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EXTREMES OF ORDER STATISTICS OF STATIONARY GAUSSIAN PROCESSES*

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Abstract. Let $\{X_i(t), t \geqslant 0\}$, $1 \leqslant i \leqslant n$, be mutually independent and identically distributed centered stationary Gaussian processes. Under some mild assumptions on the covariance function, we derive an asymptotic expansion of

$$\mathbb{P}\big(\sup_{t\in[0,xm_r(u)]} X_{(r)}(t) \leqslant u\big) \quad \text{as } u \to \infty,$$

where

$$m_r(u) = \big(\mathbb{P}(\sup_{t\in[0,1]} X_{(r)}(t) > u)\big)^{-1}\big(1 + o(1)\big),$$

and $\{X_{(r)}(t), t \geqslant 0\}$ is the rth order statistic process of $\{X_i(t), t \geqslant 0\}$, $1 \leqslant i, r \leqslant n$. As an application of the derived result, we analyze the asymptotics of supremum of the order statistic process of stationary Gaussian processes over random intervals.

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1. INTRODUCTION

Let $\{X(t) : t \geqslant 0\}$ be a centered stationary Gaussian process with continuous sample paths. One of the classical results in extreme value theory states that, under some mild conditions on the covariance function of X,

$$\lim_{u\to\infty} \mathbb{P}\big(\sup_{t\in[0,xm(u)]} X(t) \leqslant u\big) = e^{-x} \tag{1.1}$$

for $x > 0$ and $m(u) = \mathbb{P}\big(\sup_{t\in[0,1]} X(t) > u\big)^{-1}$; see, e.g., Leadbetter et al. [11], Theorem 12.3.4; Arendarczyk and Dębicki [4], Lemma 4.3; Tan and Hashorva [13], Lemma 3.3.

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Consider a vector-valued Gaussian stochastic process $\{\mathbf{X}(t) : t \geqslant 0\}$, where $\mathbf{X}(t) = \big(X_1(t), \ldots, X_n(t)\big)$ with $\{X_i(t) : t \geqslant 0\}$, $i = 1, \ldots, n$, being mutually independent copies of $\{X(t) : t \geqslant 0\}$. Denote by $\{X_{(r)}(t), t \geqslant 0\}$, $r = 1, 2, \ldots, n$, the rth smallest order statistic process, i.e., for each $t \geqslant 0$,

$$X_{(1)}(t) = \min_{1 \leqslant i \leqslant n} X_i(t) \leqslant X_{(2)}(t) \leqslant \ldots \leqslant \max_{1 \leqslant i \leqslant n} X_i(t) = X_{(n)}(t). \tag{1.2}$$

In this contribution we derive a counterpart of (1.1) for $\{X_{(r)}(t), t \geqslant 0\}$.

One of important motivations to analyze asymptotic properties of extremes of order statistic processes is their relation with the *conjunction problem*. Following [14], the set of conjunctions $C_{T,u}$ is defined as

$$C_{T,u} := \{t \in [0, T] : \min_{1 \leqslant i \leqslant n} X_i(t) > u\},$$

so

$$\mathbb{P}\left(C_{T,u} = \emptyset\right) = \mathbb{P}\big(\sup_{t \in [0,T]} \min_{1 \leqslant i \leqslant n} X_i(t) \leqslant u\big).$$

We refer to [2], [3], [6], [9], [14] for recent results on asymptotic properties of $\mathbb{P}\left(C_{T,u} \neq \emptyset\right)$.

As an application of the obtained result we provide the exact asymptotics of

$$\mathbb{P}\big(\sup_{t \in [0,\mathcal{T}]} X_{(r)}(t) > u\big) \quad \text{as } u \to \infty$$

for $\mathcal{T}$ being a nonnegative random variable independent of $\mathbf{X}(t)$. The obtained asymptotics extends the recent results of Arendarczyk and Dębicki [4].

2. PRELIMINARIES

Suppose that $\mathbf{X}(t) = \big(X_1(t), \ldots, X_n(t)\big)$ and $\{X_i(t) : t \geqslant 0\}$, $i = 1, \ldots, n$, are mutually independent centered stationary Gaussian processes with covariance function $r(t)$ satisfying the following conditions:

(A1) $r(t) = 1 - t^\alpha + o(t^\alpha)$ as $t \to 0$;

(A2) $r(t) < 1$ if $t > 0$;

(A3) $r(t) \log t \to 0$ as $t \to \infty$.

Following Dębicki et al. [9], let us introduce the *generalized Pickands constant* as

$$\mathcal{H}_{\alpha,k} = \lim_{S \to \infty} S^{-1} \mathcal{H}_{\alpha,k}(S) \in (0, \infty),$$

where

$$\mathcal{H}_{\alpha,k}(S) = \int_{R^n} \exp\big(\sum_{i=1}^{k} w_j\big) \mathbb{P}\Big(\sup_{t \in [0,S]} \min_{1 \leqslant i \leqslant k} \big(\sqrt{2} B_\alpha^{(i)}(t) - t^\alpha - w_i\big) > 0\Big) d\mathbf{w} \in (0, \infty),$$

and $B_\alpha^{(i)}$, $i = 1, \ldots, n$, are mutually independent standard fractional Brownian motions with Hurst index $\alpha/2 \in (0, 1]$, i.e., centered Gaussian processes with stationary increments and variance function t^α.

Let

$$m_r(u) := \frac{(2\pi)^{(n+1-r)/2}}{c_{n,r-1}\mathcal{H}_{\alpha,n+1-r}} u^{n+1-r-2/\alpha} \exp\left(\frac{n+1-r}{2}u^2\right), \tag{2.1}$$

where

$$c_{n,r-1} = \frac{n!}{(r-1)!(n+1-r)!}.$$

It follows from Theorem 2.2 in [8] that, for each $T > 0$ and $1 \leqslant r \leqslant n$,

$$\begin{aligned} \mathbb{P}\big(\sup_{t\in[0,T]} X_{(r)}(t) > u\big) &= c_{n,r-1}\mathcal{H}_{\alpha,n+1-r} T u^{2/\alpha} \big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big) \\ &= \frac{T}{m_r(u)}\big(1+o(1)\big) \quad \text{as } u \to \infty, \end{aligned} \tag{2.2}$$

where $\Psi(u) = \frac{1}{\sqrt{2\pi}}\int_u^\infty \exp(-x^2/2)dx$.

3. MAIN RESULTS

The following theorem constitutes the main result of this contribution.

THEOREM 3.1. *Let $\{X_j(t), t \geqslant 0\}$ be independent and identically distributed centered stationary Gaussian processes with convariance function $r(t)$ satisfying the conditions* (A1)–(A3) *and assume that $0 < A < B < \infty$ and $x > 0$. Then*

$$\mathbb{P}\big(\sup_{t\in[0,xm_r(u)]} X_{(r)}(t) \leqslant u\big) \to e^{-x} \quad \text{as } u \to \infty, \tag{3.1}$$

uniformly for $x \in [A, B]$.

Let $\mathcal{T}$ be a nonnegative random variable which is independent of $\mathbf{X}$. In the following theorem we discuss the asymptotic behavior of $\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big)$ as $u \to \infty$. It appears that the qualitative form of the asymptotics strongly depends on *heaviness* of the tail of $\mathcal{T}$.

THEOREM 3.2. *Let $\{X_j(t), t \geqslant 0\}$ be independent and identically distributed centered stationary Gaussian processes with convariance function $r(t)$ satisfying the conditions* (A1)–(A3), *and let $\mathcal{T}$ be a nonnegative random variable independent of X.*

(i) *If $\mathbb{E}\mathcal{T} < \infty$, then, as $u \to \infty$,*

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) = \mathbb{E}\mathcal{T} c_{n,r-1}\mathcal{H}_{\alpha,n+1-r} u^{2/\alpha}\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big). \tag{3.2}$$

(ii) *If* $\mathcal{T}$ *has a regularly varying tail distribution at infinity with index* $\lambda \in (0,1)$, *then*, *as* $u \to \infty$,

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) = \Gamma(1-\lambda)\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big). \tag{3.3}$$

(iii) *If* $\mathcal{T}$ *has a slowly varying tail distribution at infinity*, *then*, *as* $u \to \infty$,

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) = \mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big). \tag{3.4}$$

The proofs of Theorems 3.1 and 3.2 are given in Section 4.

4. PROOFS

Before proceeding to the proofs of Theorems 3.1 and 3.2, we give some preliminary lemmas. Let us put $\mathcal{T}_r = xm_r(u)$ and $n_r = \lfloor \mathcal{T}_r \rfloor$. For any $\varepsilon \in (0,1)$ and $1 \leqslant l \leqslant n_r$, we write $I_l = [l-1+\varepsilon, l]$ and $I_l^* = [l-1, l-1+\varepsilon]$.

LEMMA 4.1. *For each* $B > A > 0$,

$$\lim_{u\to\infty} \big|\mathbb{P}\big(\sup_{t\in[0,n_r]} X_{(r)}(t) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leqslant u\big)\big| \leqslant \rho_1(\varepsilon), \tag{4.1}$$

uniformly for $x \in [A,B]$, *where* $\rho_1(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Suppose that $x \in [A,B]$. By stationarity, Bonferroni's inequality (see, e.g., [10]) and (2.2), we have

$$\begin{aligned} 0 &\leqslant \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in[0,n_r]} X_{(r)}(t) \leqslant u\big) \\ &= \mathbb{P}\big(\sup_{t\in[0,n_r]} X_{(r)}(t) > u\big) - \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) > u\big) \\ &\leqslant \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l^*} X_{(r)}(t) > u\big) \leqslant n_r\mathbb{P}\big(\sup_{t\in[0,\varepsilon]} X_{(r)}(t) > u\big) \\ &= xm_r(u)\frac{\varepsilon}{m_r(u)}\big(1+o(1)\big) \leqslant B\varepsilon =: \rho_1(\varepsilon) \quad \text{as } u \to \infty. \end{aligned}$$

This completes the proof. ∎

LEMMA 4.2. *Let* $q = q(u) = au^{-2/\alpha}$ *for some* $a > 0$. *Then*

$$\limsup_{u\to\infty} \big|\mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t) \leqslant u\big) - \mathbb{P}\big(\max_{iq\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(iq) \leqslant u\big)\big| \leqslant \rho_2(a),$$

uniformly for $x \in [A,B]$, *where* $\rho_2(a) \to 0$ *as* $a \to 0$.

P r o o f. Since $X_i(t)$ are independent and identically distributed, we obtain

$$\begin{aligned}
&\mathbb{P}\big(\max_{iq\in I_1} X_{(r)}(iq) > u\big)\\
&= \mathbb{P}\big(\bigcup_{iq\in I_1}\bigcup_{j=n-r+1}^{n}\{\exists k_1,\ldots,k_j, X_{k_1}(iq)>u,\ldots,X_{k_j}(iq)>u\}\big)\\
&= \mathbb{P}\big(\bigcup_{iq\in I_1}\bigcup_{j=n-r+1}^{n}\{\exists k_1,\ldots,k_j, X_{k_1}(iq)>u,\ldots,X_{k_j}(iq)>u,\\
&\qquad\qquad X_k(iq)\leqslant u, k\neq k_1,\ldots,k_j\}\big)\\
&= \sum_{j=n-r+1}^{n} c_{n,j}\mathbb{P}\big(\exists_{iq\in I_1}, X_1(iq)>u,\ldots,X_j(iq)>u, X_k(iq)\leqslant u, k>j\big)\\
&= \sum_{j=n-r+1}^{n} c_{n,j}\mathbb{P}\big(\max_{iq\in I_1}\min_{1\leqslant i\leqslant j} X_i(iq)>u\big)\big(1+o(1)\big).
\end{aligned}$$

Following Dębicki et al. [8] we define

$$\mathcal{H}'_{\alpha,j}(a) = \frac{1}{a}P\Big(\max_{k\geqslant 1}\min_{1\leqslant m\leqslant j}\big(\sqrt{2}B_\alpha^{(m)}(ak)-(ak)^\alpha+\eta_m\big)\leqslant 0\Big), \tag{4.2}$$

where $j=1,2,\ldots,n$, and $\{B_\alpha^{(m)}, t\geqslant 0\}$, $m\geqslant 1$, are independent and identically distributed standard fractional Brownian motions which are further independent of independent unit exponential random variables η_m. Using analogous arguments to those in the proof of Theorem 1.1 in Dębicki et al. [8] or Lemma 1 in Albin and Choi [1], we have

$$\begin{aligned}
\mathbb{P}\big(\max_{iq\in I_1} X_{(r)}(iq)>u\big) &= \sum_{j=n-r+1}^{n}\frac{\mathcal{H}'_{\alpha,j}(a)}{\mathcal{H}_{\alpha,j}}\frac{1-\varepsilon}{m_{n+1-j}(u)}\\
&= \frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}}\frac{1-\varepsilon}{m_r(u)}\big(1+o(1)\big) \quad \text{as } u\to\infty,
\end{aligned}$$

where $\mathcal{H}'_{\alpha,k}(a)\to\mathcal{H}_{\alpha,k}$ as $a\to 0$. Therefore, by stationarity, we obtain

$$\begin{aligned}
0 &\leqslant \mathbb{P}\big(\max_{iq\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(iq)\leqslant u\big) - \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} X_{(r)}(t)\leqslant u\big)\\
&\leqslant n_r\max_{1\leqslant l\leqslant n_r}\Big(\mathbb{P}\big(\max_{iq\in I_l} X_{(r)}(iq)\leqslant u\big) - \mathbb{P}\big(\sup_{t\in I_l} X_{(r)}(t)\leqslant u\big)\Big)\\
&\leqslant n_r\mathbb{P}\left(X_{(r)}(0)>u\right) + n_r\mathbb{P}\big(\sup_{t\in[0,1-\varepsilon]} X_{(r)}(t)>u\big)\\
&\quad - n_r\mathbb{P}\big(\max_{iq\in[0,1-\varepsilon]} X_{(r)}(iq)>u\big)\\
&= x m_r(u)\left(o\left(\frac{1}{m_r(u)}\right)+\frac{1-\varepsilon}{m_r(u)}-\frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}}\frac{1-\varepsilon}{m_r(u)}\right)\big(1+o(1)\big)\\
&\leqslant B\left(1-\frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}}\right) =: \rho_2(a),
\end{aligned}$$

where the penultimate expression is due to (2.2). Since $\rho_2(a) \to 0$ as $a \to 0$, the proof is completed. ■

For each $1 \leqslant j \leqslant n$, let $\{X_j^{(k)}(t), t \geqslant 0\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed centered stationary Gaussian processes that satisfy the conditions (A1)–(A3). Define

$$Y_j(t) = X_j^{(k)}(t) \quad \text{if } t \in [k-1, k),$$

and, for $t \geqslant 0$,

$$Y_{(1)}(t) = \min_{1 \leqslant j \leqslant n} Y_j(t) \leqslant Y_{(2)}(t) \leqslant \ldots \leqslant \max_{1 \leqslant j \leqslant n} Y_j(t) = Y_{(n)}(t).$$

LEMMA 4.3. *We have*

$$\lim_{u \to \infty} \left|\mathbb{P}\big(\sup_{iq \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(iq) \leqslant u\big) - \mathbb{P}\big(\sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leqslant u\big)\right| = 0.$$

Proof. Define $A = \mathbb{N} \cap \bigcup_{l=1}^{n_r} I_l q^{-1} = \{i_1, i_2, \ldots, i_d\}$, where $1 \leqslant i_1 < i_2 < \ldots < i_d < \infty$, and observe that

$$\begin{aligned}\Delta_{(r)} &= \left|\mathbb{P}\big(\sup_{iq \in \bigcup_{l=1}^{n_r} I_l} X_{(r)}(iq) \leqslant u\big) - \mathbb{P}\big(\sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leqslant u\big)\right| \\ &= \left|\mathbb{P}\big(\sup_{i \in A} X_{(r)}(iq) \leqslant u\big) - \mathbb{P}\big(\sup_{i \in A} Y_{(r)}(iq) \leqslant u\big)\right|.\end{aligned}$$

For $i \in A$ and $1 \leqslant j \leqslant n$, we put $X_{ij} = X_j(iq)$ and $Y_{ij} = Y_j(iq) = X_j^{(\lfloor iq \rfloor + 1)}(iq)$. Note that

$$\begin{aligned}\sigma_{ij,lk}^X &= \mathbb{E}X_{ij}X_{lk} = \mathbb{E}X_j(iq)X_k(lq) = r\big((i-l)q\big)\mathbb{I}\{j=k\} := \sigma_{il}^X \mathbb{I}\{j=k\}, \\ \sigma_{ij,lk}^Y &= \mathbb{E}Y_{ij}Y_{lk} = \mathbb{E}X_j^{(\lfloor iq \rfloor + 1)}(iq)X_k^{(\lfloor lq \rfloor + 1)}(lq) \\ &= r\big((i-l)q\big)\mathbb{I}\{\lfloor iq \rfloor = \lfloor lq \rfloor\}\mathbb{I}\{j=k\} := \sigma_{il}^Y \mathbb{I}\{j=k\}.\end{aligned}$$

It follows from Theorem 2.4 in [7] that

$$\Delta_{(r)} \leqslant \frac{n(c_{n-1,r-1})^2}{(2\pi)^{n+1-r}} u^{-2(n-r)} \sum_{i,l \in A, i \neq l} |A_{il}^{(r)}| \exp\left(-\frac{(n+1-r)u^2}{1+\rho_{il}}\right),$$

where

$$\rho_{il} = \max\{|\sigma_{il}^X|, |\sigma_{il}^Y|\} = \left|r\big((i-l)q\big)\right|,$$

$$\begin{aligned}A_{il}^{(r)} &= \int_{\sigma_{il}^Y}^{\sigma_{il}^X} \frac{(1+|h|)^{2(n-r)}}{(1-h^2)^{(n+1-r)/2}} dh \\ &= \int_0^{r((i-l)q)} \frac{(1+|h|)^{2(n-r)}}{(1-h^2)^{(n+1-r)/2}} dh \mathbb{I}\{\lfloor iq \rfloor \neq \lfloor lq \rfloor\}.\end{aligned}$$

Since $\delta := \sup\{|r(t)|, t \geqslant \varepsilon\} < 1$, for $i, l \in A$ satisfying $\lfloor iq \rfloor \neq \lfloor lq \rfloor$, one has $|(i-l)q| \geqslant \varepsilon$, and $\left|r\big((i-l)q\big)\right| \leqslant \delta < 1$. Notice that the integrand in the definition of $A_{il}^{(r)}$ is continuous and bounded on $[0, \delta]$, so there exists a constant K_1 such that

$$|A_{il}^{(r)}| \leqslant K_1 \big|r\big((i-l)q\big)\big| \mathbb{I}\{\lfloor iq \rfloor \neq \lfloor lq \rfloor\}.$$

Hence,

$$\begin{aligned}
\Delta_{(r)} &\leqslant \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r}} u^{-2(n-r)} \frac{\mathcal{T}_r}{q} \sum_{\varepsilon \leqslant kq \leqslant \mathcal{T}_r} |r(kq)| \exp\left(-\frac{(n+1-r)u^2}{1+|r(kq)|}\right) \\
&= \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r}} u^{-2(n-r)} \frac{\mathcal{T}_r}{q} \sum_{\varepsilon \leqslant kq \leqslant \mathcal{T}_r^\beta} |r(kq)| \exp\left(-\frac{(n+1-r)u^2}{1+|r(kq)|}\right) \\
&\quad + \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r}} u^{-2(n-r)} \frac{\mathcal{T}_r}{q} \sum_{\mathcal{T}_r^\beta < kq \leqslant \mathcal{T}_r} |r(kq)| \exp\left(-\frac{(n+1-r)u^2}{1+|r(kq)|}\right) \\
&=: \mathbb{P}_1 + \mathbb{P}_2,
\end{aligned}$$

where $0 < \beta < (1-\delta)/(1+\delta)$.

First, we prove that $\mathbb{P}_1 \to 0$ as $u \to \infty$. Indeed,

$$\begin{aligned}
\mathbb{P}_1 &\leqslant \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r}} u^{-2(n-r)} \frac{\mathcal{T}_r^{\beta+1}}{q^2} \exp\left(-\frac{(n+1-r)u^2}{1+\delta}\right) \\
&= \frac{n(c_{n-1,r-1})^2 K_1}{(2\pi)^{n+1-r} a^2} u^{4/\alpha - 2(n-r)} \mathcal{T}_r^{\beta+1} \exp\left(-\frac{(n+1-r)u^2}{2}\right)^{2/(1+\delta)} \\
&\leqslant K_2 u^{4/\alpha - 2(n-r) + (\beta+1)(n+1-r-2/\alpha)} \exp\left(\frac{(n+1-r)u^2}{2}\right)^{\beta - (1-\delta)/(1+\delta)} \\
&\to 0 \quad \text{as } u \to \infty.
\end{aligned}$$

In order to show that $\mathbb{P}_2 \to 0$, we put $\delta(t) = \sup\{|r(s)\log s|, s \geqslant t\}$. By (A3), we have $|r(t)| \leqslant \delta(t)/\log t$ and $\delta(t) \downarrow 0$ as $t \to \infty$. Moreover,

$$\log \mathcal{T}_r = \frac{n+1-r}{2} u^2 \big(1 + o(1)\big) \quad \text{for } kq > \mathcal{T}_r^\beta.$$

Thus,

$$\begin{aligned}
\exp\left(-\frac{(n+1-r)u^2}{1+|r(kq)|}\right) &\leqslant \exp\left(-(n+1-r)u^2\left(1 - \frac{\delta(\mathcal{T}_r^\beta)}{\log \mathcal{T}_r^\beta}\right)\right) \\
&\leqslant K_3 \exp\big(-(n+1-r)u^2\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{P}_2 &\leqslant \left\{ K_4 u^{-2(n-r)} \frac{\mathcal{T}_r^2}{q^2} \exp\left(-(n+1-r)u^2\right) \frac{1}{\log \mathcal{T}_r^{\beta}} \right\} \\
&\quad \times \frac{q}{\mathcal{T}_r} \sum_{\mathcal{T}_r^{\beta} < kq \leqslant \mathcal{T}_r} |r(kq)| \log(kq) \\
&\leqslant K_5 u^{-2(n-r)} \frac{u^{2(n+1-r-2/\alpha)} \exp\left((n+1-r)u^2\right)}{u^{-4/\alpha}} \exp\left(-(n+1-r)u^2\right) \frac{1}{u^2} \\
&\quad \times \frac{q}{\mathcal{T}_r} \sum_{\mathcal{T}_r^{\beta} < kq \leqslant \mathcal{T}_r} |r(kq)| \log(kq) \\
&\leqslant K_5 \frac{q}{\mathcal{T}_r} \sum_{\mathcal{T}_r^{\beta} < kq \leqslant \mathcal{T}_r} |r(kq)| \log(kq) \to 0 \quad \text{as } u \to \infty.
\end{aligned}$$

This completes the proof. ■

LEMMA 4.4. *We have*

$$\limsup_{u\to\infty} \left| \mathbb{P}\big(\sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leqslant u \big) - \mathbb{P}\big(\sup_{t\in[0,n_r]} Y_{(r)}(t) \leqslant u \big) \right| \leqslant x\big(\rho_3(a) + \varepsilon\big),$$

where $\rho_3(a) \to 0$ *as* $a \to 0$.

P r o o f. Since I_l, $l = 1, 2, \ldots, n_r$, are disjoint, $\{Y_{(r)}(t), t \in I_l\}$ are independent, and, by stationarity,

$$\begin{aligned}
0 &\leqslant \mathbb{P}\big(\sup_{iq \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(iq) \leqslant u \big) - \mathbb{P}\big(\sup_{t \in \bigcup_{l=1}^{n_r} I_l} Y_{(r)}(t) \leqslant u \big) \\
&= \mathbb{P}\big(\sup_{iq\in[0,1-\varepsilon]} Y_{(r)}(iq) \leqslant u \big)^{n_r} - \mathbb{P}\big(\sup_{t\in[0,1-\varepsilon]} Y_{(r)}(t) \leqslant u \big)^{n_r} \\
&\leqslant n_r \Big(\mathbb{P}\big(\sup_{iq\in I_1} Y_{(r)}(iq) \leqslant u \big) - \mathbb{P}\big(\sup_{t\in I_1} Y_{(r)}(t) \leqslant u \big) \Big) \\
&\leqslant n_r \Big(\mathbb{P}\big(Y_{(r)}(0) > u \big) + \mathbb{P}\big(\sup_{iq\in[0,1-\varepsilon]} Y_{(r)}(iq) \leqslant u \big) \\
&\qquad\qquad - \mathbb{P}\big(\sup_{t\in[0,1-\varepsilon]} Y_{(r)}(t) \leqslant u \big) \Big) \\
&= x m_r(u) \left(o\left(\frac{1}{m_r(u)} \right) + \left(1 - \frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}} \right) \frac{1-\varepsilon}{m_r(u)} \right) \big(1 + o(1)\big) \\
&\leqslant x \left(1 - \frac{\mathcal{H}'_{\alpha,n+1-r}(a)}{\mathcal{H}_{\alpha,n+1-r}} \right) =: x\rho_3(a),
\end{aligned}$$

where $\rho_3(a) \to 0$ as $a \to 0$. Moreover,

$$\begin{aligned}
0 &\leqslant \mathbb{P}\big(\sup_{t\in\bigcup_{l=1}^{n_r} I_l} Y_{(r)}(t) \leqslant u\big) - \mathbb{P}\big(\sup_{t\in[0,n_r]} Y_{(r)}(t) \leqslant u\big)\\
&\leqslant \mathbb{P}\big(\sup_{t\in[0,1-\varepsilon]} Y_{(r)}(t) \leqslant u\big)^{n_r} - \mathbb{P}\big(\sup_{t\in[0,1]} Y_{(r)}(t) \leqslant u\big)^{n_r}\\
&\leqslant n_r P\big(\sup_{t\in[0,\varepsilon]} Y_{(r)}(t) > u\big)\\
&= x m_r(u)\frac{\varepsilon}{m_r(u)}\big(1+o(1)\big) = x\varepsilon\big(1+o(1)\big).
\end{aligned}$$

The combination of the above displays completes the proof. ■

LEMMA 4.5. *We have*

$$\lim_{u\to\infty} \mathbb{P}\big(\sup_{t\in[0,n_r]} Y_{(r)}(t) \leqslant u\big) = e^{-x}.$$

Proof. Since

$$\begin{aligned}
\mathbb{P}\big(\sup_{t\in[0,n_r]} Y_{(r)}(t) \leqslant u\big) &= \mathbb{P}\big(\sup_{t\in[0,1]} X_{(r)}(t) \leqslant u\big)^{n_r}\\
&= \Big(1 - \mathbb{P}\big(\sup_{t\in[0,1]} X_{(r)}(t) > u\big)\Big)^{n_r}\\
&= \big(1 - m_r(u)^{-1}\big)^{x m_r(u)}\big(1+o(1)\big) \to e^{-x},
\end{aligned}$$

the proof is completed. ■

Proof of Theorem 3.1. The proof of the theorem follows directly from Lemmas 4.1–4.5. ■

LEMMA 4.6. *For any $S > 0$, we have*

$$\mathbb{P}\big(\sup_{t\in[0,Su^{-2/\alpha}]} X_{(r)}(t) > u\big) = c_{n,r-1}\mathcal{H}_{\alpha,n+1-r}(S)\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big) \tag{4.3}$$

as $u \to \infty$.

The proof of Lemma 4.6 follows line-by-line the same reasoning as the proof of Theorem 2.2 in [8], and thus we omit it.

Proof of Theorem 3.2. (i) For any $t, u, S > 0$, let us put

$$N_t = \left\lfloor \frac{t}{Su^{-2/\alpha}} \right\rfloor \quad \text{and} \quad \Delta_k = [kSu^{-2/\alpha}, (k+1)Su^{-2/\alpha}] \text{ with } k = 0, 1, \ldots, N_t.$$

U p p e r b o u n d. By stationarity of the process $\{X_{(r)}(t), t \geqslant 0\}$ and Lemma 4.6, we obtain

$$\begin{aligned}\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) &= \int_0^\infty \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t)\\ &\leqslant \mathbb{P}\big(\sup_{s\in\Delta_0} X_{(r)}(s) > u\big)\left(\frac{u^{2/\alpha}}{S}\int_0^\infty t d\mathbb{P}(\mathcal{T}\leqslant t) + 1\right)\\ &= \frac{\mathcal{H}_{\alpha,n+1-r}(S)}{S} c_{n,r-1}\mathbb{E}\mathcal{T} u^{2/\alpha}\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big)\end{aligned}$$

as $u \to \infty$. Thus, letting $S \to \infty$, we get

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) = c_{n,r-1} s\mathcal{H}_{\alpha,n+1-r} u^{2/\alpha}\mathbb{E}\mathcal{T}\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big).$$

L o w e r b o u n d. By Bonferroni's inequality, we have

$$\begin{aligned}\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) &= \int_0^\infty \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t)\\ &\geqslant \int_0^u \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t)\\ &\geqslant \mathbb{P}\big(\sup_{s\in\Delta_0} X_{(r)}(s) > u\big)\left(\frac{u^{2/\alpha}}{S}\int_0^u t d\mathbb{P}(\mathcal{T}\leqslant t) - 1\right)\\ &\quad - \int_0^u \sum_{0\leqslant i<j\leqslant N_t} \mathbb{P}\big(\sup_{s\in\Delta_i} X_{(r)}(s) > u, \sup_{s\in\Delta_j} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T}\leqslant t)\\ &=: I_1 - I_2.\end{aligned} \tag{4.4}$$

Note that

$$I_1 = \frac{\mathcal{H}_{\alpha,n+1-r}(S)}{S} c_{n,r-1}\mathbb{E}\mathcal{T} u^{2/\alpha}\big(\Psi(u)\big)^{n+1-r}\big(1+o(1)\big)$$

as $u \to \infty$. Thus, letting $S \to \infty$, we obtain

$$I_1 \geqslant c_{n,r-1}\mathcal{H}_{\alpha,n+1-r} u^{2/\alpha}\mathbb{E}\mathcal{T}\big(\Psi(u)\big)^{n+1-r}. \tag{4.5}$$

Hence, in order to complete the proof it suffices to show that $I_2 = o(I_1)$ as $u \to \infty$.

Indeed, we have

$$
\begin{aligned}
I_2 &= \int_0^u \sum_{k=1}^{N_t} (N_t - k)\mathbb{P}\big(\sup_{s\in\Delta_0} X_{(r)}(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big) d\mathbb{P}(\mathcal{T} \leqslant t)\\
&\leqslant \frac{u^{2/\alpha}}{S} \int_0^u t d\mathbb{P}(\mathcal{T} \leqslant t) \sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} X_{(r)}(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big)\\
&\leqslant \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} X_{(r)}(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big)\\
&\leqslant c_{n,r-1} \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big)\\
&\leqslant c_{n,r-1} \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u, \sup_{s\in\Delta_k} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u\big)\\
&\quad + c_{n,r-1} \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u, \sup_{s\in\Delta_k} X_{(r)}(s) > u,\\
&\qquad\qquad \sup_{s\in\Delta_k} \min_{1\leqslant i\leqslant n+1-r} X_i(s) \leqslant u\big)\\
&=: I_{21} + I_{22}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} \min_{1\leqslant i\leqslant n+1-r} X_i(s) > u, \sup_{s\in\Delta_k} \min_{1\leqslant i\leqslant n+1-r} X_i(s) \leqslant u, \sup_{s\in\Delta_k} X_{(r)}(s) > u\big)\\
\leqslant N_u \mathbb{P}\big(\sup_{s\in\Delta_0} X_1(s) > u\big)^{n+2-r},
\end{aligned}
$$

we get $I_{22} = o(I_1)$ as $u \to \infty$. Moreover, using the relations

$$
\begin{aligned}
I_{21} &\leqslant c_{n,r-1} \frac{u^{2/\alpha}}{S} \mathbb{E}\mathcal{T} \sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} X_1(s) > u, \sup_{s\in\Delta_k} X_1(s) > u\big)^{n+r-1}\\
&\leqslant c_{n,r-1} u^{2/\alpha} \mathbb{E}\mathcal{T} \bigg(\frac{1}{S^{1/(n+r-1)}} \sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} X_1(s) > u, \sup_{s\in\Delta_k} X_1(s) > u\big)\bigg)^{n+r-1},
\end{aligned}
$$

we are left with finding a tight asymptotic bound for

$$
\frac{1}{S^{1/(n+r-1)}} \sum_{k=1}^{N_u} \mathbb{P}\big(\sup_{s\in\Delta_0} X_1(s) > u, \sup_{s\in\Delta_k} X_1(s) > u\big),
$$

which follows by the same argument as that given in the proof of Theorem D.2 in [12] (see also the proof of Theorem 3.1 in [4]), with the minor exception that the first term in the above summand is bounded by

$$\begin{aligned}&\mathbb{P}\big(\sup_{s\in\Delta_0} X_1(s) > u, \sup_{s\in\Delta_1} X_1(s) > u\big)\\&\qquad\leqslant \mathbb{P}\big(\sup_{s\in[0,Su^{-2/\alpha}]} X_1(s) > u, \sup_{\substack{[(S+S^{1/(2(n+r-1))})u^{-2/\alpha},\\(2S+S^{1/(2(n+r-1))})u^{-2/\alpha}]}} X_1(s) > u\big)\\&\qquad\quad+ \mathbb{P}\big(\sup_{s\in[0,S^{1/(2(n+r-1))}u^{-2/\alpha}]} X_1(s) > u\big).\end{aligned}$$

This completes the proof of Theorem 3.1(i).

(ii) For any $0 < A < B < \infty$ and sufficiently large u, we make the following decomposition:

$$\begin{aligned}&\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big)\\&\qquad= \Big(\int_0^{Am_r(u)} + \int_{Am_r(u)}^{Bm_r(u)} + \int_{Bm_r(u)}^{\infty}\Big)\mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big)d\mathbb{P}(\mathcal{T} \leqslant t)\\&\qquad=: I_1 + I_2 + I_3.\end{aligned}$$

We analyze I_1, I_2, I_3 separately.

Integral I_1. Since the process $\{X_{(r)}(t), t \geqslant 0\}$ is stationary, by Bonferroni's inequality, we have

$$\begin{aligned}I_1 &\leqslant \mathbb{P}\big(\sup_{s\in[0,1]} X_{(r)}(s) > u\big)\Big(\int_0^{Am_r(u)} td\mathbb{P}(\mathcal{T} \leqslant t) + 1\Big)\\&= \mathbb{P}\big(\sup_{s\in[0,1]} X_{(r)}(s) > u\big)\\&\quad\times \Big(\int_0^{Am_r(u)} \mathbb{P}(\mathcal{T} > t)dt - Am_r(u)\mathbb{P}\big(\mathcal{T} > Am_r(u)\big) + 1\Big).\end{aligned}\tag{4.6}$$

Using Karamata's theorem, we get

$$\int_0^{Am_r(u)} \mathbb{P}(\mathcal{T} > t)dt = \frac{1}{\lambda}Am_r(u)\mathbb{P}\big(\mathcal{T} > Am_r(u)\big)\big(1 + o(1)\big) \quad \text{as } u \to \infty,$$

which, combined with (4.6) and Theorem 2.2 in [8], implies that

$$\begin{aligned}I_1 &\leqslant \frac{\lambda}{1-\lambda}A\mathbb{P}\big(\mathcal{T} > Am_r(u)\big)\big(1 + o(1)\big)\\&= \frac{\lambda}{1-\lambda}A^{1-\lambda}\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1 + o(1)\big) \quad \text{as } u \to \infty.\end{aligned}$$

I n t e g r a l I_3. It is straightforward that

$$I_3 \leqslant \mathbb{P}\big(\mathcal{T} > Bm_r(u)\big)\big(1+o(1)\big) = B^{-\lambda}\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big) \quad \text{as } u \to \infty.$$

I n t e g r a l I_2. For any $\varepsilon > 0$ and sufficiently large u, applying Theorem 3.1, we get the upper bound

$$\begin{aligned}
I_2 &= \int_A^B \mathbb{P}\big(\sup_{s\in[0,xm_r(u)]} X_{(r)}(s) > u\big) d\mathbb{P}\big(\mathcal{T} \leqslant xm_r(u)\big) \\
&\leqslant (1+\varepsilon)\int_A^B (1-e^{-x}) d\mathbb{P}\big(\mathcal{T} \leqslant xm_r(u)\big) \\
&= (1+\varepsilon)\int_A^B e^{-x}\mathbb{P}\big(\mathcal{T} > xm_r(u)\big)dx - (1+\varepsilon)(1-e^{-B})\mathbb{P}\big(\mathcal{T} > Bm_r(u)\big) \\
&\quad + (1+\varepsilon)(1-e^{-A})\mathbb{P}\big(\mathcal{T} > Am_r(u)\big),
\end{aligned}$$

and similarly we obtain the lower bound

$$\begin{aligned}
I_2 &\geqslant (1-\varepsilon)\int_A^B e^{-x}\mathbb{P}\big(\mathcal{T} > xm_r(u)\big)dx - (1-\varepsilon)(1-e^{-B})\mathbb{P}\big(\mathcal{T} > Bm_r(u)\big) \\
&\quad + (1-\varepsilon)(1-e^{-A})\mathbb{P}\big(\mathcal{T} > Am_r(u)\big).
\end{aligned}$$

Since $\mathcal{T}$ has a regularly varying tail distribution at infinity, by Theorem 1.5.2 in [5], we get

$$\int_A^B e^{-x}\mathbb{P}\big(\mathcal{T} > xm_r(u)\big)dx = \mathbb{P}\big(\mathcal{T} > m_r(u)\big)\int_A^B e^{-x}x^{-\lambda}dx\big(1+o(1)\big) \quad \text{as } u \to \infty.$$

Thus, for any $\varepsilon > 0$ and $0 < A < B < \infty$, we obtain

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{I_2}{\mathbb{P}\big(\mathcal{T} > m_r(u)\big)} \\
&\qquad \leqslant (1+\varepsilon)\Big(\int_0^B x^{-\lambda}e^{-x}dx - (1-e^{-B})B^{-\lambda} + (1-e^{-A})A^{-\lambda}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{I_2}{\mathbb{P}\big(\mathcal{T} > m_r(u)\big)} \\
&\qquad \leqslant (1-\varepsilon)\Big(\int_0^B x^{-\lambda}e^{-x}dx - (1-e^{-B})B^{-\lambda} + (1-e^{-A})A^{-\lambda}\Big).
\end{aligned}$$

Therefore, letting $A \to 0$, $B \to \infty$, and $\varepsilon \to 0$, we find that I_1 and I_3 are negligible, and

$$I_2 = \Gamma(1-\lambda)\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big) \quad \text{as } u \to \infty,$$

which completes the proof of Theorem 3.2(ii).

(iii) Lower bound. From Theorem 3.1, for any given $B > 0$, it follows that

$$\begin{aligned}\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) &\geqslant \mathbb{P}\big(\sup_{s\in[0,Bm_r(u)]} X_{(r)}(s) > u\big)\mathbb{P}\big(\mathcal{T} > Bm_r(u)\big)\\ &= (1-e^{-B})\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big)\end{aligned}$$

as $u \to \infty$. Thus, letting $B \to \infty$, we obtain the asymptotic lower bound

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) \geqslant \mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big) \quad \text{as } u \to \infty.$$

Upper bound. For given $A > 0$, we get

$$\begin{aligned}&\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big)\\ &\leqslant \int_0^{Am_r(u)} \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big)d\mathbb{P}(\mathcal{T} \leqslant t) + \mathbb{P}\big(\mathcal{T} > Am_r(u)\big)\\ &= \int_0^{Am_r(u)} \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big)d\mathbb{P}(\mathcal{T} \leqslant t) + \mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big)\end{aligned}$$

as $u \to \infty$. Due to the stationarity of the process $\{X_{(r)}(t), t \geqslant 0\}$ and Bonferroni's inequality, we have

$$\begin{aligned}&\int_0^{Am_r(u)} \mathbb{P}\big(\sup_{s\in[0,t]} X_{(r)}(s) > u\big)d\mathbb{P}(\mathcal{T} \leqslant t)\\ &\qquad\leqslant \mathbb{P}\big(\sup_{s\in[0,1]} X_{(r)}(s) > u\big)\Big(\int_0^{Am_r(u)} td\mathbb{P}(\mathcal{T} \leqslant t) + 1\Big)\\ &\qquad\leqslant \mathbb{P}\big(\sup_{s\in[0,1]} X_{(r)}(s) > u\big)\Big(\int_0^{Am_r(u)} \mathbb{P}(\mathcal{T} > t)dt + 1\Big).\end{aligned} \tag{4.7}$$

From Karamata's theorem (see, e.g., Proposition 1.5.8 in [5]), we get

$$\int_0^{Am_r(u)} \mathbb{P}(\mathcal{T} > t)dt = Am_r(u)\mathbb{P}\big(\mathcal{T} > Am_r(u)\big)\big(1+o(1)\big)$$

as $u \to \infty$, which, combined with (4.7) and Theorem 2.2 in [8], implies that

$$\mathbb{P}\big(\sup_{t\in[0,\mathcal{T}]} X_{(r)}(t) > u\big) \leqslant (1+A)\mathbb{P}\big(\mathcal{T} > m_r(u)\big)\big(1+o(1)\big)$$

as $u \to \infty$. Letting $A \to 0$, we obtain (3.4). This completes the proof of Theorem 3.2. ■

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