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# EHRHARD-TYPE INEQUALITY FOR THE ISOTROPIC CAUCHY DISTRIBUTION ON THE PLANE

BY

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**Abstract.** We prove an analogue of Ehrhard's inequality for the two-dimensional isotropic Cauchy measure. In contrast to the Gaussian case, the inequality is not valid for non-convex sets. We provide the proof for rectangles which are symmetric with respect to one coordinate axis.

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## 1. INTRODUCTION

In 1971 A. Prekopa [6] provided sufficient conditions for a probability measure in $\mathbb{R}^n$ to be logarithmically concave, in terms of its density. In particular, he proved that the isotropic Gaussian measure $\gamma_n$ in $\mathbb{R}^d$ is logaritmically concave, that is, for any two convex Borel sets $A$ and $B$ and any $0 < \lambda < 1$ we have

$$\log \gamma_n((1-\lambda)A + \lambda B) \geqslant (1-\lambda)\log \gamma_n(A) + \lambda \log \gamma_n(B).$$

This result was substantially improved in 1983 by A. Ehrhard [3], who proved that, instead of the logarithm, one can use the inverse of $\Phi$, the distribution function of the standard one-dimensional Gaussian measure. Let $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt$ and let $A,\ B, \lambda$ be as above. Then

$$\Phi^{-1}(\gamma_n((1-\lambda)A + \lambda B)) \geqslant (1-\lambda)\Phi^{-1}(\gamma_n(A)) + \lambda \Phi^{-1}(\gamma_n(B)).$$

Later C. Borell [1] proved that the above inequality holds true for all Borel sets.

This property has many important consequences. For instance, it implies that $\Phi^{-1}(u(t,x))$ is a concave function of $x \in A$, where $u(t,x)$ is the Dirichlet heat kernel of a convex open set $A \subset \mathbb{R}^n$ (see [3]). This $\Phi^{-1}$-concavity of Gaussian measure also has some isoperimetric consequences. Using it, R. Latała and K. Oleszkiewicz [5] proved that among all Borel symmetric convex subsets of $\mathbb{R}^n$

with fixed measure, the symmetric strip $\{x \in \mathbb{R}^n : |x_n| < a\}$ has the smallest measure of the boundary.

In order to prove the above-mentioned inequality, Ehrhard [3] defined in $\mathbb{R}^n$ a family of $k$-symmetrizations, $k = 1, \ldots, n$. In our paper we use 2-symmetrization in $\mathbb{R}^2$; let us recall this notion for the standard Gaussian measure $\gamma_2$. Let $A$ be an open set in the plane. The 2-symmetrization of $A$ is a half-plane $H_A = \{(x,y) : x < a\}$ such that $\gamma_2(A) = \gamma_2(H_A) = \Phi(a)$, where $\Phi$ is the distribution function of $N(0,1)$. Ehrhard extensively used two important features of the standard Gaussian measure in $\mathbb{R}^n$: $\gamma_n$ is a product measure and it is rotationally invariant. These crucial properties helped Ehrhard to use a kind of induction on $k = 1, \ldots, n$.

In this paper we investigate some analogous concavity properties of the two-dimensional isotropic Cauchy distribution $\mu_2$ with density $f(x,y) = (2\pi(1+x^2+y^2)^{3/2})^{-1}$. Let $F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$ be the distribution function of the standard one-dimensional Cauchy distribution and let $F^{-1}(x) = -\cot(\pi x)$ be its inverse. Our main theorem states that $\kappa(b,c,d) = \cot(\pi\mu_2((-b,b)\times(c,d)))$ is a convex function of $(b,c,d)$ for $b > 0$ and $c < d$.

In the proof we use 2-symmetrization of rectangles for the standard Cauchy measure $\mu_2$ on the plane. Namely, given a rectangle $\mathcal{P}$ on the plane we can find a half-plane $H_{\mathcal{P}} = \{(x,y) : x < p\}$ such that $\mu_2(\mathcal{P}) = \mu_2(H_{\mathcal{P}}) = \frac{1}{2} + \frac{1}{\pi}\arctan(p)$. Unfortunately, $\mu_2$ is not a product measure, hence Ehrhard's method cannot be applied in this case. Let us only mention that in [2] we examined important properties of 1-symmetrization for the standard Cauchy measure in $\mathbb{R}^n$.

The convexity of the function $\kappa(x,y,z)$ implies the following:

PROPOSITION 1.1. *Let $\mu_2$ be the standard Cauchy measure on the plane and let $A$, $B$ be two rectangles symmetric with respect to the same axis. Then for all $0 < \lambda < 1$,*

$$F^{-1}(\mu_2((1-\lambda)A + \lambda B)) \geqslant (1-\lambda)F^{-1}(\mu_2(A)) + \lambda F^{-1}(\mu_2(B)),$$

*where $F^{-1}(x) = -\cot(\pi x)$.*

*Proof.* For a given rectangle $(-b,b)\times(c,d)$ there is a unique function $\kappa(b,c,d)$ (see Lemma 2.3 below) such that

$$\int_{-b}^{b}\int_{c}^{d} \frac{1}{2\pi(1+x^2+y^2)^{3/2}}\,dy\,dx = \int_{-\infty}^{\kappa(b,c,d)} \frac{1}{\pi(1+x^2)}\,dx.$$

In Theorem 2.1 we will prove that $\kappa(x,y,z) = \cot(\pi\mu_2((-x,x)\times(y,z)))$ is convex as a function of $(x,y,z)$ for $x > 0$ and $y < z$. This implies that for all $(b_1,c_1,d_1)$ and $(b_2,c_2,d_2)$ in this domain and all $\lambda \in (0,1)$,

$$\kappa\big((1-\lambda)(b_1,c_1,d_1) + \lambda(b_2,c_2,d_2)\big) \leqslant (1-\lambda)\kappa(b_1,c_1,d_1) + \lambda\kappa(b_2,c_2,d_2).$$

Now we show that this implies Proposition 1.1.

Let $\mathcal{P}_1 = (-b_1, b_1) \times (c_1,\, d_1)$ and $\mathcal{P}_2 = (-b_2, b_2) \times (c_2,\, d_2)$ be two rectangles, symmetric with respect to the $y$-axis. For all $\lambda \in (0, 1)$ we have

$$(1-\lambda)\mathcal{P}_1 + \lambda\mathcal{P}_2$$
$$= (-(1-\lambda)b_1 - \lambda b_2, (1-\lambda)b_1 + \lambda b_2) \times ((1-\lambda)c_1 + \lambda c_2, (1-\lambda)d_1 + \lambda d_2).$$

Then, by the concavity of $-\kappa(x, y, z)$,

$$\begin{aligned} F^{-1}(\mu_2((1-\lambda)\mathcal{P}_1 + \lambda\mathcal{P}_2)) &= -\kappa\big((1-\lambda)(b_1, c_1, d_1) + \lambda(b_2, c_2, d_2)\big) \\ &\geqslant -(1-\lambda)\kappa(b_1, c_1,\, d_1) - \lambda\kappa(b_2, c_2,\, d_2) \\ &= (1-\lambda)F^{-1}(\mu_2(\mathcal{P}_1)) + \lambda F^{-1}(\mu_2(\mathcal{P}_2)). \quad \blacksquare \end{aligned}$$

We also state the following:

CONJECTURE 1.1. *Let $\mu_n$ be the isotropic Cauchy measure in $\mathbb{R}^n$ and let $A,\ B$ be convex Borel sets in $\mathbb{R}^n$. Then for all $0 < \lambda < 1$ the inequality from Proposition 1.1 holds true.*

REMARK 1.1. Observe that in dimension 1, $\cot(\pi \int_a^b \frac{1}{\pi(1+x^2)}\, dx) = \frac{1+ab}{b-a}$ is a convex function of $(a, b)$, $a < b$.

REMARK 1.2. Let $A(r, R) = \{(x, y) : r^2 \leqslant x^2 + y^2 \leqslant R^2\}$ be an annulus. Computing the Hessian, one can easily check that $\cot(\pi\mu_2(A(r, R)))$ is not a convex function of $(r, R)$, $0 < r < R < \infty$. This shows that convexity of sets is crucial in the above conjecture and this is in sharp contrast to the Gaussian case.

As a first step towards proving the above conjecture, we will prove it for rectangles symmetric with respect to one of the axes of the plane. It is in analogy with the beautiful proof of the classical Brunn–Minkowski theorem, given in 1957 by Hadwiger and Ohmann (compare [4, Theorem 4.1]), where the first step was to prove the theorem for rectangular parallelepipeds whose sides are parallel to the coordinate hyperplanes.

## 2. MAIN RESULT

We begin with some notations. Let $F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x$ for $x \in \mathbb{R}$ be the distribution function of the standard one-dimensional Cauchy measure. Then for $0 < x < 1$ we have $F^{-1}(x) = -\cot(\pi x)$. Let $\mu_2$ be the two-dimensional Cauchy measure with density $f(x, y) = (2\pi(1 + x^2 + y^2)^{3/2})^{-1}$. Consider a rectangle $\mathcal{R}$ that is symmetric with respect to one of the coordinate axes of the plane. Without loss of generality we can assume that $\mathcal{R} = (-b, b) \times (c, d)$, where $b > 0$ and $d > c$. Instead of $F^{-1}(x) = -\cot(\pi x)$ we will use $\cot(\pi x)$ and prove the convexity of $\cot(\pi\mu_2(\mathcal{R}))$. Here is our main theorem.

THEOREM 2.1 (Cotangent-convexity of Cauchy distribution on the plane). *Let $\mathcal{R} = (-b, b) \times (c, d)$, $b > 0$, $c < d$, be a rectangle symmetric with respect to the $y$-axis and let $\mu_2$ be the Cauchy measure on the plane with density $f(x, y) = (2\pi(1 + x^2 + y^2)^{3/2})^{-1}$. The Hessian of $\cot(\pi\mu_2(\mathcal{R}))$ is positive-definite with respect to $(b, c, d)$ for $b > 0$ and $c < d$, which means that this function is convex on this set.*

REMARK 2.1. The measure $\mu_2$ is rotationally invariant so that the above theorem is valid for all rectangles that are symmetric with respect to any straight line through the origin.

We will examine the function

$$\kappa(b, c, d) := \cot(\pi\mu_2((-b, b) \times (c, d)))$$

and prove its convexity for $b > 0$ and $d > c$. Observe that the measure $\mu_2$ is symmetric so that if $d < c < 0$ then $\kappa(b, d, c) = \kappa(b, -c, -d)$, and in what follows we will assume that either $0 < c < d$ or $c < 0 < d$. We start by computing $\mu_2((-b, b) \times (c, d))$.

LEMMA 2.1. *We have*

$$\int_0^b \int_0^d \frac{dt\, ds}{(1 + s^2 + t^2)^{3/2}} = \int_0^b \frac{d}{1 + s^2} \frac{ds}{\sqrt{1 + s^2 + d^2}} = \operatorname{arc\,cot}\left(-\frac{b\, d}{\sqrt{1 + b^2 + d^2}}\right) + \frac{\pi}{2}.$$

For all $x, y$ with $0 < x, y < \infty$ we have the following formula (an immediate consequence of the formula for the cotangent of the difference of two angles):

$$\operatorname{arc\,cot}(x) - \operatorname{arc\,cot}(y) = \operatorname{arc\,cot} \frac{1 + xy}{y - x}.$$

Taking into account the above formulas we obtain, for every $b > 0$ and every $c < d$:

LEMMA 2.2 (Measure of a symmetric rectangle).

$$\begin{aligned}
&\mu_2((-b, b) \times (c, d)) \\
&= \int_{-b}^{b} \int_c^d \frac{dt\, ds}{2\pi(1 + s^2 + t^2)^{3/2}} = \int_0^b \int_c^d \frac{dt\, ds}{\pi(1 + s^2 + t^2)^{3/2}} \\
&= \frac{1}{\pi}\left(\operatorname{arc\,cot}\left(-\frac{b\, d}{\sqrt{1 + b^2 + d^2}}\right) - \operatorname{arc\,cot}\left(-\frac{bc}{\sqrt{1 + b^2 + c^2}}\right)\right) \\
&= \frac{1}{\pi} \operatorname{arc\,cot} \frac{b^2 cd + \sqrt{1 + b^2 + c^2}\sqrt{1 + b^2 + d^2}}{bd\sqrt{1 + b^2 + c^2} - bc\sqrt{1 + b^2 + d^2}}.
\end{aligned}$$

By our definition $\kappa(b,c,d) = \cot(\pi\mu_2((-b,b)\times(c,d)))$ so that

$$\kappa(b,c,d) = \cot\left(\pi\int_{-b}^{b}\int_{c}^{d}\frac{dt\,du}{2\pi(1+u^2+t^2)^{3/2}}\right).$$

The above lemma and the definition of $\kappa$ imply the following.

LEMMA 2.3 (Formula for $\kappa(b,c,d)$). *For all $b>0$ and $c<d$ we have*

$$\kappa(b,c,d) = \frac{b^2cd+\sqrt{1+b^2+c^2}\sqrt{1+b^2+d^2}}{bd\sqrt{1+b^2+c^2}-bc\sqrt{1+b^2+d^2}}.$$

Our aim is to prove that $\kappa(b,c,d)$ is a convex function of $(b,c,d)$. We obtain this by checking that the Hessian matrix of $\kappa$ is positive-definite. We begin by computing the second order derivatives of $\kappa$:

$$\frac{\partial^2\kappa}{\partial d^2} = \frac{(1+b^2)^2(1+c^2)((3d^2+2(1+b^2))\sqrt{1+b^2+c^2}-3cd\sqrt{1+b^2+d^2})}{b(1+b^2+d^2)^{3/2}(d\sqrt{1+b^2+c^2}-c\sqrt{1+b^2+d^2})^3},$$

$$\frac{\partial^2\kappa}{\partial d\partial c} = \frac{-2(1+b^2)^2(b^2cd+\sqrt{1+b^2+c^2}\sqrt{1+b^2+d^2})}{b\sqrt{1+b^2+c^2}\sqrt{1+b^2+d^2}(d\sqrt{1+b^2+c^2}-c\sqrt{1+b^2+d^2})^3},$$

$$\frac{\partial^2\kappa}{\partial d\partial b} = \frac{(1+b^2)(1+c^2)}{b^2(1+b^2+d^2)^{3/2}\sqrt{1+b^2+c^2}} \times\frac{2b^2cd\sqrt{1+b^2+d^2}+(1+d^2+b^2-b^2d^2)\sqrt{1+b^2+c^2}}{(d\sqrt{1+b^2+c^2}-c\sqrt{1+b^2+d^2})^2},$$

$$\frac{\partial^2\kappa}{\partial c^2} = \frac{(1+b^2)^2(1+d^2)((3c^2+2b^2+2)\sqrt{1+b^2+d^2}-3cd\sqrt{1+b^2+c^2})}{b(1+b^2+c^2)^{3/2}(d\sqrt{1+b^2+c^2}-c\sqrt{1+b^2+d^2})^3},$$

$$\frac{\partial^2\kappa}{\partial c\partial b} = \frac{-(1+b^2)(1+d^2)}{b^2(1+b^2+c^2)^{3/2}\sqrt{1+b^2+d^2}} \times\frac{2b^2cd\sqrt{1+b^2+c^2}+(1+c^2+b^2-b^2c^2)\sqrt{1+b^2+d^2}}{(d\sqrt{1+b^2+c^2}-c\sqrt{1+b^2+d^2})^2},$$

$$\frac{\partial^2\kappa}{\partial b^2} = \frac{(1+c^2)(1+d^2)}{b^3(1+b^2+c^2)^{3/2}(1+b^2+d^2)^{3/2}} \times\left(\frac{2(1+b^2)(1+b^2+c^2)(1+b^2+d^2)}{d\sqrt{1+b^2+c^2}-c\sqrt{1+b^2+d^2}} + \frac{b^2((1+b^2)^2-d^2c^2)+b^2cd\sqrt{1+b^2+c^2}\sqrt{1+b^2+d^2}}{d\sqrt{1+b^2+c^2}-c\sqrt{1+b^2+d^2}}\right).$$

As we see, the formulas are complicated. The level of complexity will increase significantly when trying to establish the positivity of the minors of the Hessian. Therefore we introduce the notation

$$x=\sqrt{1+b^2+c^2},\quad y=\sqrt{1+b^2+d^2},\quad \theta=dx-cy,$$

which gives

$$\kappa = \frac{b^2cd + xy}{b\theta}.$$

Moreover, we write

$$H_c = \frac{2b}{x(1+c^2)}, \qquad H_d = \frac{2b}{y(1+d^2)}, \qquad H_b = \frac{\theta}{xy(1+b^2)}$$

together with $\sigma = b(xy - cd)$ and $r = (1+b^2)x^2y^2/(bxy)$, and rewrite the second order derivatives of $\kappa$ in the following way:

$$\begin{aligned}
\frac{\partial^2\kappa}{\partial c^2} &= \frac{1+\kappa^2}{2\theta} H_c^2 \left(\frac{3(1+c^2)}{2b^2}\sigma - r_{cc}\right), & \frac{\partial^2\kappa}{\partial c \partial d} &= \frac{1+\kappa^2}{2\theta} H_c H_d (\sigma - r), \\
\frac{\partial^2\kappa}{\partial d^2} &= \frac{1+\kappa^2}{2\theta} H_d^2 \left(\frac{3(1+d^2)}{2b^2}\sigma - r_{dd}\right), & \frac{\partial^2\kappa}{\partial c \partial b} &= \frac{1+\kappa^2}{2\theta} H_c H_b (2\sigma - 2r_{cb}), \\
\frac{\partial^2\kappa}{\partial b^2} &= \frac{1+\kappa^2}{2\theta} H_b^2 (-2\sigma + 2(r + r_{bb})), & \frac{\partial^2\kappa}{\partial d \partial b} &= \frac{1+\kappa^2}{2\theta} H_d H_b (-2\sigma + 2r_{db}).
\end{aligned}$$

The $r$-functions appearing above are

$$\begin{aligned}
r_{cc} &= \frac{y^2(1+c^2)(1+b^2)}{2bxy}, & r_{dd} &= \frac{x^2(1+d^2)(1+b^2)}{2bxy}, \\
r_{bb} &= \frac{(1+b^2)(b^2(x^2+y^2) + x^2y^2)}{bxy}, & & \\
r_{cb} &= \frac{y^2(1+b^2)(1+c^2+2b^2)}{2bxy}, & r_{db} &= \frac{x^2(1+b^2)(1+d^2+2b^2)}{2bxy}.
\end{aligned}$$

The structure of the derivatives allows us to write the Hessian of $\kappa$ as

$$\left(\frac{1+\kappa^2}{2\theta}\right)^3 A \begin{bmatrix} \frac{3(1+c^2)}{2b^2}\sigma - r_{cc} & \sigma - r & 2\sigma - 2r_{cb} \\ \sigma - r & \frac{3(1+d^2)}{2b^2}\sigma - r_{dd} & -2\sigma + 2r_{db} \\ 2\sigma - 2r_{cb} & -2\sigma + 2r_{db} & -2\sigma + 2(r + r_{bb}) \end{bmatrix} A$$

with $A = \operatorname{diag}(H_c, H_d, H_b)$. The positivity of $\frac{1+\kappa^2}{2\theta}$ and of $H_c, H_d, H_b$ allows us to omit them in what follows.

We now provide some crucial algebraic properties of the functions introduced. Write

$$s_c = \frac{b^2(1+b^2)y^2}{bxy}, \qquad s_d = \frac{b^2(1+b^2)x^2}{bxy}, \qquad s = s_c + s_d = \frac{b^2(1+b^2)(x^2+y^2)}{bxy}$$

and

$$m_{cd} = \frac{(1+b^2)^2}{b^2}(1+c^2)(1+d^2), \qquad m_{c+d} = (1+b^2)^2(2+c^2+d^2),$$
$$m_b = (1+b^2)^2 b^2.$$

These functions are the basic bricks in the $r$-functions algebra presented below.

We have

(2.1) $$r_{cb} = r + s_c, \qquad r_{db} = r + s_d, \qquad r_{bb} = r + s,$$

(2.2) $$r_{cc} + r_{dd} = r - \tfrac{1}{2}s, \qquad r_{cb} + r_{db} = r + \tfrac{1}{2}s,$$

(2.3) $$(1+c^2)r_{dd} + (1+d^2)r_{cc} = \frac{(1+c^2)(1+d^2)}{2b^2}s,$$

(2.4) $$(1+c^2)r_{db} + (1+d^2)r_{cb} = (2+c^2+d^2)r - \frac{(1+c^2)(1+d^2)}{2b^2}s.$$

As we can see, the $r$-functions and the specified linear combinations of them can be represented in terms of $r$- and $s$-functions. Analogous formulas for products of $r$-functions in terms of $m$-functions are

(2.5) $$r^2 = \frac{(1+b^2)^2x^2y^2}{b^2} = m_{cd} + m_{c+d} + m_b,$$

(2.6) $$rs = (1+b^2)^2(x^2+y^2) = m_{c+d} + 2m_b,$$

(2.7) $$r_{cc}r_{dd} = \frac{(1+b^2)^2}{4b^2}(1+c^2)(1+d^2) = \tfrac{1}{4}m_{cd},$$

(2.8) $$r_{cb}r_{db} = \frac{(1+b^2)^2}{4b^2}(1+c^2+2b^2)(1+d^2+2b^2) = \tfrac{1}{4}m_{cd} + \tfrac{1}{2}m_{c+d} + m_b,$$

(2.9) $$r_{cc}r_{db} + r_{dd}r_{cb} = \tfrac{1}{2}m_{cd} + \tfrac{1}{2}m_{c+d}.$$

We begin with the positivity of the smallest minor and prove

LEMMA 2.4. *For all $c < d$ and $b > 0$ we have*

$$\frac{3(1+c^2)}{2b^2}\sigma - r_{cc} \geqslant 0.$$

*Proof.* Indeed, we can just write

$$\frac{3(1+c^2)}{2b^2}\sigma - r_{cc} = \frac{1+c^2}{2bxy}\left[3xy(xy-cd) - y^2(1+b^2)\right]$$

and observe that the expression in brackets is equal to

$$xy(xy-cd) + y^2(x^2+c^2) - 2xycd,$$

which is non-negative since $x \geqslant |c|$, $y \geqslant |d|$ and $x^2+c^2 \geqslant 2x|c|$. ■

The second step is to prove the positivity of the determinant of the second minor:

LEMMA 2.5. *For all $c < d$ and $b > 0$ we have*

$$\begin{vmatrix} \frac{3(1+c^2)}{2b^2}\sigma - r_{cc} & \sigma - r \\ \sigma - r & \frac{3(1+d^2)}{2b^2}\sigma - r_{dd} \end{vmatrix} = \frac{3(1+c^2)(1+d^2)\theta^2}{4b^2}\left(2-\frac{cd}{xy}\right)+\theta^2 > 0.$$

*Proof.* The proof is based on the formulas

$$(1+b^2)^2(1+c^2)(1+d^2) - (xy+b^2cd)^2 = b^2\theta^2, \tag{2.10}$$

$$2xy(xy-cd) - (1+b^2)(x^2+y^2) = \theta^2. \tag{2.11}$$

To prove the first one, observe that

$$\begin{aligned}(1+b^2)^2(1+c^2)(1+d^2) &= (1+c^2)(1+d^2) + 2b^2(1+c^2+d^2+c^2d^2) \\ &\quad + b^4(1+c^2+d^2+c^2d^2), \\ x^2y^2 + b^4c^2d^2 &= (1+c^2)(1+d^2) + b^2(2+c^2+d^2) + b^4(1+c^2d^2),\end{aligned}$$

and consequently

$$\begin{aligned}&(1+b^2)^2(1+c^2)(1+d^2) - (xy+b^2cd)^2 \\ &\qquad = b^2(c^2+d^2+2c^2d^2)b^4(c^2+d^2) - 2b^2xycd \\ &\qquad = b^2(d^2x^2+c^2y^2) - 2b^2xycd = b^2(dx-cy)^2 = b^2\theta^2.\end{aligned}$$

The second one follows immediately from

$$2x^2y^2 = (1+b^2)(x^2+y^2) + d^2x^2 + c^2y^2.$$

Going back to the determinant

$$\frac{9(1+c^2)(1+d^2)}{4b^4}\sigma^2 - \frac{3\sigma}{2b^2}((1+c^2)r_{dd} + (1+d^2)r_{cc}) + r_{cc}r_{dd} - (\sigma - r)^2$$

and using (2.3) together with (2.7) we obtain the expression

$$\begin{aligned}\frac{9(1+c^2)(1+d^2)}{4b^4}\sigma^2 &- \frac{3(1+c^2)(1+d^2)(1+b^2)}{4b^2}\frac{x^2+y^2}{bxy}\sigma \\ &- \frac{3(1+b^2)^2(1+c^2)(1+d^2)}{4b^2} \\ &+ \frac{(1+b^2)^2(1+c^2)(1+d^2)}{b^2} - \frac{(xy+b^2cd)^2}{b^2}.\end{aligned}$$

By (2.10) the sum of the last two terms is just $\theta^2$. Extracting from the first three the common factor $3(1+c^2)(1+d^2)/(4b^2)$, we get

$$\frac{3\sigma^2}{b^2} - \frac{xy-cd}{xy}(1+b^2)(x^2+y^2) - (1+b^2)^2.$$

Since

$$\begin{aligned}\sigma^2 &= b^2(xy-cd)^2 = -2b^2xycd + b^2(x^2y^2+c^2d^2)\\ &= b^2(dx-cy)^2 + b^2(x^2y^2+c^2d^2-d^2x^2-c^2y^2) = b^2\theta^2 + b^2(1+b^2)^2,\end{aligned}$$

we have

$$\begin{aligned}&3\theta^2 + 2(1+b^2)^2 - \frac{xy-cd}{xy}(1+b^2)(x^2+y^2)\\ &\qquad= 3\theta^2 + 2(1+b^2)^2 - (1+b^2)(x^2+y^2) + \frac{cd}{xy}(1+b^2)(x^2+y^2)\\ &\qquad= 2\theta^2 + \theta^2 - (1+b^2)(c^2+d^2) + \frac{cd}{xy}(1+b^2)(x^2+y^2)\\ &\qquad= 2\theta^2 - 2cd(xy-cd) + \frac{cd}{xy}(1+b^2)(x^2+y^2)\\ &\qquad= 2\theta^2 - \frac{cd}{xy}[2xy(xy-cd) - (2+b^2)(x^2+y^2)].\end{aligned}$$

The last expression in brackets is equal to $\theta^2$ by (2.11), and altogether we arrive at the final form of the determinant:

$$\frac{3(1+c^2)(1+d^2)\theta^2}{4b^2}\left(2 - \frac{cd}{xy}\right) + \theta^2.$$

Its positivity is now straightforward since $xy > cd$. ■

Finally, we prove the following:

LEMMA 2.6. *For all $c < d$ and $b > 0$ we have*

$$\begin{vmatrix} \frac{3(1+c^2)}{2b^2}\sigma - r_{cc} & \sigma - r & 2\sigma - 2r_{cb}\\ \sigma - r & \frac{3(1+d^2)}{2b^2}\sigma - r_{dd} & -2\sigma + 2r_{db}\\ 2\sigma - 2r_{cb} & -2\sigma + 2r_{db} & -2\sigma + 2(r + r_{bb}) \end{vmatrix} > 0.$$

*Proof.* We begin by expanding the determinant as a polynomial in $\sigma$:

$$F_0 + F_1\sigma + F_2\sigma^2 + F_3\sigma^3$$

with

$$
\begin{aligned}
F_0 &= 2(r + r_{bb})(r_{cc}r_{dd} - r^2) + 8rr_{cb}r_{db} + 4r_{dd}r_{cb}^2 + 4r_{cc}r_{db}^2,\\
F_1 &= 2r^2 - 2r_{cc}r_{dd} - \frac{r + r_{bb}}{b^2}\big(3(1+c^2)r_{dd} + 3(1+d^2)r_{cc} - 4rb^2\big)\\
&\quad - 8(r_{cb}r_{db} + r(r_{cb} + r_{db}))\\
&\quad - \frac{2}{b^2}\big(3(1+c^2)r_{db}^2 + 3(1+d^2)r_{cb}^2 + 4b^2r_{dd}r_{cb} + 4b^2r_{cc}r_{db}\big),\\
F_2 &= 2(r + r_{bb})\left(\frac{9(1+c^2)(1+d^2)}{4b^4} - 1\right)\\
&\quad + \frac{3}{b^2}\big((1+c^2)r_{dd} + (1+d^2)r_{cc}\big) - 4r\\
&\quad + 8(r_{cb} + r_{db} + r) + 4(r_{cc} + r_{dd}) + \frac{12}{b^2}\big((1+c^2)r_{db} + (1+d^2)r_{cb}\big),\\
F_3 &= -\frac{9}{2b^4}(1+c^2)(1+d^2) - 6 - \frac{6}{b^2}(2 + c^2 + d^2).
\end{aligned}
$$

The second step is to compute the coefficients $F_i$, for $i = 0, 1, 2, 3, 4$, in terms of

$$
F = \frac{3}{4b^6}\big(3(1+c^2)(1+d^2) + 4b^2(2 + c^2 + d^2 + b^2)\big). \tag{2.12}
$$

Starting with $F_0$, since

$$
r_{dd}r_{cb}^2 + r_{cc}r_{db}^2 = (r_{cb} + r_{db})(r_{cc}r_{db} + r_{dd}r_{cb}) - (r_{cc} + r_{dd})r_{cb}r_{db}
$$

we can write

$$
\begin{aligned}
\tfrac{1}{2}F_0 &= (r + r_{bb})(r_{cc}r_{dd} - r^2) + 2r_{cb}r_{db}(2r - (r_{cc} + r_{dd}))\\
&\quad + 2(r_{cb} + r_{db})(r_{cc}r_{db} + r_{dd}r_{cb}).
\end{aligned}
$$

Then, using (2.1) and (2.2), we arrive at

$$
\tfrac{1}{2}F_0 = \big(r + \tfrac{1}{2}s\big)\big(2r_{cc}r_{dd} - 2r^2 + 2r_{cb}r_{db} + 2(r_{cc}r_{db} + r_{dd}r_{cb})\big) = 0,
$$

where the last equality is a consequence of (2.5) and (2.7)–(2.9).

The computation of $F_1$ is also based on $m$-functions. First we note that

$$
\begin{aligned}
(1+c^2)r_{db}^2 + (1+d^2)r_{cb}^2 &= (r_{cb} + r_{db})\big((1+c^2)r_{db} + (1+d^2)r_{cb}\big)\\
&\quad - (2 + c^2 + d^2)r_{cb}r_{db}.
\end{aligned}
$$

Using (2.1) and (2.2) we get $r + r_{bb} = 2r + s = 2(r_{cb} + r_{db})$ and both observations lead to

$$
\begin{aligned}
\frac{1}{2}F_1 &= \frac{3(2 + c^2 + d^2)}{b^2}r_{cb}r_{db}\\
&\quad - \frac{3(r_{cb} + r_{db})}{b^2}[(1+c^2)r_{dd} + (1+d^2)r_{cc} + (1+c^2)r_{db} + (1+d^2)r_{cb}]\\
&\quad + r^2 - r_{cc}r_{dd} - 4r_{cb}r_{db} - 4(r_{cc}r_{db} + r_{dd}r_{cb}).
\end{aligned}
$$

Then, using (2.3) and (2.4) we can reduce the second component above and write $F_1/2$ as

$$\frac{3(2+c^2+d^2+b^2)}{b^2}(r_{cb}r_{db} - r(r_{cb}+r_{db})) + \tilde{F_1},$$

where

$$\begin{aligned}\tilde{F_1} &= r^2 + 3r(r_{cb}+r_{db}) - (r_{cc}r_{dd} + 7r_{cb}r_{db} + 4(r_{cc}r_{db} + r_{dd}r_{cb}))\\ &= 4r^2 - \tfrac{3}{2}rs - \tfrac{1}{4}m_{cd} - 7\big(\tfrac{1}{4}m_{cd} + \tfrac{1}{2}m_{c+d} + m_b\big) - 4\big(\tfrac{1}{2}m_{cd} + \tfrac{1}{2}m_{c+d}\big)\\ &= 4r^2 - \tfrac{3}{2}rs - 4(m_{cd} + m_{c+d} + m_b) - \tfrac{3}{2}(m_{c+d} + 2m_b) = 0.\end{aligned}$$

Here we use (2.2) and (2.7)–(2.9) in the first equality, and the last one follows from (2.5) and (2.6). Finally, since

$$\begin{aligned}r_{cb}r_{db} - r(r_{cb}+r_{db}) &= r_{cb}r_{db} - r^2 - rs/2\\ &= \tfrac{1}{4}m_{cd} + \tfrac{1}{2}m_{c+d} + m_b - (m_{cd} + m_{c+d} + m_b) - \big(\tfrac{1}{2}m_{c+d} + m_b\big)\\ &= -\frac{(1+b^2)^2}{4b^2}(3(1+c^2)(1+d^2) + 4b^2(2+c^2+d^2+b^2))\\ &= -\frac{b^4(1+b^2)^2}{3}F,\end{aligned}$$

we obtain the final form of $F_1$:

$$F_1 = -2b^2(1+b^2)^2(2+c^2+d^2+b^2)F. \tag{2.13}$$

The formula for $F_2$ together with $r + r_{bb} = 2r + s$, $r_{cb} + r_{db} = r + \frac{1}{2}s$, $r_{cc} + r_{dd} = r - \frac{1}{2}s$ and the equalities (2.3) and (2.4) give

$$\begin{aligned}\frac{1}{2}F_2 &= \left[\frac{9(1+c^2)(1+d^2)}{4b^4} - 1\right]2r + \left[\frac{9(1+c^2)(1+d^2)}{4b^4} - 1\right]s\\ &\quad + \frac{3(1+c^2)(1+d^2)}{4b^2}s + 8r + s\\ &\quad + \frac{6}{b^2}(2+c^2+d^2)r - \frac{6(1+c^2)(1+d^2)}{2b^4}s\\ &= \left[\frac{9(1+c^2)(1+d^2)}{4b^4} + 3 + \frac{3(2+c^2+d^2)}{2b^2}\right]2r.\end{aligned}$$

Recalling (2.12) we arrive at

$$F_2 = \frac{3xy(1+b^2)}{b^5}(3(1+c^2)(1+d^2) + 4b^2(2+c^2+d^2+b^2)) = 4bxy(1+b^2)F.$$

It is easy to see that $F_3 = -2b^2F$. Altogether we get the following formula for the determinant:

$$\begin{aligned}&F_0 + F_1\sigma + F_2\sigma^2 + F_3\sigma^3\\&\quad = 2\sigma F\big[-b^2(1+b^2)^2(2+c^2+d^2+b^2) + 2bxy\sigma(1+b^2) - b^2\sigma^2\big].\end{aligned}$$

To finish the proof we need to compute the last bracket term. First recall that

$$\sigma^2 = b^2\theta^2 + b^2(1+b^2)^2.$$

We also have

$$\begin{aligned}2cd\sigma &= 2cdb(xy-cd) = -b((dx-cy)^2 - d^2x^2 - c^2y^2) - 2bc^2d^2\\&= -b\theta^2 + b(d^2x^2 + c^2y^2 - 2c^2d^2) = -b\theta^2 + b(1+b^2)(c^2+d^2).\end{aligned}$$

These two formulas give

$$\begin{aligned}2b(1+b^2)xy\sigma &= 2b^2(1+b^2)xy(xy-cd) = 2(1+b^2)\sigma^2 + 2cd\sigma b(1+b^2)\\&= 2(1+b^2)(b^2\theta^2 + b^2(1+b^2)^2) + ((1+b^2)(c^2+d^2) - \theta^2)b^2(1+b^2)\\&= (1+b^2)b^2\theta^2 + 2b^2(1+b^2)^3 + b^2(1+b^2)^2(c^2+d^2)\\&= b^2\theta^2 + b^2[b^2\theta^2 + (1+b^2)^2(x^2+y^2)]\\&= b^2\theta^2 + b^2[\sigma^2 + (1+b^2)^2(2+c^2+d^2+b^2)].\end{aligned}$$

This allows us to compute that

$$-b^2(1+b^2)^2(2+c^2+d^2+b^2) + 2bxy\sigma(1+b^2) - b^2\sigma^2 = b^2\theta^2,$$

and consequently the determinant is

$$\frac{3(dx-cy)^2(xy-cd)}{2b^3}\big(3(1+c^2)(1+d^2) + 4b^2(2+c^2+d^2+b^2)\big),$$

which is obviously positive. This ends the proof. ■

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