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ON THE SEMI-MITTAG-LEFFLER DISTRIBUTIONS

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Abstract. The semi-Mittag-Leffler (SML) distribution arises as the marginal of a stationary Markovian process, and is a generalization of the well-known Mittag-Leffler (ML) or positive Linnik distribution. Unlike the ML distribution, which has been well established, few properties of the SML distribution are discussed in the literature. In this paper, we derive some more characterizations of the SML and related distributions. By using stochastic inequalities, we further extend some characterizations, including Pitman and Yor's (2003) result about the hyperbolic sine distribution.

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1. INTRODUCTION

The semi-Mittag-Leffler distribution arises as the marginal of a stationary Markovian process, and generalizes the well-known Mittag-Leffler (or positive Linnik) distribution. The latter is well established in the literature, while few properties of the former are known. We start with the definition.

Let F be the distribution function of a nonnegative random variable X, denoted $X \sim F_X = F$. We say that F is a *semi-Mittag-Leffler* (SML) *distribution function* with exponent $\alpha \in (0,1]$ and order $b \in (0,1)$ if its Laplace–Stieltjes transform (LS-transform) $\hat{F}$ is of the form

$$(1.1)\qquad \hat{F}(s) = \mathsf{E}[\exp(-sX)] = (1+\eta(s))^{-1}, \quad s \in \mathbb{R}_+ := [0,\infty),$$

with $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ increasing (nondecreasing) and satisfying the equation

$$(1.2)\qquad \eta(s) = b^{-\alpha}\eta(bs), \quad s \in \mathbb{R}_+$$

(Pillai [27]; Jayakumar and Pillai [11]). It is clear that $\eta(0^+) = \lim_{s\to 0^+} \eta(s) = 0$ (since $\hat{F}(0^+) = 1$) and that $\eta \in \mathcal{C}^\infty(0,\infty)$. For convenience, we also say that F, X

and $\hat{F}$ are SML(α, b) distribution function, random variable, and (LS-)transform, respectively.

We record some observations: (i) The degenerate distribution at zero is SML with $\eta = 0$. (ii) If X is SML(α, b), then so is cX for any $c > 0$ (positive scale invariance). (iii) If the function η in (1.1) satisfies (1.2), then $(1 + \eta(s))^{-1}$ and $(1 + c\eta(s))^{-1}$ with constant $c > 1$ are SML(α, b) transforms. (iv) To better understand (1.2), define $h(x) = \eta(\mathrm{e}^x)/\mathrm{e}^{\alpha x}$ for $x \in \mathbb{R}$. Then $\eta(s) = s^\alpha h(\log s)$ and we can rewrite (1.1) as

$$\hat{F}(s) = (1 + s^\alpha h(\log s))^{-1}, \quad s > 0, \tag{1.3}$$

where the nonnegative function h has a period, $-\log b$, due to the scaling property (1.2). Clearly, $h \in \mathcal{C}^\infty(\mathbb{R})$.

On the other hand, the order of an SML distribution defined above is not unique. For example, an SML(α, b) distribution is also SML(α, b^2). But if it is SML(α, b_1) and SML(α, b_2) with $\log b_1/\log b_2$ irrational, then the h in (1.3) will be a constant function, say $h(x) = \lambda \geqslant 0$ for $x \in \mathbb{R}$. In this case, the SML(α, b) distribution reduces to a Mittag-Leffler (ML) distribution with LS-transform

$$\hat{F}(s) = (1 + \lambda s^\alpha)^{-1}, \quad s \in \mathbb{R}_+, \tag{1.4}$$

which was first investigated by Gnedenko [5]. The $\hat{F}$ in (1.4) is simply called an ML transform with exponent α, or ML(α) transform.

The ML distribution has the following properties (for simplicity, we consider here the case $\lambda = 1$ in (1.4)):

1. Each ML distribution function has (a) an explicit form and (b) a completely monotone derivative (Pillai [28]; Lin [19]). More precisely, the ML distribution function F with LS-transform $\hat{F}(s) = (1 + s^\alpha)^{-1}$, $s \geqslant 0$, is of the form

$$F(x) = 1 - E_\alpha(-x^\alpha), \quad x \geqslant 0, \tag{1.5}$$

and has the density

$$f(x) = x^{\alpha-1} E_{\alpha,\alpha}(-x^\alpha), \quad x > 0.$$

Here, the one-parameter and two-parameter Mittag-Leffler functions are defined as follows (we consider only real-valued functions with positive parameters α, β):

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad t \in \mathbb{R}. \tag{1.6}$$

2. The tail of the ML distribution function F in (1.5) satisfies

$$F(x) \sim x^\alpha/(1 + x^\alpha) \quad \text{as } x \to \infty \text{ (Lin [19]).}$$

3. Any ML(α) distribution is geometrically infinitely divisible (GID) and hence infinitely divisible (ID), so we can consider the ML(α) process (with LS-transform $(1+s^{\alpha})^{-t}$ at time $t>0$) (Pillai [29]; Lin [19]).

4. Any ML(α) distribution is a generalized Gamma convolution (GGC), and it has a hyperbolically completely monotone (HCM) density function if and only if $\alpha \leqslant 1/2$ (Jedidi and Simon [13, p. 1835]). In this connection, see Bondesson [2].

For more properties of the ML distribution, see the survey paper by Jayakumar and Suresh [12] and the references therein.

Recently, Kataria and Vellaisamy [15] considered the convolution of Mittag-Leffler distributions and its applications to (state dependent) fractional point processes, while Jose et al. [14] pointed out possible applications of generalized Mittag-Leffler distributions and processes to astrophysics and time series modeling. The multivariate ML distributions have been investigated by Albrecher et al. [1]. On the other hand, the ML function E_{α} in (1.6) is an important special function related to fractional calculus. In particular, it is a Fox H-function (and hence a transcendental function), and the function $g(t)=E_{\alpha}(\lambda t^{\alpha})$, $t>0$, satisfies the fractional-order integral equation

$$\lambda(J^{\alpha}_{0+}g)(t)=g(t)-1, \quad t>0,\ \lambda\in\mathbb{R}.$$

Here, the Riemann–Liouville (left-sided) fractional integral $(J^{\alpha}_{0+}g)(t)$ (of order α) of a measurable function g is defined by

$$(J^{\alpha}_{0+}g)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{g(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \quad t>0,$$

provided the integral exists. For more generalizations of the Mittag-Leffler function and their applications to physical and applied sciences as well as information and communication problems, see the monograph by Gorenflo et al. [6] and the survey papers by Gorenflo et al. [7], Haubold et al. [8], Mainardi [22], [23] and Mathai [24].

By contrast, as mentioned above, few properties of the SML distribution are available in the literature although it is a natural extension of the ML distribution and is closely related to the well-established positive semi-stable (SS) distribution (Maejma [21]). More precisely, the SML distribution is the compound-exponential of a positive SS distribution (and hence is ID) as described below. Rewrite the SML(α,b) transform (1.1) as follows:

$$\hat{F}(s)=\frac{1}{1+\eta(s)}=\frac{1}{1-\log\hat{G}(s)}=\int_0^{\infty}(\hat{G}(s))^{y}\mathrm{e}^{-y}\,\mathrm{d}y, \quad s\geqslant 0, \tag{1.7}$$

where $\eta(s)$ satisfies (1.2) and $\hat{G}(s)=\exp(-\eta(s))$, $s\geqslant 0$, is the LS-transform of a positive SS distribution with exponent α and order b. In practice, an SML(α, b) random variable generates a stationary-independent-increments stochastic (composition) process, which is the subordination of the positive SS(α, b) process to the standard gamma process. This constitutes a one-to-one correspondence between the positive SS and SML distributions (see, e.g., Kozubowski [17] and Steutel and van Harn [31, Chapters I and II]). The latter relation can help us construct some explicit SML transforms. Moreover, each SML(α, b) distribution is in fact a *positive* semi-α-Laplace distribution and it belongs to the domain of partial attraction of the corresponding positive SS distribution (see the explanation in Remark 2.1 below). A possible application to the household income problem was given by Pillai [26]. These facts suggest the importance of studying the SML distribution.

When $\alpha = 1$, the SML(α, b) distribution reduces to the exponential one with mean $\mu = \eta'(0^+) \geqslant 0$ (including the degenerate case); in other words, the η in (1.1) is a linear function: $\eta(s) = \mu s$, $s \geqslant 0$ (see Hu and Lin [9, p. 145]).

The SML distribution can also arise as a solution to the stability problem in a geometric compounding model (see Theorem 3.1 below or Hu and Lin [9, Theorem 2]). In the next section, we first give a decomposition characterization of the SML distribution (Theorem 2.1) and then apply it to elaborate on two stationary stochastic processes of which the SML distribution is a marginal. In Section 3, we give some more characterizations of the SML and related distributions (Theorems 3.1–3.3). Finally, by using stochastic inequalities, in Section 4 we further extend some characterizations (Theorems 4.1–4.3), including Pitman and Yor's [30] result about the hyperbolic sine distribution.

2. SML DISTRIBUTIONS AS MARGINALS OF STATIONARY STOCHASTIC PROCESSES

We first present a decomposition characterization of the SML distribution and then apply it to the stochastic processes in question.

THEOREM 2.1. *Let* $\alpha \in (0, 1]$ *and* $b \in (0, 1)$ *be constants and let* X, X_1, X_2 *be nonnegative random variables with the same distribution function* F. *Assume further that* B *is a Bernoulli random variable with* $\Pr(B = 1) = b^\alpha$ *and that* X_1, X_2, B *are independent. Then the distributional equation*

$$X \overset{\mathrm{d}}{=} bX_1 + (1 - B)X_2 \tag{2.1}$$

holds if and only if F *is an SML*(α, b) *distribution function.*

Proof. Let $\hat{F}$ be the LS-transform of X and let $\eta(s) = 1/\hat{F}(s) - 1$ for $s \geqslant 0$. Then rewrite (2.1) in terms of LS-transforms as

$$\hat{F}(s) = \hat{F}(bs)[b^\alpha + (1 - b^\alpha)\hat{F}(s)], \qquad s \geqslant 0. \tag{2.2}$$

Consequently,

$$\frac{1}{\hat{F}(bs)} = (1-b^\alpha) + \frac{b^\alpha}{\hat{F}(s)}, \qquad s \geqslant 0,$$

or equivalently $\eta(bs) = b^\alpha \eta(s),\ s \geqslant 0$. This proves the theorem. ∎

REMARK 2.1. In Theorem 2.1, if we consider general random variables X, X_1, X_2 taking values in $\mathbb{R}$ (instead of $\mathbb{R}_+$), then all solutions to (2.1) are so-called semi-α-Laplace distributions (Pillai [26, Theorem 1]) and hence they are GID distributions as defined by Klebanov et al. [16] (see, e.g., Mohan et al. [25, pp. 174–175]). Therefore, we may say that each SML(α, b) distribution is a positive semi-α-Laplace distribution. Our proof here for the nonnegative case is much simpler. Moreover, the SML(α, b) distribution belongs to the (strict-sense) domain of partial attraction of the corresponding positive SS distribution (with the same exponent α and order b) along similar lines to Pillai's [26, Theorem 2]. For some properties and applications of the semi-α-Laplace distribution and its multivariate cases, see Divanji [3] and Yeh [32].

REMARK 2.2. Note that in the decomposition (2.1), both X and bX_1 are SML(α, b), while $Y := (1-B)X_2$ has a mixture distribution of two SML(α, b)'s. All the three random variables are GID. When $\alpha = 1$, the distributional equation (2.1) characterizes the exponential distribution. In this connection, see Lin and Hu [20, Theorem 3] for a further extension.

We are ready to clarify and elaborate on two stochastic processes considered in Jayakumar and Pillai [11]. Both processes have SML marginals under stationarity. We need some notations. Let

$$\{\varepsilon_n\}_{n=-\infty}^{\infty} = \{\varepsilon_n : n = 0, \pm 1, \pm 2, \ldots\}$$

be a set of i.i.d. nonnegative random variables with the same distribution function F_ε and LS-transform $\hat{F}_\varepsilon$, and let $\{B_n\}_{n=-\infty}^{\infty}$, independent of $\{\varepsilon_n\}_{n=-\infty}^{\infty}$, be a set of i.i.d. Bernoulli random variables with $\Pr(B_n = 1) = b^\alpha$, where $\alpha \in (0, 1]$ and $b \in (0, 1)$ are constants. The next result is an immediate consequence of Theorem 2.1 (compare (2.1)–(2.3)).

COROLLARY 2.1 (Stationary Markovian process). *In the above setting, consider the Markovian process starting from X_0:*

$$(2.3) \qquad X_0 = \varepsilon_0, \qquad X_n = bX_{n-1} + (1-B_n)\varepsilon_n, \qquad n \in \mathbb{N} := \{1, 2, 3, \ldots\}.$$

Suppose that $\{X_n\}_{n=0}^{\infty}$ is stationary, namely, all X_n have the same distribution and hence $X_n \stackrel{\mathrm{d}}{=} \varepsilon_n$ for all $n \geqslant 0$. Then each X_n is an SML(α, b) random variable.

We next turn to the following first-order autoregressive process (AR(1) model, (2.4)), in which X_0 depends on $B_n, \varepsilon_n, n = 0, -1, -2, \ldots$; explicitly,

$$X_0 = \sum_{n=0}^{\infty} b^n (1 - B_{-n}) \varepsilon_{-n}.$$

Note that $X_0 \neq \varepsilon_0$ in general; see Gaver and Lewis [4] for examples of this kind of AR(1) models. Applying again (2.1) and (2.2), we have the following.

COROLLARY 2.2 (AR(1) model). *In the above setting, consider the process*

$$X_n = bX_{n-1} + (1 - B_n)\varepsilon_n, \quad n = 0, \pm 1, \pm 2, \ldots. \tag{2.4}$$

(i) *If $\{X_n\}_{n=-\infty}^{\infty}$ is stationary, then the common LS-transform $\hat{F}$ of X_n satisfies*

$$\frac{\hat{F}(s)}{\hat{F}(bs)} = b^\alpha + (1 - b^\alpha)\hat{F}_\varepsilon(s), \quad s \geqslant 0.$$

(ii) *If, in addition, $X_n \stackrel{\mathrm{d}}{=} \varepsilon_n$, then each X_n is an SML(α, b) random variable.*

3. CHARACTERIZATIONS OF SML AND RELATED DISTRIBUTIONS

Let $X, X_1, X_2, \ldots$ be a sequence of i.i.d. nonnegative random variables with distribution function F. For each $p \in (0, 1)$, let $N_1(p)$ be a geometric random variable, independent of $\{X_n\}_{n=1}^{\infty}$, with $\Pr(N_1(p) = n) = p(1-p)^{n-1}$, $n \in \mathbb{N}$. Consider the random summation (geometric compounding model)

$$S_{N_1(p)} := \sum_{n=1}^{N_1(p)} X_n. \tag{3.1}$$

Then, in addition to Theorem 2.1, we have the following characterization result.

THEOREM 3.1 (Hu and Lin [9]). *Let $p \in (0, 1)$ and $\alpha \in (0, 1]$ be constants. Then in the above setting, the distributional equation*

$$X \stackrel{\mathrm{d}}{=} p^{1/\alpha} S_{N_1(p)}$$

holds if and only if X has an SML(α, b) distribution with $b = p^{1/\alpha}$.

There are two kinds of geometric random variables: one takes values in $\mathbb{N}$, the other in $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. In view of Theorem 3.1, it is natural to ask: What will happen if in the random summation $S_{N_1(p)}$ (defined in (3.1)) we use instead a geometric random variable taking values in $\mathbb{N}_0$? We now consider the geometric random variable N_0 defined by

$$\Pr(N_0 = n) = \frac{p}{1+p}\left(\frac{1}{1+p}\right)^n, \quad n \in \mathbb{N}_0, \tag{3.2}$$

and have instead the following characteristic property of the GID mixture of SML distributions, where $S_0 := 0$. The other characterization result follows.

THEOREM 3.2. *Let $p \in (0,1)$ and $\alpha \in (0,1]$ be constants, and let $N_0 = N_0(p/(1+p))$ be the geometric random variable defined in* (3.2). *Further, assume that $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent copies of $0 \leqslant X \sim F$ and it is independent of N_0. Then the distributional equation*

$$X \stackrel{\mathrm{d}}{=} p^{1/\alpha} S_{N_0} := p^{1/\alpha} \sum_{n=1}^{N_0} X_n \tag{3.3}$$

holds if and only if X has the GID mixture distribution function

$$F(x) = p + (1-p) F_{\alpha,b}(x), \qquad x \geqslant 0, \tag{3.4}$$

where $F_{\alpha,b}$ is an SML(α, b) distribution function with $b = p^{1/\alpha}$.

Proof. It is clear that (3.3) and (3.4) each imply $\chi := \Pr(X = 0) > 0$. If X is degenerate at zero, i.e., $\chi = 1$, then both (3.3) and (3.4) hold, and we have $\hat{F}_{\alpha,b}(s) = [1 + \eta(s)]^{-1}$ with $\eta(s) = 0$ for $s \geqslant 0$. Therefore, it remains to consider the case $\chi \neq 1$, i.e., $\chi < 1$.

Using LS-transform, (3.3) and (3.4) are equivalent to (3.5) and (3.6) below respectively (recall that $S_0 = 0$):

$$\hat{F}(s) = \sum_{n=0}^{\infty} \frac{p}{1+p} \left(\frac{1}{1+p} \right)^n (\hat{F}(p^{1/\alpha} s))^n \tag{3.5}$$

$$= \left[1 + \frac{1 - \hat{F}(p^{1/\alpha} s)}{p} \right]^{-1}, \qquad s \geqslant 0;$$

$$\hat{F}(s) = p + (1-p) \hat{F}_{\alpha,b}(s), \qquad s \geqslant 0. \tag{3.6}$$

We will prove the equivalence of (3.5) and (3.6) with $b = p^{1/\alpha}$.

(Sufficiency) Suppose that (3.6) holds with $b = p^{1/\alpha}$, so

$$\hat{F}(s) = p + (1-p) \frac{1}{1 + \eta(s)}, \qquad s \geqslant 0, \tag{3.7}$$

where $\eta(s) = b^{-\alpha} \eta(bs) = p^{-1} \eta(p^{1/\alpha} s)$. We want to prove that $\hat{F}$ satisfies (3.5). This can be done by just plugging (3.7) into the RHS of (3.5) and carrying out the calculations, applying the identity

$$1 - \hat{F}(p^{1/\alpha} s) = \frac{(1-p) p\, \eta(s)}{1 + p\, \eta(s)}, \qquad s \geqslant 0.$$

(Necessity) Suppose that (3.5) holds true. We want to prove (3.6) with $b = p^{1/\alpha}$. Recall that $\lim_{s \to \infty} \hat{F}(s) = \Pr(X = 0) = \chi > 0$. Letting $s \to \infty$

in (3.5) yields $(\chi - 1)(\chi - p) = 0$. This implies $\chi = p$ because X is not degenerate at zero by assumption. Since the LS-transform $\hat{F}$ is completely monotone on $(0, \infty)$, we have $\chi = p < \hat{F}(s) \leqslant 1$ for $s \geqslant 0$. Write

$$\hat{F}(s) = p + (1-p)\frac{1}{1+g(s)}, \qquad s \geqslant 0, \tag{3.8}$$

or equivalently

$$g(s) = \frac{1-\hat{F}(s)}{\hat{F}(s) - p}, \qquad s \geqslant 0,$$

which is well-defined. Then plugging (3.8) into (3.5) and carrying out the calculations, we finally have $g(s) = p^{-1}g(p^{1/\alpha}s)$ for $s \geqslant 0$. That is, $\phi(s) = [1+g(s)]^{-1}$, $s \geqslant 0$, is an SML$(\alpha, p^{1/\alpha})$ transform. This, together with (3.8), implies that (3.6) holds true with $b = p^{1/\alpha}$. ■

THEOREM 3.3. *Let $p \in (0,1)$ and $\alpha \in (0,1]$ be constants. Suppose that B is a Bernoulli random variable, independent of X, X_1, X_2 defined above, and $\Pr(B = 1) = 1/(1+p)$. Then the distributional equation*

$$X \stackrel{\mathrm{d}}{=} B \cdot (X_1 + p^{1/\alpha} X_2) \tag{3.9}$$

holds if and only if X has the GID mixture distribution function

$$F(x) = p + (1-p)F_{\alpha,b}(x), \qquad x \geqslant 0,$$

where $F_{\alpha,b}$ is an SML(α, b) distribution function with $b = p^{1/\alpha}$.

Proof. Using LS-transform, (3.9) is equivalent to (3.5). The conclusion follows immediately from Theorem 3.2. ■

Clearly, if $X \sim F$ is degenerate at zero, i.e., $X = 0$ a.s., then the stability condition (3.3) is satisfied by all $p \in (0,1)$. Conversely, if (3.3) holds for any two distinct $p \in (0,1)$, then $X = 0$ a.s. This is an application of (3.5), as shown below. A similar result holds for (3.9).

COROLLARY 3.1. *Let p_1 and p_2 be distinct real numbers in $(0,1)$. In addition to the assumptions in Theorem 3.2, if the stability condition (3.3) is satisfied by $p = p_1, p_2$, then $X = 0$ a.s.*

Proof. By the assumptions, (3.5) holds for $p = p_1, p_2$, and hence

$$\frac{1-\hat{F}(p_1^{1/\alpha}s)}{p_1} = \frac{1-\hat{F}(p_2^{1/\alpha}s)}{p_2}, \qquad s \geqslant 0.$$

Letting $s \to \infty$ yields

$$\frac{1-\Pr(X=0)}{p_1} = \frac{1-\Pr(X=0)}{p_2}. \tag{3.10}$$

Suppose, on the contrary, that X is not degenerate at zero; then $\Pr(X = 0) < 1$ and by (3.10), $p_1 = p_2$, a contradiction. Therefore, $X = 0$ a.s. ■

4. EXTENSIONS

We improve some characterization results by using stochastic inequalities. For random variables X and Y, we say that X is no greater than Y in the stochastic order, denoted $X \leqslant_{\rm st} Y$, if $\Pr(X > x) \leqslant \Pr(Y > x)$ for all $x \in \mathbb{R}$. The first result extends the case $\alpha = 1$ of Theorem 3.2. For convenience, the proofs and the needed lemmas are given in the appendix.

THEOREM 4.1. *Let $0 \leqslant X \sim F$, $p \in (0,1)$, $N_0 = N_0(p/(1+p))$ and $\{X_n\}_{n=1}^{\infty}$ be as in Theorem 3.2. Then the stochastic inequality*

$$pS_{N_0} = p \sum_{n=1}^{N_0} X_n \leqslant_{\rm st} X \tag{4.1}$$

holds if and only if X has the GID mixture distribution function

$$F(x) = p + (1-p)(1 - \exp(-\lambda x)),\ x \geqslant 0, \quad \textit{for some } \lambda \geqslant 0. \tag{4.2}$$

Similarly, we can extend the special case ($\alpha = 1$) of Theorem 3.3 to the following. The proof is omitted.

THEOREM 4.2. *Under the assumptions of Theorem 3.3 with $\alpha = 1$, the stochastic inequality $B\,(X_1 + pX_2) \leqslant_{\rm st} X$ holds if and only if X has the GID mixture distribution function in (4.2).*

Moreover, we improve a result of Pitman and Yor [30]:

THEOREM 4.3. *Let $0 \leqslant X \sim F_X = F$ have mean $\mu = \mathsf{E}[X] \in (0,\infty)$ and let $T \geqslant 0$ have distribution function $F_T(x) = 2\sqrt{x} - x,\ 0 \leqslant x \leqslant 1$. Assume further that a random variable $Z \geqslant 0$ has the length-biased distribution function induced by F:*

$$F_Z(z) = \frac{1}{\mu} \int_0^z x \,\mathrm{d}F(x), \qquad z \geqslant 0,$$

and that the random variables X, T, Z are independent. Then we have the following:

(i) *If*

$$Z \leqslant_{\rm st} X + TZ, \tag{4.3}$$

then $\mathsf{E}[X^2] < \infty$ and the variation coefficient $\mathsf{CV}(X) = \sqrt{\mathsf{Var}(X)}/\mu$ does not exceed $1/\sqrt{5}$.

(ii) *If, in addition, $\mathsf{CV}(X) = 1/\sqrt{5}$, then the LS-transform of X is*

$$\hat{F}(s) = \left(\sqrt{3\mu s}/\sinh\sqrt{3\mu s}\,\right)^2, \qquad s \geqslant 0.$$

REMARK 4.1. Under the assumptions of Theorem 4.3 with $\mu = 2/3$, Pitman and Yor [30, Proposition 12] proved that the distributional equation $Z \stackrel{\rm d}{=} X + TZ$ holds if and only if $\hat{F}(s) = (\sqrt{2s}/\sinh\sqrt{2s}\,)^2$, $s \geqslant 0$.

Appendix. We finally provide the needed lemmas and the proofs of Theorems 4.1 and 4.3. Here, Lemmas 4.1 and 4.2 are well-known (see, e.g., Lin and Hu [20, p. 94] and Lin [18]), while Lemma 4.3 is a novel powerful tool (compare with Hu and Lin [10, Lemma 2.1(a)]). Lemma 4.4 extends Lemma 4.3 if the underlying distribution function F has finite mean.

LEMMA 4.1. *Let X, Y be nonnegative random variables.*

(i) *If $X \leqslant_{\mathrm{st}} Y$, then their LS-transforms satisfy $\hat{F}_Y(s) \leqslant \hat{F}_X(s)$, $s \geqslant 0$.*

(ii) *If $X \leqslant_{\mathrm{st}} Y$ and $\mathsf{E}[X] = \mathsf{E}[Y] < \infty$, then $X \stackrel{\mathrm{d}}{=} Y$.*

LEMMA 4.2. *Let $0 \leqslant X \sim F$. Then for each integer $n \geqslant 1$,*

$$\mathsf{E}[X^n] = \lim_{s \to 0^+} (-1)^n \hat{F}^{(n)}(s) = (-1)^n \hat{F}^{(n)}(0^+) \quad (\textit{finite or infinite}).$$

LEMMA 4.3. *Let $p \in (0, 1)$. Suppose that $0 \leqslant X \sim F$ satisfies*

$$\hat{F}(s) \leqslant \left[1 + \frac{1 - \hat{F}(ps)}{p}\right]^{-1}, \quad 0 \leqslant s \leqslant s_0, \quad \textit{for some } s_0 > 0. \tag{4.4}$$

Then we have the following:

(i) *$\mu = \mathsf{E}[X]$ is finite.*

(ii) *There exists an $s_1 \in (0, s_0)$ such that $\hat{F}(s) \leqslant p + (1-p)[1 + \mu s/(1-p)]^{-1}$ for $s \in [0, s_1]$, and hence $\mathsf{E}[X^2] \leqslant 2\mu^2/(1-p) < \infty$.*

Proof. (i) Since $\hat{F}$ is right continuous at zero and $\hat{F}(0) = 1$, there exists a constant $s_1 \in (0, s_0)$ such that $p < \hat{F}(s) \leqslant 1$ for $s \in [0, s_1]$. Hence we can define

$$g(s) = \frac{1 - \hat{F}(s)}{\hat{F}(s) - p}, \quad s \in [0, s_1],$$

or equivalently

$$\hat{F}(s) = p + (1-p)\frac{1}{1 + g(s)}, \quad s \in [0, s_1]. \tag{4.5}$$

Moreover, $\hat{F}$ is completely monotone on $(0, \infty)$, so that by (4.5) and Lemma 4.2,

$$\hat{F}'(0^+) = -(1-p)g'(0^+). \tag{4.6}$$

On the other hand, it follows from (4.4) that

$$\frac{g(s)}{s} \geqslant \frac{g(ps)}{ps}, \quad s \in (0, s_1].$$

By iteration, we further get, for fixed $s \in (0, s_1]$,

$$\frac{g(s)}{s} \geqslant \frac{g(ps)}{ps} \geqslant \frac{g(p^2 s)}{p^2 s} \geqslant \cdots \geqslant \frac{g(p^n s)}{p^n s}, \quad \forall n \geqslant 1. \tag{4.7}$$

Letting $n \to \infty$ in (4.7) yields

$$\infty > \frac{g(s)}{s} \geqslant g'(0^+), \quad s \in (0, s_1]. \tag{4.8}$$

This, together with (4.6), implies that $\mathsf{E}[X] = -\hat{F}'(0^+) = (1-p)g'(0^+) < \infty$.

(ii) It follows from (4.6) and (4.8) that

$$\frac{g(s)}{s} \geqslant \frac{\mu}{1-p}, \quad s \in (0, s_1],$$

and hence $\hat{F}(s) \leqslant p + (1-p)[1 + \mu s/(1-p)]^{-1}$ for $s \in [0, s_1]$. Equivalently,

$$\frac{\hat{F}(s) - 1 + \mu\, s}{s^2} \leqslant \frac{\mu^2}{1 - p + \mu\, s}, \quad s \in (0, s_1].$$

Letting $s \to 0^+$ and using Lemma 4.2, we conclude that $\frac{1}{2}\mathsf{E}[X^2] = \frac{1}{2}\hat{F}''(0^+) \leqslant \mu^2/(1-p) < \infty$. ■

LEMMA 4.4. *Let $X \sim F$ be a nonnegative random variable with LS-transform $\hat{F}$ and mean $\mu = \mathsf{E}[X] \in (0, \infty)$. Let $0 \leqslant T \leqslant 1$ be a random variable with distribution function F_T and mean $\mathsf{E}[T] \in [0, 1)$. Define the Bernstein function*

$$\sigma(s) = \int_0^1 \frac{1 - \hat{F}(ts)}{t}\, \mathrm{d}F_T(t), \quad s \geqslant 0,$$

where the integrand $(1 - \hat{F}(ts))/t$ is defined for $t = 0$ by continuity to be equal to μs. Suppose that for some constant $s_0 > 0$,

$$\hat{F}(s) \leqslant [1 + \sigma(s)]^{-1}, \quad s \in [0, s_0). \tag{4.9}$$

Then for any $p \in [0, 1)$ such that $F_T(p) \in (0, 1]$,

$$\hat{F}(s) \leqslant 1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda}\, \frac{1}{1 + \lambda s}, \quad s \in [0, s_0), \tag{4.10}$$

where $\lambda = \frac{\mu}{F_T(p)(1-p)} \geqslant \mu$, and hence $\mathsf{E}[X^2] \leqslant 2\lambda\mu < \infty$.

Proof. Assume that the random variable $X_* \sim F_*$ has the equilibrium distribution function induced by F, namely,

$$F_*(x) = \frac{1}{\mu} \int_0^x \bar{F}(t)\, \mathrm{d}t, \quad x \geqslant 0,$$

where $\bar{F}(t) = \mathsf{P}[X > t] = 1 - F(t)$ for $t \geqslant 0$. Recall the relations

$$\hat{F}_*(0) = \hat{F}_{X_*}(0) = 1, \quad \hat{F}_*(s) = \hat{F}_{X_*}(s) = \frac{1 - \hat{F}(s)}{\mu\, s} > 0, \quad s > 0.$$

Then it follows from (4.9) that

$$\frac{1}{\hat{F}_*(s)} \leqslant \mu s + \frac{1}{\int_0^1 \hat{F}_*(ts)\,\mathrm{d}F_T(t)}, \quad s \in [0, s_0). \tag{4.11}$$

Using the Cauchy–Schwarz inequality

$$\int_0^1 \hat{F}_*(ts)\,\mathrm{d}F_T(t) \cdot \int_0^1 \frac{1}{\hat{F}_*(ts)}\,\mathrm{d}F_T(t) \geqslant 1, \quad s \geqslant 0,$$

we further have, by (4.11), for $s \in [0, s_0)$,

$$\begin{aligned}\frac{1}{\hat{F}_*(s)} &\leqslant \mu s + \int_0^1 \frac{1}{\hat{F}_*(ts)}\,\mathrm{d}F_T(t) = \mu s + \int_0^p \frac{1}{\hat{F}_*(ts)}\,\mathrm{d}F_T(t) + \int_p^1 \frac{1}{\hat{F}_*(ts)}\,\mathrm{d}F_T(t)\\ &\leqslant \mu s + F_T(p)\frac{1}{\hat{F}_*(ps)} + (1 - F_T(p))\frac{1}{\hat{F}_*(s)}.\end{aligned}$$

Consequently,

$$\frac{1}{\hat{F}_*(s)} \leqslant \frac{\mu s}{F_T(p)} + \frac{1}{\hat{F}_*(ps)}, \quad s \in [0, s_0). \tag{4.12}$$

Setting $\beta = \mu/F_T(p)$ and iterating (4.12), we have a chain of inequalities: for fixed $s \in [0, s_0)$,

$$\begin{aligned}\frac{1}{\hat{F}_*(s)} &\leqslant \beta s + \frac{1}{\hat{F}_*(ps)} \;\leqslant\; \beta s + \beta p s + \frac{1}{\hat{F}_*(p^2 s)}\\ &\leqslant \cdots\\ &\leqslant \beta s(1 + p + p^2 + \cdots + p^{n-1}) + \frac{1}{\hat{F}_*(p^n s)}\\ &= \frac{\beta s(1 - p^n)}{1 - p} + \frac{1}{\hat{F}_*(p^n s)}, \quad \forall n \geqslant 1.\end{aligned}$$

Letting $n \to \infty$ yields

$$\frac{1}{\hat{F}_*(s)} \leqslant \frac{\beta s}{1 - p} + 1, \quad s \in [0, s_0),$$

or equivalently

$$\frac{\mu s}{1-\hat{F}(s)} \leqslant \lambda s + 1, \qquad s \in (0, s_0),$$

where $\lambda = \beta/(1-p) = \mu/[F_T(p)(1-p)]$. Therefore,

$$\hat{F}(s) \leqslant 1 - \frac{\mu s}{1+\lambda s} = 1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda}\frac{1}{1+\lambda s}, \qquad s \in [0, s_0).$$

This proves (4.10). The last claim follows from (4.10) and the facts that

$$\lim_{s\to 0^+} \frac{\hat{F}(s) - 1 + \mu s}{s^2} = \frac{1}{2}\hat{F}''(0^+) = \frac{1}{2}\mathsf{E}[X^2],$$
$$\lim_{s\to 0^+} \frac{1}{s^2}\left(-\frac{\mu}{\lambda} + \frac{\mu}{\lambda}\frac{1}{1+\lambda s} + \mu s\right) = \lambda\mu. \ \blacksquare$$

REMARK 4.2. When $T = p \in (0,1)$ a.s., (4.9) reduces to (4.4). If $T = 0$ a.s., then, by definition, (4.9) becomes $\hat{F}(s) \leqslant (1+\mu s)^{-1}$ for $s \in [0, s_0)$. On the other hand, $F_T(p) = 1$ for all $p \in [0,1)$, and the RHS of (4.10) (as a function of p) has the minimum $(1+\mu s)^{-1}$, for each $s \in [0, s_0)$. Therefore, the functional upper bound of $\hat{F}(s)$ in (4.10) is sharp in this sense.

Proof of Theorem 4.1. The sufficiency part is a consequence of Theorem 3.2 with $\alpha = 1$, because the SML(α, b) distribution function in (3.4) reduces to an exponential one in (4.2). To prove the necessity part, suppose $X \sim F$ satisfies (4.1). Then, by Lemma 4.1(i),

$$\hat{F}(s) \leqslant \left[1 + \frac{1-\hat{F}(ps)}{p}\right]^{-1}, \qquad s \geqslant 0,$$

and hence condition (4.4) in Lemma 4.3 is satisfied. Thus $\mathsf{E}[X] < \infty$. Moreover, $\mathsf{E}[pS_{N_0}] = \mathsf{E}[X] < \infty$, and hence $pS_{N_0} \stackrel{\mathrm{d}}{=} X$ by Lemma 4.1(ii). Finally, the conclusion follows from Theorem 3.2 with $\alpha = 1$. $\blacksquare$

Proof of Theorem 4.3. (a) We first prove part (i). From (4.3) it follows that

$$\hat{F}_Z(s) \geqslant \hat{F}(s)\hat{F}_{TZ}(s), \qquad s \geqslant 0, \tag{4.13}$$

where $\hat{F}_{TZ}(s) = \mathsf{E}[\exp(-sTZ)]$ for $s \geqslant 0$ (see Lemma 4.1(i)). Recall that

$$\hat{F}_Z(s) = \mathsf{E}[\mathrm{e}^{-sZ}] = \frac{-\hat{F}'(s)}{\mu}, \qquad \int_0^s \hat{F}_Z(x)\,\mathrm{d}x = \frac{1-\hat{F}(s)}{\mu}, \qquad s \geqslant 0,$$

and define

$$\sigma(s) := \mu\int_0^s \hat{F}_{TZ}(x)\,\mathrm{d}x = \int_0^1 \frac{1-\hat{F}(ts)}{t}\,\mathrm{d}F_T(t), \qquad s \geqslant 0.$$

Then, by (4.13),

$$\hat{F}(s) \leqslant \exp(-\sigma(s)) \leqslant [1+\sigma(s)]^{-1}, \quad s \geqslant 0.$$

Lemma 4.4 applies and hence $\mathsf{E}[X^2] < \infty$, or equivalently $\mathsf{E}[Z] < \infty$. By (4.3) again, we have

$$\mathsf{E}[Z] \leqslant \mathsf{E}[X] + \mathsf{E}[TZ] = \mathsf{E}[X] + \mathsf{E}[T]\mathsf{E}[Z], \tag{4.14}$$

from which $\mathsf{CV}(X) = \sqrt{\mathsf{Var}(X)}/\mu \leqslant 1/\sqrt{5}$, because $\mathsf{E}[Z] = \mathsf{E}[X^2]/\mu$. This proves (i).

(b) Suppose, in addition, $\mathsf{CV}(X) = 1/\sqrt{5}$. Then the equality in (4.14) holds true: $\mathsf{E}[Z] = \mathsf{E}[X] + \mathsf{E}[TZ]$. This, together with (4.3), implies that $Z \stackrel{\mathrm{d}}{=} X + T\,Z$ (see Lemma 4.1(ii)). Therefore,

$$\hat{F}(s) = \left(\sqrt{3\mu s}/\sinh\sqrt{3\mu s}\right)^2, \quad s \geqslant 0,$$

by Pitman and Yor [30] (see Remark 4.1 above). Here, we apply the fact that for each $c > 0$, the random variable cZ obeys the length-biased distribution induced by $cX \sim F_{cX}(x) = F_X(x/c)$, where $x \geqslant 0$. Namely,

$$F_{cZ}(z) = F_Z(z/c) = \frac{1}{\mu}\int_0^{z/c} x\,\mathrm{d}F_X(x) = \frac{1}{c\mu}\int_0^{z} x\,\mathrm{d}F_{cX}(x), \quad z \geqslant 0.$$

The proof is complete. ■

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