and r_T satisfies assumption* (2.1) *and $T^{1/p} = O(r_T)$. Then*

$$\lim_{T \to \infty} \frac{r_T}{T^{1/2}} I_S(T, r_T) = \mathcal{C} \quad \text{a.s.},$$
$$\lim_{T \to \infty} \frac{r_T}{T^{1/2}} I_S^0(T, r_T) = \mathcal{C} \quad \text{a.s.},$$

where $\mathcal{C} \in (0, \infty)$ is the constant from Theorem 2.1.

Moreover, if the variables X_j have a finite exponential moment, then the above mentioned equalities are true under assumptions (2.1) *and $\ln T = o(r_T)$.*

3. A TOOL FOR TRANSFER TO RANDOM WALK

We now recall the main tool for the transfer of the results known for a Wiener process to a random walk.

Let $X = \{X_1, X_2, \dots\}$ be a sequence of independent random variables with finite second moments and let $Y = \{Y_1, Y_2, \dots\}$ be a sequence of independent Gaussian random variables such that each Y_j has the same expectation and variance as X_j. We want to construct (on some common probability space) the sequences $\bar{X} = \{\bar{X}_1, \bar{X}_2, \dots\}$ and $\bar{Y} = \{\bar{Y}_1, \bar{Y}_2, \dots\}$ equidistributed with X and Y, respectively, so that the discrepancy

$$\max_{1 \leqslant k \leqslant n} \Big| \sum_{j=1}^{k} \bar{X}_j - \sum_{j=1}^{k} \bar{Y}_j \Big|$$

is small in the sense of a.s. behavior: as $n \to \infty$, with probability one the discrepancy should increase no faster than some known function; the latter is defined by the moment characteristics of the sequence X. Such closeness of two sequences is called *strong approximation*.

The optimal rate of strong approximation for sums of i.i.d. random variables was obtained by J. Komlos, P. Major and G. Tusnady. Here is a statement of their result, called the KMT-theorem.

THEOREM 3.1. *Let $X = \{X_1, X_2, \dots\}$ be a sequence of i.i.d. random variables having finite moment of order $p > 2$. Then one may construct on some probability space a sequence $\bar{X} = \{\bar{X}_1, \bar{X}_2, \dots\}$ equidistributed with X and a sequence $\{\bar{Y}_1, \bar{Y}_2, \dots\}$ of independent Gaussian random variables having the same expectation and variance such that*

$$\sum_{j=1}^{n} \bar{X}_j - \sum_{j=1}^{n} \bar{Y}_j = o(n^{1/p}) \quad \textit{a.s.}$$

If the variables X_j have a finite exponential moment, then one can obtain

$$\sum_{j=1}^{n} \bar{X}_j - \sum_{j=1}^{n} \bar{Y}_j = O(\ln n) \quad \textit{a.s.}$$

For the first claim of the theorem, see [14, p. 214] for $2 < p < 3$ and [8] for $p > 2$. For the second claim see [7, p. 112]. We also refer to [17], [21] for various extensions of the KMT-approach.

4. PROOFS

4.1. Proof of Theorem 2.3. Before starting the proof, recall the following useful technical result. Let $m(T, r)$ denote the median of the random variable $I_W(T, r)$. The concentration of $I_W(T, r)$ and the limiting properties of $m(T, r)$ are described in the following lemma [12, Corollary 3.2 and p. 408].

LEMMA 4.1. (a) *For all $T, r, \rho > 0$,*

$$\mathbb{P}\left(|I_W(T, r) - m(T, r)| \geqslant \rho\right) \leqslant \exp\{-\rho^2/2\}. \tag{4.1}$$

(b) *If a function $T \mapsto r_T$ satisfies $\lim_{T\to\infty} r_T/T^{1/2} = 0$, then*

$$\lim_{T\to\infty} \frac{r_T}{T^{1/2}} m(T, r) = \mathcal{C}.$$

Now we are ready to start proving (2.2) and (2.3).

STEP 1: *the proof of* (2.2). Consider first an exponentially growing sequence of time instants $T_k := a^k$, where $a > 1$ is an arbitrary fixed number, as well as the corresponding sequence of band widths r_{T_k}. By using the concentration inequality (4.1) together with (2.1), for arbitrary $\varepsilon, M > 0$ and sufficiently large k_0 we have

$$\sum_{k=k_0}^{\infty} \mathbb{P}\biggl(\frac{r_{T_k}}{T_k^{1/2}}|I_W(T_k, r_{T_k}a^{1/2}) - m(T_k, r_{T_k}a^{1/2})| > \varepsilon\biggr)$$
$$\leqslant \sum_{k=k_0}^{\infty} \exp\biggl\{-\frac{T_k\varepsilon^2}{2r_{T_k}^2}\biggr\} \leqslant \sum_{k=k_0}^{\infty} \exp\{-\varepsilon^2 M \ln\ln T_k/2\}$$
$$= \sum_{k=k_0}^{\infty} \exp\{-\varepsilon^2 M(\ln\ln a + \ln k)/2\},$$

and this sum turns out to be finite if we choose $M = M(\varepsilon) > 2\varepsilon^{-2}$. By the Borel–Cantelli lemma we obtain

$$\lim_{k\to\infty} \frac{r_{T_k}}{T_k^{1/2}}\bigl(I_W(T_k, r_{T_k}a^{1/2}) - m(T_k, r_{T_k}a^{1/2})\bigr) = 0 \quad \text{a.s.} \tag{4.2}$$

Next, by the assumption of our theorem, $a^{1/2} r_{T_k}/T_k^{1/2} \to 0$, and therefore Lemma 4.1 yields the convergence of medians,

$$\frac{r_{T_k}}{T_k^{1/2}} m(T_k, a^{1/2} r_{T_k}) \to a^{-1/2}\mathcal{C}.$$

Taking into account (4.2), we obtain

$$\lim_{k\to\infty} \frac{r_{T_k}}{T_k^{1/2}} I_W(T_k, a^{1/2} r_{T_k}) = a^{-1/2}\mathcal{C} \quad \text{a.s.} \tag{4.3}$$

Similarly, considering the sequence $a^{-1/2} r_{T_k}$, we get

$$\lim_{k\to\infty} \frac{r_{T_k}}{T_k^{1/2}} I_W(T_k, a^{-1/2} r_{T_k}) = a^{1/2}\mathcal{C} \quad \text{a.s.} \tag{4.4}$$

Let us now consider arbitrary values of the time parameter T. By the assumption of our theorem, the function

$$T \mapsto \frac{r_T}{T^{1/2}} = \frac{r_T(\ln\ln T)^{1/2}}{T^{1/2}} \cdot \frac{1}{(\ln\ln T)^{1/2}}$$

is non-increasing. Therefore, for every $T \in [T_{k-1}, T_k]$,

$$\frac{r_{T_k}}{T_k^{1/2}} \leqslant \frac{r_T}{T^{1/2}} \leqslant \frac{r_{T_{k-1}}}{T_{k-1}^{1/2}},$$

hence

$$a^{-1/2} r_{T_k} \leqslant r_T \leqslant a^{1/2} r_{T_{k-1}}.$$

By using the first of these inequalities as well as the fact that $I_W(\cdot, r)$ is non-decreasing, while $I_W(T, \cdot)$ is non-increasing, we obtain the bound

$$I_W(T, r_T) \leqslant I_W(T_k, a^{-1/2} r_{T_k}).$$

Since r_T is non-decreasing, the limiting relation (4.4) yields

$$\limsup_{T\to\infty} \frac{r_T I_W(T, r_T)}{T^{1/2}} \leqslant \limsup_{k\to\infty} \frac{r_{T_k} I_W(T_k, a^{-1/2} r_{T_k})}{(T_k/a)^{1/2}} \leqslant a\mathcal{C}.$$

Similarly, on the other hand,

$$I_W(T, r_T) \geqslant I_W(T_{k-1}, a^{1/2} r_{T_{k-1}}),$$

and it follows from (4.3) that

$$\liminf_{T\to\infty} \frac{r_T I_W(T, r_T)}{T^{1/2}} \geqslant \liminf_{k\to\infty} \frac{r_{T_{k-1}} I_W(T_{k-1}, a^{1/2} r_{T_{k-1}})}{(aT_{k-1})^{1/2}} \geqslant a^{-1}\mathcal{C}.$$

By letting a tend to 1 in the asymptotic bounds obtained, we get (2.2).

STEP 2: *the proof of* (2.3). We will now check the convergence of $I_W^0(T, r_T)$. It follows from the definition that

$$I_W(T, r) \leqslant I_W^0(T, r) \quad \text{for all } T, r > 0.$$

This yields the lower bound in (2.3):

$$\liminf_{T\to\infty} \frac{r_T}{T^{1/2}} I_W^0(T, r_T) \geqslant \lim_{T\to\infty} \frac{r_T}{T^{1/2}} I_W(T, r_T) = \mathcal{C}.$$

The proof of the upper bound requires another approach because $I_W^0(\cdot, r)$ is not necessarily monotone.

Let us fix a $\delta \in (0, 1/3)$. For each interval length T we decompose $T = T_* + L_T$, where $L_T \approx r_T^2 \ll T$ is small compared to T. The choice of L_T will be made precise later on.

Let h_1 be the taut string at which $I_W(T_*, (1-3\delta)r_T)$ is attained, i.e. $h_1(0) = 0$ and

$$\|h_1 - W\|_{T_*} \leqslant (1-3\delta) r_T, \tag{4.5}$$

$$|h_1|_{T_*} = I_W(T_*, (1-3\delta)r_T) \leqslant I_W(T, (1-3\delta)r_T). \tag{4.6}$$

Further, let us introduce an auxiliary Wiener process

$$\widetilde{W}(s) := W(T_* + s) - W(T_*), \qquad 0 \leqslant s \leqslant T - T_* = L_T,$$

and approximate it with the taut string h_2 at which $I_{\widetilde{W}}(L_T, \delta r_T)$ is attained, i.e. $h_2(0) = 0$ and

$$\|h_2 - \widetilde{W}\|_{L_T} \leqslant \delta r_T, \tag{4.7}$$

$$|h_2|_{L_T} = I_{\widetilde{W}}(L_T, \delta r_T). \tag{4.8}$$

Finally, we define an approximation function h with fixed end $h(T) = W(T)$ by

$$h(t) := \begin{cases} h_1(t), & 0 \leqslant t \leqslant T_*, \\ h_1(T_*) + h_2(t - T_*) + (t - T_*)\nu, & T_* \leqslant t \leqslant T, \end{cases}$$

where the constant ν is found from the equation

$$h(T) = h_1(T_*) + h_2(L_T) + L_T\nu = W(T),$$

i.e.

$$\nu = \frac{W(T) - h_1(T_*) - h_2(L_T)}{L_T}.$$

Let us remark that h is continuous at the transition point T_*, since $h_2(0) = 0$. Moreover, h is *absolutely* continuous, being the result of continuous gluing of two absolutely continuous pieces.

Furthermore, by (4.5) and (4.7) we have

$$\begin{aligned} |\nu| &= \frac{|[W(T_*) - h_1(T_*)] + [W(T) - W(T_*) - h_2(L_T)]|}{L_T} \\ &\leqslant \frac{|W(T_*) - h_1(T_*)| + |\widetilde{W}(L_T) - h_2(L_T)|}{L_T} \\ &\leqslant \frac{(1-3\delta)r_T + \delta r_T}{L_T} = \frac{(1-2\delta)r_T}{L_T}. \end{aligned} \tag{4.9}$$

Let us evaluate the uniform distance between h and W. For $0 \leqslant t \leqslant T_*$, by using (4.5) we have

$$|h(t) - W(t)| = |h_1(t) - W(t)| \leqslant \|h_1 - W\|_{T_*} \leqslant (1-3\delta)r_T.$$

For $T_* \leqslant t \leqslant T$, we use an identity that is easy to verify,

$$h(t) - W(t) = h_2(t - T_*) - \widetilde{W}(t - T_*) - \big[L_T - (t - T_*)\big]\nu + \widetilde{W}(L_T) - h_2(L_T).$$

By using (4.7) and (4.9), we obtain

$$|h(t) - W(t)| \leqslant 2|h_2 - \widetilde{W}|_{L_T} + L_T|\nu| \leqslant 2\delta r_T + (1-2\delta)r_T \leqslant r_T.$$

By merging the estimates for both intervals, we find

$$\|h - W\|_T \leqslant r_T.$$

Let us now evaluate the energy of the function h. In view of (4.6), (4.8), and (4.9), we have

$$\begin{aligned} |h|_T^2 &= \int_0^{T_*} h'(t)^2\,dt + \int_{T_*}^{T} h'(t)^2\,dt \\ &= \int_0^{T_*} h_1'(t)^2\,dt + \int_{T_*}^{T} (h_2'(t - T_*) + \nu)^2\,dt \\ &\leqslant \int_0^{T_*} h_1'(t)^2\,dt + 2\int_{T_*}^{T} h_2'(t - T_*)^2\,dt + 2\nu^2 L_T \\ &\leqslant I_W(T, (1-3\delta)r_T)^2 + 2I_{\widetilde{W}}(L_T, \delta r_T)^2 + \frac{2r_T^2}{L_T}. \end{aligned}$$

We may conclude that

$$I_W^0(T, r_T)^2 \leqslant |h|_T^2 \leqslant I_W(T, (1-3\delta)r_T)^2 + 2I_{\widetilde{W}}(L_T, \delta r_T)^2 + \frac{2r_T^2}{L_T}.$$

Since by (2.2),

$$\lim_{T\to\infty} \frac{r_T^2}{T} I_W(T, (1-3\delta)r_T)^2 = (1-3\delta)^{-2}\mathcal{C}^2 \quad \text{a.s.},$$

and δ can be chosen arbitrarily small, to get the upper bound in (2.3) it is enough to show that

$$\lim_{T\to\infty} \frac{r_T^2}{T} I_{\widetilde{W}}(L_T, \delta r_T)^2 = 0 \quad \text{a.s.,} \tag{4.10}$$

$$\lim_{T\to\infty} \frac{r_T^2}{T}\,\frac{r_T^2}{L_T} = 0. \tag{4.11}$$

Our next goal is a reduction to a discrete set of time instants. To this end, we construct a time sequence inductively by letting $T_0 := 1$, $T_{k+1} := T_k + r_{T_k}^2$. We denote $r_k := r_{T_k}$ and introduce the auxiliary Wiener processes by $\widetilde{W}_k(s) := W(T_k + s) - W(T_k)$.

From the regularity conditions of our theorem, it easily follows that

$$\lim_{k\to\infty} T_k = \infty, \qquad \lim_{k\to\infty} \frac{T_{k+1}}{T_k} = 1, \qquad \lim_{k\to\infty} \frac{r_{k+1}}{r_k} = 1.$$

Let us now specify the parameters of the previous construction by letting, for $T \in [T_{k+1}, T_{k+2})$,

$$T_* = T_*(T) := T_k, \qquad L_T := T - T_k.$$

Then for all sufficiently large k we have

$$\frac{r_T}{T^{1/2}} \leqslant \frac{2r_k}{T_k^{1/2}}$$

and

$$L_T \geqslant T_{k+1} - T_k = r_k^2 > r_{k+2}^2/2 \geqslant r_T^2/2,$$

and (4.11) follows from (2.1). On the other hand, for all large k,

$$L_T \leqslant T_{k+2} - T_k = r_k^2 + r_{k+1}^2 < 3r_k^2,$$

which in combination with (4.10) yields

$$I_{\widetilde{W}_k}(L_T, \delta r_T) \leqslant I_{\widetilde{W}_k}(L_T, \delta r_k) \leqslant I_{\widetilde{W}_k}(3r_k^2, \delta r_k).$$

Therefore, to prove (4.10) it is sufficient to verify the relation

$$\lim_{k\to\infty} \frac{r_k}{T_k^{1/2}} I_{\widetilde{W}_k}(3r_k^2, \delta r_k)^2 = 0 \quad \text{a.s.}$$

which only concerns the discrete sequence of time instants.

Notice that the variables $I_{\widetilde{W}_k}(3r_k^2, \delta r_k)$ are equidistributed with $I_W(3,\delta)$ by the self-similarity of the Wiener process. According to the Borel–Cantelli lemma, it is enough to check that for every $\varepsilon > 0$,

$$\sum_k \mathbb{P}\biggl(I_W(3,\delta) \geqslant \varepsilon \frac{T_k^{1/2}}{r_k} \biggr) < \infty. \tag{4.12}$$

By using the Gaussian concentration estimate (4.1) we only need to check that for every $h > 0$,

$$\sum_k \exp\biggl\{ -\frac{hT_k}{r_k^2} \biggr\} < \infty.$$

By using the fact that the functions $T \mapsto r_T$ and $T \mapsto T/r_T^2$ are non-decreasing, we get

$$\begin{aligned}\int_{T_{k-1}}^{T_k} \exp\biggl\{ -\frac{hT}{r_T^2} \biggr\} r_T^{-2}\, dT &\geqslant (T_k - T_{k-1}) \exp\biggl\{ -\frac{hT_k}{r_k^2} \biggr\} r_k^{-2} \\ &= \frac{r_{k-1}^2}{r_k^2} \exp\biggl\{ -\frac{hT_k}{r_k^2} \biggr\} = \exp\biggl\{ -\frac{hT_k}{r_k^2} \biggr\} (1 + o(1)).\end{aligned}$$

It remains to prove that

$$\int^{\infty} \exp\biggl\{ -\frac{hT}{r_T^2} \biggr\} r_T^{-2}\, dT < \infty.$$

Let us write the integrand as

$$\exp\biggl\{ -\frac{hT}{r_T^2} \biggr\} r_T^{-2} = \exp\biggl\{ -\frac{hT}{r_T^2} \biggr\} h \frac{T}{r_T^2} \cdot h^{-1} T^{-1} = \exp\{-u\} u \cdot h^{-1} T^{-1},$$

where $u := hT/r_T^2$. Recall that, according to (2.1), we have $u > 2 \ln\ln T$ for sufficiently large T. Therefore,

$$\exp\{-u\} u \leqslant 2 \exp\{-2 \ln\ln T\} \ln\ln T = 2 (\ln T)^{-2} \ln\ln T,$$

and we arrive at an estimate having the form of a convergent integral,

$$2h^{-1} \int^{\infty} \frac{\ln\ln T}{(\ln T)^2} T^{-1}\, dT < \infty.$$

Thus (4.12) is proved, and hence (4.10) also follows. ■

4.2. Proof of Proposition 2.1. Our reasoning is based on the following elementary bound: for all $T, r > 0$ and every function $w \in C[0,T]$,

$$I_w(T,r)^2 \geqslant \frac{[|w(T)-w(0)|-r]_+^2}{T}. \tag{4.13}$$

Indeed, suppose $\|h-w\|_T \leqslant r$ and $h(0)=w(0)$. Then the Hölder inequality yields

$$\begin{aligned} |h|_T^2 &= \int_0^T h'(s)^2\,ds \geqslant \frac{(\int_0^T h'(s)\,ds)^2}{T} \\ &= \frac{(h(T)-h(0))^2}{T} \geqslant \frac{[|w(T)-w(0)|-r]_+^2}{T}, \end{aligned}$$

and (4.13) follows.

Applying now (4.13) to $w=W, r=r_T$ in the framework of our proposition, and using the law of the iterated logarithm for a Wiener process [11, Ch. 17], we obtain

$$\begin{aligned} \limsup_{T\to\infty} \frac{r_T}{T^{1/2}} I_W(T,r_T) &\geqslant M \limsup_{T\to\infty} \frac{1}{(\ln\ln T)^{1/2}} \frac{[|W(T)|-r_T]_+}{T^{1/2}} \\ &= M \limsup_{T\to\infty} \frac{|W(T)|}{(T\ln\ln T)^{1/2}} = M\sqrt{2}. \blacksquare \end{aligned}$$

4.3. Proof of Theorem 2.4. Recall that for every integer $k \in [0,T]$ we consider the partial sum S_k of i.i.d. random variables X_j, with zero expectations, unit variances and finite moment of order $p > 2$. We approximate these sums by the partial sums W_k of independent standard normal random variables $\bar{Y}_j$ satisfying the assumptions of Theorem 3.1. The random broken line S is defined by the nodes (k, S_k) as follows:

$$S(t) = \begin{cases} S_k, & t=k,\ k=0,1,\dots,T, \\ (k+1-t)S_k + (t-k)S_{k+1}, & t \in (k,k+1),\ k=0,1,\dots,T-1. \end{cases}$$

The random broken line $W(t)$ is defined analogously for the nodes (k, W_k). Then Theorem 3.1 yields $S_k - W_k = o(k^{1/p})$ with probability one. This means that, as $T \to \infty$, we have

$$\|S-W\|_T = \sup_{0\leqslant t\leqslant T} |S(t)-W(t)| = \max_{0\leqslant k\leqslant T} |S_k - W_k| = o(T^{1/p}) \quad \text{a.s.} \tag{4.14}$$

Now we build a Wiener process $\widetilde{W}(t)$ upon the broken line $W(t)$, by adding independently a Brownian bridge $W_k^0(t-k)$ to every segment, connecting the nodes (k, W_k) and $(k+1, W_{k+1})$. Let $I_{\widetilde{W}}(T, r_T)^2$ denote the energy of the taut string running in the band of width r_T around the process $\widetilde{W}$ on the time interval $[0,T]$.

An upper bound for the energy. First of all, notice that (4.14) and the assumptions of our theorem provide

$$\|S-W\|_T = o(T^{1/p}) = o(r_T).$$

Let $\rho_T := r_T - \|S-W\|_T = r_T(1+o(1))$.

Choose a string h accompanying the Wiener process $\widetilde{W}$ so that $\|h-\widetilde{W}\|_T \leqslant \rho_T$ and $|h|_T = I_{\widetilde{W}}(T,\rho_T)$. Let $\widehat{h}$ be the random broken line corresponding to the nodes $(k,h(k))$, $0 \leqslant k \leqslant T$. Then

$$\begin{aligned}\|\widehat{h}-S\|_T &\leqslant \|\widehat{h}-W\|_T + \|W-S\|_T \leqslant \|h-\widetilde{W}\|_T + \|W-S\|_T \\ &\leqslant \rho_T + \|W-S\|_T = r_T\end{aligned}$$

and

$$\begin{aligned}|\widehat{h}|_T^2 &= \sum_{k=0}^{T-1}\int_k^{k+1} \widehat{h}'(s)^2\,ds = \sum_{k=0}^{T-1}(\widehat{h}(k+1)-\widehat{h}(k))^2 = \sum_{k=0}^{T-1}(h(k+1)-h(k))^2 \\ &= \sum_{k=0}^{T-1}\Big(\int_k^{k+1} h'(s)\,ds\Big)^2 \leqslant \sum_{k=0}^{T-1}\int_k^{k+1} h'(s)^2\,ds = |h|_T^2.\end{aligned}$$

By applying Theorem 2.3 and Remark 2.1, we obtain

$$\begin{aligned}I_S(T,r_T) &\leqslant |\widehat{h}|_T \leqslant |h|_T = I_{\widetilde{W}}(T,\rho_T) \\ &= \mathcal{C}\frac{T^{1/2}}{\rho_T}(1+o(1)) = \mathcal{C}\frac{T^{1/2}}{r_T}(1+o(1)),\end{aligned}$$

which provides the required upper bound.

A lower bound for the energy. Here, we must additionally take into account that

$$\|\widetilde{W}-W\|_T = \max_{0\leqslant k<T}\max_{0\leqslant s\leqslant 1}|W_k^0(s)| = O((\ln T)^{1/2}) \quad \text{a.s.}$$

This follows from the well known bounds for the maxima of Gaussian processes [11, Ch. 12]

$$\mathbb{P}\Big(\max_{0\leqslant s\leqslant 1}|W_k^0(s)| \geqslant x\Big) = \exp\{-2x^2(1+o(1))\}, \qquad x\to\infty,$$

combined with the Borel–Cantelli lemma. In particular, under the assumptions of our theorem we have

$$\|\widetilde{W}-W\|_T = O((\ln T)^{1/2}) = o(T^{1/p}) = o(r_T).$$

Let

$$\rho_T := r_T + \|S - W\|_T + \|\widetilde{W} - W\|_T$$
$$= r_T + o(T^{1/p}) + O((\ln T)^{1/2}) = r_T(1 + o(1)).$$

Consider the string h accompanying the random broken line S such that $\|h - S\|_T \leqslant r_T$ and $|h|_T = I_S(T, r_T)$. Then

$$\|h - \widetilde{W}\|_T \leqslant \|h - S\|_T + \|S - W\|_T + \|W - \widetilde{W}\| \leqslant \rho_T.$$

Therefore, Theorem 2.3 and Remark 2.1 yield

$$I_S(T, r_T) = |h|_T \geqslant I_{\widetilde{W}}(T, \rho_T) = \mathcal{C}\frac{T^{1/2}}{\rho_T}(1 + o(1)) = \mathcal{C}\frac{T^{1/2}}{r_T}(1 + o(1)),$$

which yields the required lower bound.

The asymptotic behavior of $I_S^0(T, r_T)$ is investigated in the same way. One should additionally apply Remark 2.2 to Theorem 2.3 with $x_T := S(T) - W(T)$.

The second part of the theorem is proved exactly as the first one but refers to the second part of Theorem 3.1, i.e. instead of (4.14) we use

$$\|S - W\|_T = \sup_{0 \leqslant t \leqslant T} |S(t) - W(t)| = \max_{0 \leqslant k \leqslant T} |S_k - W_k| = O(\ln T) \quad \text{a.s.} \ \blacksquare$$

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