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ON THE MONOTONICITY OF TAIL PROBABILITIES*

BY

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Abstract. Let S and X be independent random variables, assuming values in the set of non-negative integers, and suppose further that both $\mathbb{E}(S)$ and $\mathbb{E}(X)$ are integers satisfying $\mathbb{E}(S) \geqslant \mathbb{E}(X)$. We establish a sufficient condition for the tail probability $\mathbb{P}(S \geqslant \mathbb{E}(S))$ to be larger than the tail $\mathbb{P}(S+X \geqslant \mathbb{E}(S+X))$, when the mean of S is equal to the mode.

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1. MAIN RESULT

We are interested in the comparison between the tails $\mathbb{P}(S \geqslant \mathbb{E}(S))$ and $\mathbb{P}(S+X \geqslant \mathbb{E}(S+X))$, where S and X are independent random variables. In everyday language, suppose an enterprise S is successful if the result exceeds the mean; would it be beneficial to include one more enterprise X? In many applications, S is a sum of independent random variables and X adds one more to the sum. By the central limit theorem, $\mathbb{P}(S \geqslant \mathbb{E}(S))$ converges to $1/2$. Therefore, if $\mathbb{P}(S \geqslant \mathbb{E}(S)) > 1/2$ (the enterprise is favorably skewed), one would expect that adding one more term to the sum would lower this probability.

All random variables under consideration take values in $\mathbb{N} \cup \{0\}$. We establish an inequality that applies to random variables that satisfy certain "skewness" conditions. Throughout the text, given a positive integer n, we denote the set $\{1, \ldots, n\}$ by $[n]$.

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DEFINITION 1.1 (Right-skewness). Assume that S is unimodal with mode s. Then we say that S is *right-skewed* if

$$\mathbb{P}(S = s - i) \leqslant \mathbb{P}(S = s + i - 1) \quad \text{for all } i \in [s].$$

In our definition, we allow that the mode is not unique. It is possible that $\mathbb{P}(S = s - 1) = \mathbb{P}(S = s)$ and that is why we put the $\leqslant$ sign. If the inequality is strict, then the inequality in our main result is also strict.

DEFINITION 1.2 (Left-loadedness). Let X be a random variable such that $m := \mathbb{E}(X)$ is an *integer*. For $i \in [m]$, set $\alpha_i := \mathbb{P}(X \leqslant m - i) - \mathbb{P}(X \geqslant m + i)$. Then we say that X is *left-loaded* if either of the following two conditions holds true:

(L_1): The sequence $\{\alpha_i\}_{i=1}^m$ changes sign once from positive to negative, i.e., there exists $\ell \in [m]$ such that $\alpha_i \geqslant 0$ for $i \leqslant \ell$, and $\alpha_i \leqslant 0$ for $i > \ell$.

(L_2): $\sum_{i=1}^k \alpha_i \geqslant 0$ for all $k \in [m]$.

A random variable can be both right-skewed and left-loaded. For instance, if $\mathbb{E}(S) = 1$ then it is not hard to prove that S is left-loaded. If such an S is unimodal, such as the binomial distribution $Bin(n, 1/n)$, then it is also right-skewed. Another example is a geometric random variable with parameter $1/n$. Our main result reads as follows.

THEOREM 1.1. *Let $s \geqslant m$ be two positive integers. Suppose that S and X are* independent *random variables, assuming values in the set of non-negative integers, that satisfy the following conditions:*

- *S is right-skewed with mode s.*
- *X is left-loaded with mean m.*

Then $\mathbb{P}(S \geqslant s) \geqslant \mathbb{P}(S + X \geqslant s + m)$.

Note that we have replaced the mean of S by its mode. If S is binomial or Poisson with integer mean, then the mean is equal to the mode. We will show that Poisson random variables with integer mean are both right-skewed and left-loaded, and that binomial random variables are right-skewed if $p \leqslant 1/2$. We conjecture that a binomial random variable is left-loaded if it has integer mean and $p \leqslant 1/2$. This seems to be hard to prove and is related to an old inequality of Simmons [6].

Our inequality is well-established for standard random variables. Let $Poi(\lambda)$ denote a Poisson random variable of mean λ. Teicher [7] showed that

$$\mathbb{P}(Poi(k) \geqslant k) \geqslant \mathbb{P}(Poi(k + 1) \geqslant k + 1) \quad \text{for all } k \geqslant 1, \tag{1.1}$$

which follows from our result if we take $S \sim Poi(k)$ and $X \sim Poi(1)$. Let $Bin(m, p)$ denote a binomial random variable of parameters m and $p \in (0, 1)$.

Chaundy and Bullard [1] showed that for every fixed positive integer $n \geqslant 1$ and probability $p = 1/n$,

$$\mathbb{P}(Bin(nk, p) \geqslant k) \geqslant \mathbb{P}(Bin(n(k+1), p) \geqslant k+1) \quad \text{for all } k \geqslant 1. \tag{1.2}$$

This follows from our result if we take $S \sim Bin(nk, p)$ and $X \sim Bin(n, p)$ for $p = 1/n$. We remark that both inequalities (1.1) and (1.2) concern the monotonicity of tail probabilities of the form $\mathbb{P}(S_k \geqslant \mathbb{E}(S_k))$, where S_k is a sum of k independent random variables of mean 1. These results have been extended to the case of integer means (see [3, Theorem 2.1] and [4, Theorem 2.3]), and several of those extensions can be deduced from our main result. However, Theorem 1.1 provides a bit more, since it allows one to convolute different distributions. For example, it follows from the results in Section 3 that Theorem 1.1 implies that $\mathbb{P}(S \geqslant s) \geqslant \mathbb{P}(S + X \geqslant \mathbb{E}(S + X))$ for $S \sim Bin(n, s/n)$ with $n \geqslant 2s$, and $X \sim Poi(m)$ with $s \geqslant m$, a result which may be seen as a “mixture” of (1.1) and (1.2).

The tail probability $\mathbb{P}(S \geqslant \mathbb{E}(S))$ has been extensively studied for Poisson random variables, motivated by a conjecture by Ramanujan that was eventually settled by Flajolet. This research is ongoing and results continue to be sharpened and extended; see [2] for recent progress and further references. It is not possible to deduce such refined results for parametrized families from our inequality, which puts relatively weak constraints on the distributions of S and X.

2. PROOF OF MAIN RESULT

We begin with an observation.

LEMMA 2.1. *Let X be a random variable, assuming non-negative integer values, such that $m := \mathbb{E}(X)$ is an* integer. *Then*

$$\sum_{i=1}^{m} \big(\mathbb{P}(X \leqslant m - i) - \mathbb{P}(X \geqslant m + i)\big) = \sum_{i \geqslant m+1} \mathbb{P}(X \geqslant m + i).$$

In particular, $\sum_{i=1}^{m} \big(\mathbb{P}(X \leqslant m - i) - \mathbb{P}(X \geqslant m + i)\big) \geqslant 0.$

Proof. Notice that

$$m = \sum_{i=1}^{m} \mathbb{P}(X \geqslant i) + \sum_{i=m+1}^{2m} \mathbb{P}(X \geqslant i) + \sum_{i \geqslant 2m+1} \mathbb{P}(X \geqslant i),$$

which, upon transferring the first two sums on the right to the other side, is equivalent to

$$\sum_{i=1}^{m} (\mathbb{P}(X \leqslant m - i) - \mathbb{P}(X \geqslant m + i)) = \sum_{i \geqslant m+1} \mathbb{P}(X \geqslant m + i). \quad \blacksquare$$

We now prove our main result, which applies to random variables that are skewed to the right. One would expect that there exists a corresponding result for variables that are skewed to the left. However, our proof does not easily transfer to this case. One problem is that the inequality $\sum_{i=1}^{m}\big(\mathbb{P}(X \leqslant m-i) - \mathbb{P}(X \geqslant m+i)\big) \geqslant 0$ holds for all random variables. It does not change sign if we skew the random variable to the left.

Proof of Theorem 1.1. If we condition on S we have

$$\begin{aligned}\mathbb{P}(S+X \geqslant s+m) &= \sum_{i\geqslant 0} \mathbb{P}(X \geqslant s+m-i)\cdot \mathbb{P}(S=i)\\ &= \mathbb{P}(S \geqslant s+m) + \sum_{i=0}^{s+m-1} \mathbb{P}(X \geqslant s+m-i)\cdot \mathbb{P}(S=i).\end{aligned}$$

Hence $\mathbb{P}(S+X \geqslant s+m) \leqslant \mathbb{P}(S \geqslant s)$ is equivalent to

$$\sum_{i=0}^{s+m-1} \mathbb{P}(X \geqslant s+m-i)\cdot \mathbb{P}(S=i) \leqslant \sum_{i=s}^{s+m-1} \mathbb{P}(S=i),$$

which can be rearranged as

$$\sum_{i=0}^{s-1} \mathbb{P}(S=i)\cdot \mathbb{P}(X \geqslant s+m-i) \leqslant \sum_{i=s}^{s+m-1} \mathbb{P}(S=i)\cdot \mathbb{P}(X \leqslant s+m-i-1).$$

This is equivalent to

$$\sum_{i=1}^{s} \mathbb{P}(S=s-i)\cdot\mathbb{P}(X \geqslant m+i) \leqslant \sum_{i=1}^{m} \mathbb{P}(S=s+i-1)\cdot\mathbb{P}(X \leqslant m-i). \tag{2.1}$$

Let L and R denote the left-hand side and the right-hand side of (2.1). Since S is unimodal with mode $s \geqslant m$, we can estimate L as follows:

$$\begin{aligned} L &\leqslant \sum_{i=1}^{m} \mathbb{P}(S=s-i)\cdot \mathbb{P}(X \geqslant m+i)\\ &\quad + \mathbb{P}(S=s-m-1)\cdot \sum_{i=m+1}^{s} \mathbb{P}(X \geqslant m+i)\\ &=: \ell_1 + \ell_2,\end{aligned}$$

with the convention that ℓ_2 is equal to 0 when $s=m$. Now, since S is right-skewed, we have

$$\ell_1 \leqslant \sum_{i=1}^{m} \mathbb{P}(S=s+i-1)\cdot \mathbb{P}(X \geqslant m+i) =: R_1. \tag{2.2}$$

Using again the right-skewness of S and Lemma 2.1, we have

$$\ell_2 \leqslant \mathbb{P}(S = s+m) \cdot \Big(\sum_{i=1}^{m} (\mathbb{P}(X \leqslant m-i) - \mathbb{P}(X \geqslant m+i))\Big) =: R_2. \tag{2.3}$$

It follows from (2.1)–(2.3) that it is enough to show that $R_1 + R_2 \leqslant R$, or equivalently

$$\sum_{i=1}^{m} \big(\mathbb{P}(S = s+i-1) - \mathbb{P}(S = s+m)\big) \cdot \big(\mathbb{P}(X \leqslant m-i) - \mathbb{P}(X \geqslant m+i)\big) \geqslant 0. \tag{2.4}$$

For each $i \in [m]$, let $\Delta_i := \mathbb{P}(S = s+i-1) - \mathbb{P}(S = s+m)$ as well as $\alpha_i := \mathbb{P}(X \leqslant m-i) - \mathbb{P}(X \geqslant m+i)$, and note that (2.4) is equivalent to

$$\sum_{i=1}^{m} \Delta_i \cdot \alpha_i \geqslant 0. \tag{2.5}$$

The unimodality of S implies that $\Delta_1 \geqslant \cdots \geqslant \Delta_m \geqslant 0$. We distinguish two cases.

Suppose first that X satisfies condition (L_1). Let $\ell \in [m]$ be such that $\alpha_i \geqslant 0$ for $i \leqslant \ell$, and $\alpha_i \leqslant 0$ for $i > \ell$. Then, since $\{\Delta_i\}_{i\in[m]}$ is non-increasing, it follows that

$$\sum_{i=1}^{m} \Delta_i \cdot \alpha_i \geqslant \Delta_\ell \sum_{i=1}^{\ell} \alpha_i + \Delta_\ell \sum_{i=\ell+1}^{m} \alpha_i = \Delta_\ell \sum_{i\in[m]} \alpha_i \geqslant 0,$$

where the last estimate follows from the second statement in Lemma 2.1. Hence we obtain (2.5) and the result follows.

Now assume that X satisfies condition (L_2). Set $\Sigma_i := \sum_{j=1}^{i} \alpha_j$ for $i \in [m]$, and notice that $\Sigma_i \geqslant 0$ by assumption. Using summation by parts, we have

$$\sum_{i=1}^{m} \Delta_i \cdot \alpha_i = \Delta_m \cdot \Sigma_m + \sum_{i=1}^{m-1} (\Delta_i - \Delta_{i+1}) \cdot \Sigma_i \geqslant 0.$$

Hence, we obtain (2.5) and the result follows. ∎

3. SKEWNESS OF RANDOM VARIABLES

The standard examples of non-negative random variables that take values in $\mathbb{N} \cup \{0\}$ are Poisson, binomial, or negative binomial. We examine their "skewness" properties.

LEMMA 3.1. *Fix a positive integer s, and let $S \sim Poi(s)$. Then S is right-skewed.*

Proof. Since s is a positive integer it follows that the mode of S is equal to s. For $i \in [s]$, let $\beta_i = \frac{\mathbb{P}(S=s-i)}{\mathbb{P}(S=s+i-1)}$. Since the mode of S is equal to s, it follows that

$\beta_1 \leqslant 1$. Next, note that $\beta_i \geqslant \beta_{i+1}$ is equivalent to $s^2 \geqslant s^2 - i^2$, which is clearly correct for each $i \in [s]$. Hence, the sequence $\{\beta_i\}_{i=1}^s$ is non-increasing, and the fact that $\beta_1 \leqslant 1$ finishes the proof. ∎

LEMMA 3.2. *Fix a positive integer s, and let $S \sim Bin(n, p)$ for some $n \geqslant 2s$ with $p = s/n$. Then S is right-skewed.*

Proof. The proof is similar to the proof of Lemma 3.1. Let $\beta_i = \frac{\mathbb{P}(S=s-i)}{\mathbb{P}(S=s+i-1)}$ for $i \in [s]$. Since S is unimodal with mode s, we have $\beta_1 \leqslant 1$. Furthermore, $\beta_i \geqslant \beta_{i+1}$ is equivalent to

$$s^2 \cdot ((n-s+1)^2 - i^2) \geqslant (n-s)^2 \cdot (s^2 - i^2). \tag{3.1}$$

Now observe that (3.1) holds true when $s^2 \cdot ((n-s)^2 - i^2) \geqslant (n-s)^2 \cdot (s^2 - i^2)$ and the latter is equivalent to $n - s \geqslant s$, which is true by assumption. Hence (3.1) holds true and we conclude that the sequence $\{\beta_i\}_{i\in[s]}$ is non-decreasing. The result follows. ∎

We denote the negative binomial distribution by $NB(r, p)$ where $r \in \mathbb{N}$ is the number of failures and $p \in (0, 1)$ is the probability of success. If $S \sim NB(r, p)$ then $\mathbb{P}(S = k) = \binom{k+r-1}{r-1} p^k q^r$ with $q = 1 - p$ the probability of failure. If $q = 1/n$, the negative binomial has mean $r(n-1)$ and mode $(r-1)(n-1)$.

LEMMA 3.3. *Let $S \sim NB(r, p)$ with $p = 1 - 1/n$ for some integer $n > 1$. Then S is right-skewed.*

Proof. Let $a_k = \mathbb{P}(S = k)$. Then

$$\frac{a_{k+1}}{a_k} = \frac{(k+r)p}{k+1}$$

is $\leqslant 1$ if and only if $k + 1 \geqslant p(r-1)/q$. In particular, S is unimodal with mode $\lfloor p(r-1)/q \rfloor$, which is equal to $s = (n-1)(r-1)$ for our choice of p. To prove that S is right-skewed, it suffices to show that $\frac{a_{s-i-1}}{a_{s-i}} \leqslant \frac{a_{s+i}}{a_{s+i-1}}$, in other words,

$$\frac{s-i}{(s+r-1-i)p} \leqslant \frac{(s+r-1+i)p}{s+i}.$$

For our choice of p, this is equivalent to

$$\frac{s-i}{s-ip} \leqslant \frac{s+ip}{s+i},$$

which obviously holds true. ∎

We have thus established the right-skewness of standard non-negative discrete random variables for certain parameters. Left-loadedness is more difficult to verify. We will prove that a Poisson random variable with integer mean is left-loaded.

Simmons [6] proved that a binomial random variable X with integer mean m satisfies $\mathbb{P}(X \leqslant m-1) > \mathbb{P}(X \geqslant m+1)$ if $n > 2m$. This has been generalized to other distributions by Perrin and Redside [5, Proposition 3.3].

LEMMA 3.4. *Let X be a random variable with integer mean m. Then*

$$\mathbb{P}(X \leqslant m-1) > \mathbb{P}(X \geqslant m+1)$$

if X is Poisson.

LEMMA 3.5. *Fix a positive integer $m \geqslant 3$, and let $X \sim Poi(m)$. Then*

$$\mathbb{P}(X \geqslant 2m) > \mathbb{P}(X = 0).$$

Proof. It is enough to show that $\mathbb{P}(X = 2m) > \mathbb{P}(X = 0)$, or equivalently that $m^{2m} > (2m)!$. This holds if $m = 3$ and we proceed by induction:

$$\begin{aligned}(m+1)^{2(m+1)} &= \left(\frac{m+1}{m}\right)^{2m} \cdot (m+1)^2 \cdot m^{2m} \\ &> 4(m+1)^2 \cdot (2m)! > (2(m+1))!. \quad \blacksquare\end{aligned}$$

A sequence $\{a_i\}_{i=1}^m$ of real numbers is said to be *U-shaped* if there exists $\ell \in [m]$ such that $a_1 \geqslant \cdots \geqslant a_\ell$ and $a_\ell \leqslant \cdots \leqslant a_m$.

LEMMA 3.6. *Let $m \geqslant 3$ be an integer, and let $X \sim Poi(m)$. Then X is left-loaded.*

Proof. We show that X satisfies condition (L_1). Recall that $\alpha_i = \mathbb{P}(X \leqslant m-i) - \mathbb{P}(X \geqslant m+i)$. We have to show that $\{\alpha_i\}_{i=1}^m$ changes sign once. Lemma 3.4 implies that $\alpha_1 > 0$ and Lemma 3.5 implies that $\alpha_m \leqslant 0$, and it suffices to show that the sequence $\{\alpha_i\}_{i=1}^m$ is U-shaped. Since for every $i \in [m-1]$ we have

$$\alpha_{i+1} = \alpha_i - \mathbb{P}(X = m-i) + \mathbb{P}(X = m+i),$$

it is enough to show that the sequence $\{b_i\}_{i=1}^m$, where $b_i := \mathbb{P}(X = m-i) - \mathbb{P}(X = m+i)$, changes sign once. To this end, for $i \in [m]$, let

$$\beta_i = \frac{\mathbb{P}(X = m+i)}{\mathbb{P}(X = m-i)}.$$

Then $\beta_i \geqslant \beta_{i+1}$ is equivalent to $i^2 + i \leqslant m$. Since the sequence $\{i^2+i\}_{i=1}^m$ is increasing, it follows that the sequence $\{\beta_i\}_{i=1}^m$ is U-shaped. Now note that $\beta_1 < 1$, and the proof of Lemma 3.5 implies that $\beta_m \geqslant 1$. Since $\{\beta_i\}_{i=1}^m$ is U-shaped, there exists a unique $k \in [m]$ such that $\beta_i < 1$ for $i \leqslant k$, and $\beta_i \geqslant 1$ for $i \geqslant k+1$, which in turn yields $b_i > 0$ for $i \leqslant k$, and $b_i \leqslant 0$ for $i \geqslant k+1$. In other words, the sequence $\{b_i\}_{i=1}^m$ changes sign once, as desired. $\blacksquare$

LEMMA 3.7. *Let $X \sim Poi(m)$ for a natural number m. Then X is left-loaded.*

Proof. We need to verify the remaining two cases of $m = 1$ and $m = 2$. If $m = 1$, then the second statement in Lemma 2.1 implies that X satisfies condition (L_2). If $m = 2$, then Lemma 3.4 and the second statement in Lemma 2.1 imply that X satisfies condition (L_2). If $m \geqslant 3$ then Lemma 3.6 implies that X satisfies condition (L_1). The result follows. ∎

In a similar way, one can show that a $Bin(n, m/n)$ random variable is left-loaded for a certain range of parameters. More precisely, it satisfies condition (L_2) when $m \in \{1, 2\}$, and condition (L_1) when $4 \leqslant m \leqslant n/3$, but numerical experiments suggest that it is left-loaded for $m \leqslant n/2$ (see the conjecture below). The same appears to be true for a negative binomial distribution with parameter $p = 1 - 1/n$.

4. CONCLUDING REMARKS

We expect that a binomial random variable is left-loaded if $p \leqslant 1/2$. More specifically, we conjecture the following.

CONJECTURE 4.1. *Fix positive integers n, m such that $n \geqslant 2m$, and let $X \sim Bin(n, m/n)$. Then X is left-loaded.*

Condition (L_2) says that $\sum_{i=1}^{k} \alpha_i \geqslant 0$ for all $1 \leqslant k \leqslant m$. Note that our conjecture extends Simmons' inequality (see [6] and [5]).

We have established the right-skewness of random variables for a limited set of parameter values. It is likely that this parameter range can be considerably extended.

The main restriction on our result is that $\mathbb{E}(X)$ is an integer. This is used in Lemma 2.1, which is just a rearrangement of terms. To extend our result to X with non-integer mean, one needs to find a way around this lemma.

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