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FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY G-BROWNIAN MOTION WITH DELAYS

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Abstract. This paper consists of two parts. In part I, existence and uniqueness of solution for fractional stochastic differential equations driven by G-Brownian motion with delays (G-FSDEs for short) is established. In part II, the averaging principle for this type of equations is given. We prove under some assumptions that the solution of G-FSDE can be approximated by solution of its averaged stochastic system in the sense of mean square.

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1. INTRODUCTION

Fractional stochastic differential equations (FSDEs for short) have been applied to describe problems that arise in a variety of fields with memory effect, including finance, physics and optimal control. Among the important theoretical results on FSDEs, we cite the existence and uniqueness of solution, stability results, and the averaging principle for fractional systems with perturbations [3, 2, 4, 12].

On the other hand, the theory of sublinear expectation has been of interest to many researchers due to important potenial applications in uncertainty problems. The concept of uncertainty in fluctations was studied by Peng [9, 10, 11], who established a new stochastic process called G-Brownian motion as a way to incorporate the unknown volatility into financial models. Denis and Martini [5] suggested a structure based on quasi-sure analysis from abstract potential theory to construct a similar structure using a tight family of possibly mutually singular probability measures. To date, problems with uncertainty based on G-Brownian motion have been widely studied by several authors.

The solution properties of FSDEs have also been widely studied. Zhang et al. [16] studied the existence and uniqueness of solution for SDEs of fractional order

$q > 1$ with finite delays. Moghaddam and Zhang [8] studied sufficient conditions for existence and uniqueness of solutions of fractional stochastic delay differential equations. The averaging method is a powerful tool to strike a balance between complex models that are more realistic and simpler models that are more amenable to analysis and simulation. For the averaging principle for FSDEs we refer to [1, 7, 13, 14].

In this paper we study, under suitable assumptions, the existence and uniqueness of solution, as well as the theory of averaging, for the following G-FSDE:

$$\begin{cases} D_t^\alpha X(t) = b(t, X_t)dt + h(t, X_t)d\langle B\rangle_t + \sigma(t, X_t)dB_t, \quad t \in [0, T], \\ X_0 = \Psi := \{\Psi(\theta) : -\tau \leqslant \theta \leqslant 0\}, \end{cases} \tag{1.1}$$

where $X_t = \{X(t+\theta) : -\tau \leqslant \theta \leqslant 0\}$, $\tau \in [0, +\infty[$, and D_t^α is the Caputo fractional derivative with $\alpha \in (1/2, 1)$. The coefficients b, h, σ are in the space $\mathcal{M}_G^2([0, T]; \mathbb{R})$ and $\{\langle B\rangle_t, t \geqslant 0\}$ is the quadratic variation process of G-Brownian motion. We denote by $BC([-\tau, 0]; \mathbb{R})$ the family of bounded continuous $\mathbb{R}$-valued mappings ϕ defined on $[-\tau, 0]$ with norm $\|\phi\| = \sup_{-\tau\leqslant\theta\leqslant 0} |\phi(\theta)|$.

This article is organized as follows. In the next section, we give some preliminaries. In Section 3, we present the existence and uniqueness of solution for fractional stochastic differential equation driven by G-Brownian motion. In the last section, the averaging principle for this type of equation is given.

2. PRELIMINARIES ON SUBLINEAR EXPECTATION

In this section, we introduce notations and preliminary results in the G-framework which we need. Further details can be found in [6, 9, 10, 11].

Let $C_{\mathrm{b,lip}}(\mathbb{R}^n)$ be the space of all bounded and Lipschitz continuous functions on $\mathbb{R}^n$. Let $T \in \mathbb{R}^+$ be a fixed time. Consider the space Ω of all real valued continuous functions on $[0, T]$ such that $\omega(0) = 0$ equipped with the following distance:

$$\rho(\omega^1, \omega^2) = \sum_{n=1}^{\infty} 2^n \Big(\max_{t=[0,n]} |\omega_t^1 - \omega_t^2| \wedge 1 \Big), \quad \omega^1, \omega^2 \in \Omega,$$

and consider the canonical process $B_t(\omega) = \omega_t$ for $t \in [0, \infty)$ and $\omega \in \Omega$. Let

$$\mathrm{Lip}(\Omega_t) := \{\phi(B_{t_1}, B_{t_2} - B_{t_n}) : t_1, \ldots, t_n \in [0, t],\, \phi \in C_{\mathrm{b,lip}}(\mathbb{R}^n)\},$$
$$\mathrm{Lip}(\Omega) := \bigcup_{n=1}^{\infty} \mathrm{Lip}(\Omega_n).$$

We have $\mathrm{Lip}(\Omega_t) \subset \mathrm{Lip}(\Omega_T)$ for each $t \in [0, T]$.

A functional $\widehat{\mathbb{E}} : \mathcal{H} := \mathrm{Lip}(\Omega) \to \mathbb{R}$ is a *consistent sublinear expectation* on the lattice $\mathcal{H}$ of real functions if it satisfies:

- Monotonicity: for all $X, Y \in \mathcal{H}$, $X \geqslant Y \Rightarrow \widehat{\mathbb{E}}[X] \geqslant \widehat{\mathbb{E}}[Y]$.
- Constant preserving: for all $c \in \mathbb{R}$, $\widehat{\mathbb{E}}[c] = c$.
- Subadditivity: for all $X, Y \in \mathcal{H}$, $\widehat{\mathbb{E}}[X+Y] \leqslant \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$.
- Positive homogeneity: for all $\lambda \geqslant 0, Y \in \mathcal{H}$, $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$.

The triplet $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is a *sublinear expectation space*.

DEFINITION 2.1. Let $Y = (Y_1, \ldots, Y_n)$ be an n-dimensional random vector on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$. It is said to be *independent* of an m-dimensional random vector $X = (X_1, \ldots, X_m)$ if for each $\varphi \in C_{\mathrm{b,lip}}(\mathbb{R}^{n+m})$,

$$\widehat{\mathbb{E}}[\varphi(X,Y)] = \widehat{\mathbb{E}}\big[\widehat{\mathbb{E}}[\varphi(x,Y)]_{x=X}\big].$$

DEFINITION 2.2 (G-Brownian motion). The canonical process $(B_t)_{t\geqslant 0}$ on $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ is called a *G-Brownian motion* if the following properties are satisfied:

(1) $B_0 = 0$.

(2) For any $t, s \geqslant 0$ the increment $B_{t+s} - B_t$ is $\mathcal{N}(0, [s\underline{\sigma}^2, s\overline{\sigma}^2])$-distributed.

(3) $(B_{t_1}, \ldots, B_{t_n})$ is independent of B_t for all $n \geqslant 1$ and $t_1, \ldots, t_n \in [0, t]$.

We denote by $L_G^p(\Omega_T)$ ($p \geqslant 1$) the Banach space completion of $\mathrm{Lip}(\Omega_T)$ under the natural norm $\|X\|_p := \widehat{\mathbb{E}}[|X|^p]^{1/p}$ and we consider the following simple process $\mathcal{M}_G^{p,0}(0,T)$: for a given partition $\pi_T = \{t_0, t_1, \ldots, t_N\}$ of $[0,T]$,

$$\Big\{\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)\mathbb{I}_{[t_j, t_{j+1})}(t),\ \xi_j \in L_G^p(\Omega_{t_j})\Big\}.$$

Let us denote by $\mathcal{M}_G^p(0,T)$ the completion of $\mathcal{M}_G^{p,0}(0,T)$ under the norm

$$\|\eta\|_{\mathcal{M}_G^p(0,T)} := \Big[\int_0^T \widehat{\mathbb{E}}[|\eta_s|^p]\,ds\Big]^{1/p}, \qquad p \geqslant 1.$$

For each $\eta \in \mathcal{M}_G^{2,0}(0,T)$, the related Ito integral of $(B_t)_{t\geqslant 0}$ is defined by

$$I(\eta) = \int_0^T \eta(s)\,dB_s := \sum_{j=0}^{N-1} \eta_j (B_{t_{j+1}} - B_{t_j}),$$

where the mapping $I : \mathcal{M}_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is continuously extended to $\mathcal{M}_G^2(0,T)$. The quadratic variation process $\langle B\rangle_t$ of $(B_t)_{t\geqslant 0}$, defined by

$$\langle B\rangle_t := B_t^2 - 2\int_0^t B_s\,dB_s.$$

For each $\eta \in \mathcal{M}_G^{1,0}(0,T)$, let $J : \mathcal{M}_G^{1,0}(0,T) \to L_G^1(\Omega_T)$ be given by

$$J(\eta) = \int_0^T \eta(t)\, d\langle B\rangle_t := \sum_{j=0}^{N-1} \xi_j(\langle B\rangle_{t_{j+1}} - \langle B\rangle_{t_j}).$$

Then J can be extended continuously to $\mathcal{M}_G^1(0,T)$.

DEFINITION 2.3. We define the *capacity* $\mathbb{C}$ associated with $\widehat{\mathbb{E}}$ by putting

$$\mathbb{C}(A) := \sup_{p\in\mathcal{P}} P(A), \qquad A \in \mathcal{B}(\Omega).$$

We will say that a set $A \in \mathcal{B}(\Omega)$ is *polar* if $\mathbb{C}(A) = 0$. We say that a property holds *quasi-surely* (*q.s.* for short) if it holds outside a polar set.

LEMMA 2.1 ([11]). *Let $X \in L_G^p(\Omega)$. Then for each $\alpha > 0$,*

$$\mathbb{C}(|X| > \alpha) \leqslant \frac{\widehat{\mathbb{E}}[|X|^p]}{\alpha^p}.$$

The following two lemmas are the G-BDG type inequalities with respect to the quadratic variation process $\langle B\rangle_t$ and B_t respectively.

LEMMA 2.2. *Let $p \geqslant 1$, $\eta \in \mathcal{M}_G^p(0,T)$ and $0 \leqslant s \leqslant t \leqslant u \leqslant T$. Then*

$$\widehat{\mathbb{E}}\Big[\sup_{s\leqslant t\leqslant u}\Big|\int_s^t \eta_v\, d\langle B\rangle_v\Big|^p\Big] \leqslant C_1|u-s|^{p-1}\int_s^u \widehat{\mathbb{E}}[|\eta_v|^p]\, dv,$$

where C_1 is a positive constant independent of η.

LEMMA 2.3. *Let $p \geqslant 2$, $\eta \in \mathcal{M}_G^p(0,T)$ and $0 \leqslant s \leqslant t \leqslant u \leqslant T$. Then*

$$\widehat{\mathbb{E}}\Big[\sup_{s\leqslant t\leqslant u}\Big|\int_s^t \eta_v\, dB_v\Big|^p\Big] \leqslant C_2|u-s|^{p/2-1}\int_s^u \widehat{\mathbb{E}}[|\eta_v|^p]\, dv,$$

where C_2 is a positive constant independent of η.

We will need the generalized Gronwall lemma:

THEOREM 2.1 ([15]). *Let $\beta > 0$ and $b \geqslant 0$. Assume that a, u are nonnegative and locally integrable functions defined on $[0,T]$ such that*

$$u(t) \leqslant a(t) + b\int_0^t (t-s)^{\beta-1}u(s)\, ds.$$

Then

$$u(t) \leqslant a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\Big]\, ds \quad \textit{for each } 0 \leqslant t < T.$$

COROLLARY 2.1 ([15]). *Under the hypotheses of Theorem* 2.1*, let* $a(t)$ *be a nondecreasing function on* $[0, T)$*. Then*

$$u(t) \leqslant a(t)E_\beta(b\Gamma(\beta)t^\beta),$$

where E_β *is the Mittag-Leffler function defined by* $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

3. EXISTENCE AND UNIQUENESS

The objective of this section is to prove, under suitable conditions, existence and uniqueness of the solution of equation (1.1), where the functions $b(\cdot, x), h(\cdot, x)$ and $\sigma(\cdot, x)$ are in $\mathcal{M}_G^2([0, T])$ for each $x \in \mathbb{R}$. To this end, we make the following assumptions:

(H_1) The functions $J = b, h, \sigma : [0, T] \times BC([-\tau, 0]; \mathbb{R}) \times \Omega \to \mathbb{R}$ satisfy the Lipschitz condition with respect to x, uniformly in t, that is, for any $x, y \in BC([-\tau, 0]; \mathbb{R})$,

$$|J(t, x) - J(t, y)|^2 \leqslant D\|x - y\|^2 \quad \text{q.s.},$$

(H_2) $|J(t, 0)|^2 \leqslant P$ q.s.

uniformly with respect to t, where D and P are positive constants.

DEFINITION 3.1. We say that the process $X \in \mathcal{M}_G^2([-\tau, T])$ is a *solution* of equation (1.1) with initial condition Ψ if for all $t \in [0, T]$,

$$\begin{aligned} X(t) = \Psi(0) &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s, X_s)\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, X_s)\, d\langle B\rangle_s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, X_s)\, dB_s \end{aligned} \tag{3.1}$$

and $X(t) = \Psi(t)$ for all $t \in [-\tau, 0]$.

THEOREM 3.1. *Let assumptions* (H_1) *and* (H_2) *be satisfied. Then the G-FSDE* (1.1) *has a unique solution in* $\mathcal{M}_G^2([-\tau, T])$.

Proof. **Uniqueness.** Let $X, Z \in \mathcal{M}_G^2([-\tau, T])$ be two solutions of (1.1) with the same initial condition Ψ. First, observe that for all $0 \leqslant s \leqslant T$,

$$\|X_s - Z_s\| \leqslant \sup_{u\in[-\tau,s]} |X(u) - Z(u)| \leqslant \sup_{u\in[0,s]} |X(u) - Z(u)|.$$

It is clear that for $0 \leqslant t \leqslant T$,

$$\begin{aligned} X(t) - Z(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[b(s,X_s) - b(s,Z_s)]\, ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[h(s,X_s) - h(s,Z_s)]\, d\langle B\rangle_s \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\sigma(s,X_s) - \sigma(s,Z_s)]\, dB_s, \end{aligned}$$

so that

$$\begin{aligned} \sup_{t\in[0,T]} |X(t) - Z(t)|^2 &\leqslant \frac{3}{\Gamma(\alpha)^2} \sup_{t\in[0,T]} \left| \int_0^t (t-s)^{\alpha-1}[b(s,X_s) - b(s,Z_s)]\, ds \right|^2 \\ &+ \frac{3}{\Gamma(\alpha)^2} \sup_{t\in[0,T]} \left| \int_0^t (t-s)^{\alpha-1}[h(s,X_s) - b(s,Z_s)]\, d\langle B\rangle_s \right|^2 \\ &+ \frac{3}{\Gamma(\alpha)^2} \sup_{t\in[0,T]} \left| \int_0^t (t-s)^{\alpha-1}[\sigma(s,X_s) - \sigma(s,Z_s)]\, dB_s \right|^2. \end{aligned}$$

Applying the Cauchy–Schwarz, Hölder and G-BDG inequalities under hypothesis (H_1), we easily obtain the following estimate:

$$\begin{aligned} \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |X(t) - Z(t)|^2\Big] &\leqslant \frac{3T^{2\alpha-1}}{\Gamma(\alpha)^2} \int_0^T (D\widehat{\mathbb{E}}[\|X_s - Z_s\|^2])\, ds \\ &+ \frac{3T^{2\alpha-1}C_1}{\Gamma(\alpha)^2} \int_0^T (D\widehat{\mathbb{E}}[\|X_s - Z_s\|^2])\, ds \\ &+ \frac{3C_2T^{2\alpha-2}}{\Gamma(\alpha)^2} \int_0^T (D\widehat{\mathbb{E}}[\|X_s - Z_s\|^2])\, ds \\ &\leqslant r_1 \int_0^T \widehat{\mathbb{E}}[\|X_s - Z_s\|^2]\, ds \\ &\leqslant r_1 \int_0^T \widehat{\mathbb{E}}\Big[\sup_{u\in[0,s]} |X(u) - Z(u)|^2\Big]\, ds, \end{aligned}$$

where $r_1 = \frac{3DT^{2\alpha-2}(T+TC_1+C_2)}{\Gamma(\alpha)^2}$. It follows, by the classical Gronwall lemma, that

$$\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |X(s) - Z(s)|^2\Big] = 0,$$

which implies that $X(s) = Z(s)$ q.s. for any $s \in [0,T]$ and then $X(s) = Z(s)$ q.s. for all $s \in [-\tau, T]$.

Existence. Let $X^0(t) = 0$ for any $t \in [-\tau, T]$. Define the following Picard sequence: For each $n \geqslant 1$, we set $X_0^n = \Psi$ and

$$X^n(t) = \Psi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s, X_s^{n-1})\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s, X_s^{n-1})\, d\langle B\rangle_s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s, X_s^{n-1})\, dB_s. \tag{3.2}$$

The existence will be proved in three steps.

Step 1: We prove that

$$X^n(t) \in L_G^2(\Omega) \quad \textit{for all } t \in [0,T].$$

We claim that $X^n \in \mathcal{M}_G^2([-\tau, T])$. Indeed, we have

$$|X^n(t)|^2 \leqslant 4|\Psi(0)|^2 + \frac{4}{\Gamma(\alpha)^2} \Big| \int_0^t (t-s)^{\alpha-1} b(s, X_s^{n-1})\, ds \Big|^2 + \frac{4}{\Gamma(\alpha)^2} \Big| \int_0^t (t-s)^{\alpha-1} h(s, X_s^{n-1})\, d\langle B\rangle_s \Big|^2 + \frac{4}{\Gamma(\alpha)^2} \Big| \int_0^t (t-s)^{\alpha-1} \sigma(s, X_s^{n-1})\, dB_s \Big|^2.$$

Applying the Cauchy–Schwarz, Hölder and G-BDG inequalities, we get

$$\widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]} |X^n(t)|^2 \Big) \leqslant 4|\Psi(0)|^2 + \frac{4T}{\Gamma(\alpha)^2} \widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]} \int_0^t (t-s)^{2\alpha-2} |b(s, X_s^{n-1})|^2\, ds \Big) + \frac{4TC_1}{\Gamma(\alpha)} \widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]} \int_0^t (t-s)^{2\alpha-2} |h(s, X_s^{n-1})|^2\, ds \Big) + \frac{4C_2}{\Gamma(\alpha)} \widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]} \int_0^t (t-s)^{2\alpha-2} |\sigma(s, X_s^{n-1})|^2\, ds \Big).$$

From (H_1) and (H_2), we derive

$$|J(s,x)|^2 \leqslant 2|J(s,x) - J(s,0)|^2 + 2|J(s,0)|^2 \leqslant 2D\|x\|^2 + 2P.$$

It follows that

$$
\begin{aligned}
\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}|X^n(t)|^2\Big] &\leqslant 4\|\Psi\|^2 \\
&\quad + \frac{8T}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}(D\widehat{\mathbb{E}}[\|X_s^{n-1}\|^2]+P)\,ds \\
&\quad + \frac{8TC_1}{\Gamma(\alpha)}\int_0^T (t-s)^{2\alpha-2}(D\widehat{\mathbb{E}}[\|X_s^{n-1}\|^2]+P)\,ds \\
&\quad + \frac{8C_2}{\Gamma(\alpha)}\int_0^T (t-s)^{2\alpha-2}(D\widehat{\mathbb{E}}[\|X_s^{n-1}\|^2]+P)\,ds \\
&\leqslant 4\|\Psi\|^2 + \frac{8PT^{2\alpha-1}(T+TC_1+C_2)}{(2\alpha-1)\Gamma(\alpha)^2} \\
&\quad + \frac{8D(T+TC_1+C_2)}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\widehat{\mathbb{E}}[\|X_s^{n-1}\|^2]\,ds.
\end{aligned}
$$

Noting that

$$
\|X_s^{n-1}\|^2 \leqslant \sup_{u\in[-\tau,s]}|X^{n-1}(u)|^2 \leqslant \|\Psi\|^2 + \sup_{u\in[0,s]}|X^{n-1}(u)|^2,
$$

it follows that

$$
\begin{aligned}
&\widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]}|X^n(t)|^2\Big) \\
&\quad \leqslant 4\|\Psi\|^2 + \frac{8PT^{2\alpha-1}(T+TC_1+C_2)}{(2\alpha-1)\Gamma(\alpha)^2} \\
&\qquad + \frac{8D(T+TC_1+C_2)}{\Gamma(\alpha)^2}\int_0^T (t-s)^{2\alpha-2}\widehat{\mathbb{E}}\Big[\|\Psi\|^2 + \sup_{u\in[0,s]}|X^{n-1}(u)|^2\Big]\,ds \\
&\quad \leqslant r_2 + r_3\int_0^T (t-s)^{(2\alpha-1)-1}\widehat{\mathbb{E}}\Big(\sup_{u\in[0,s]}|X^{n-1}(u)|^2\Big)\,ds,
\end{aligned}
$$

where

$$
\begin{aligned}
r_2 &= 4\|\Psi\|^2\left(1+\frac{2DT^{2\alpha-1}(T+TC_1+C_2)}{(2\alpha-1)\Gamma(\alpha)^2}\right) + \frac{8PT^{2\alpha-1}(T+TC_1+C_2)}{(2\alpha-1)\Gamma(\alpha)^2}, \\
r_3 &= \frac{8D(T+TC_1+C_2)}{\Gamma(\alpha)^2}.
\end{aligned}
$$

On the other hand, for any $k \geqslant n$, we have

$$\max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]} |X^n(t)\Big)|^2)$$
$$\leqslant r_2 + r_3 \int_0^T (t-s)^{2\alpha-2} \max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{u\in[0,s]} |X^{n-1}(u)\Big)|^2)\, ds.$$

Moreover,

$$\max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{u\in[0,s]} |X^{n-1}(u)|^2\Big) \leqslant \max\Big\{\widehat{\mathbb{E}}\|\Psi\|^2, \max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{u\in[0,s]} |X^n(u)|^2\Big)\Big\}$$
$$\leqslant \|\Psi\|^2 + \max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{u\in[0,s]} |X^n(u)|^2\Big).$$

Therefore,

$$\max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]} |X^n(t)|^2\Big)$$
$$\leqslant r_2 + r_3 \int_0^T (t-s)^{2\alpha-2}\Big(\|\Psi\|^2 + \max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{u\in[0,s]} |X^n(u)|^2\Big)\Big)\, ds$$
$$\leqslant r_4 + r_3 \int_0^T (t-s)^{2\alpha-2} \max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{u\in[0,s]} |X^n(u)|^2\Big)\, ds,$$

where $r_4 = r_2 + r_3 \frac{T^{2\alpha-1}}{(2\alpha-1)}\|\Psi\|^2$. Now by Corollary 2.1, we get

$$\max_{1\leqslant n\leqslant k} \widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]} |X^n(t)|^2\Big) \leqslant r_4 E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}).$$

We deduce that

$$\widehat{\mathbb{E}}\Big(\sup_{t\in[0,T]} |X^n(t)|^2\Big) \leqslant r_4 E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}), \tag{3.3}$$

so that

$$\widehat{\mathbb{E}}(|X^n(t)|^2) \leqslant r_4 E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}) \quad \text{for each } t\in[0,T],$$

which implies that $X^n(t) \in L^2_G(\Omega)$. We deduce that

$$\|X^n\|^2_{\mathcal{M}^2_G([-\tau,T])} = \int_{-\tau}^T \widehat{\mathbb{E}}(|X^n_s|^2)\, ds$$
$$= \int_{-\tau}^0 \|\Psi\|^2 ds + \int_0^T \widehat{\mathbb{E}}(|X^n_s|^2)\, ds$$
$$\leqslant \tau\|\Psi\|^2 + T r_4 E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}),$$

which means that $X^n \in \mathcal{M}^2_G([-\tau,T])$.

Step 2: We prove that $(X^n)_{n\in\mathbb{N}}$ *is a Cauchy sequence in* $\mathcal{M}_G^2([0,T])$. Consider the space

$$H_T := \Big\{ X \in \mathcal{M}_G^2([0,T]) : \widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |X(s)|^2\Big] < \infty \Big\}$$

equipped with the norm

$$N(X) = \Big(\widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |X(s)|^2\Big]\Big)^{1/2}.$$

It follows from (3.2) that

$$\begin{aligned} X^1(t) - X^0(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} b(s,0)\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s,0)\, d\langle B\rangle_s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s,0)\, dB_s. \end{aligned}$$

Similarly to the proof of uniqueness, we have

$$\begin{aligned} \widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |X^1(t) - X^0(t)|^2\Big] &\leqslant \frac{3}{\Gamma(\alpha)^2} \sup_{t\in[0,T]} \Big|\int_0^t (t-s)^{\alpha-1} b(s,0)\, ds\Big|^2 \\ &\quad + \frac{3}{\Gamma(\alpha)^2} \Big|\sup_{t\in[0,T]} \int_0^t (t-s)^{\alpha-1} h(s,0)\, d\langle B\rangle_s\Big|^2 \\ &\quad + \frac{3}{\Gamma(\alpha)^2} \sup_{t\in[0,T]} \Big|\int_0^t (t-s)^{\alpha-1} \sigma(s,0)\, dB_s\Big|^2 \\ &\leqslant \frac{3T}{\Gamma(\alpha)^2} \int_0^T (t-s)^{2\alpha-2} |b(s,0)|^2\, ds \\ &\quad + \frac{3TC_1}{\Gamma(\alpha)^2} \int_0^T (t-s)^{2\alpha-2} |h(s,0)|^2\, ds \\ &\quad + \frac{3C_2}{\Gamma(\alpha)^2} \int_0^T (t-s)^{2\alpha-2} |\sigma(s,0)|^2\, ds. \end{aligned}$$

Then from (H_2) we get

$$\begin{aligned} &\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |X^1(t) - X^0(t)|^2\Big] \\ &\leqslant \frac{3TP}{\Gamma(\alpha)^2} \int_0^T (t-s)^{2\alpha-2} ds + \frac{3TC_1P}{\Gamma(\alpha)^2} \int_0^T (t-s)^{2\alpha-2} ds + \frac{3C_2P}{\Gamma(\alpha)^2} \int_0^T (t-s)^{2\alpha-2}\, ds \\ &\leqslant \frac{3(T + TC_1 + C_2)PT^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}. \end{aligned}$$

It follows that

$$N(X^1 - X^0) \leqslant K, \tag{3.4}$$

where $K = \big(\frac{3(T+TC_1+C_2)PT^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}\big)^{1/2}$. Now for any $n \geqslant 1$ and $t \in [0,T]$ we have

$$\begin{aligned} X^{n+1}(t) - X^n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[b(s,X_s^n) - b(s,X_s^{n-1})]\, ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[h(s,X_s^n) - h(s,X_s^{n-1})]\, d\langle B\rangle_s \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\sigma(s,X_s^n) - \sigma(s,X_s^{n-1})]\, dB_s. \end{aligned}$$

Similarly to the proof of uniqueness, we get

$$\begin{aligned} &\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]} |X^{n+1}(t) - X^n(t)|^2\Big] \\ &\qquad \leqslant \frac{3}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\Big|\int_0^t (t-s)^{\alpha-1}[b(s,X_s^n) - b(s,X_s^{n-1})]\, ds\Big|^2\Big] \\ &\qquad + \frac{3}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\Big|\int_0^t (t-s)^{\alpha-1}[h(s,X_s^n) - h(s,X_s^{n-1})]\, d\langle B\rangle_s\Big|^2\Big] \\ &\qquad + \frac{3}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,T]}\Big|\int_0^t (t-s)^{\alpha-1}[\sigma(s,X_s^n) - \sigma(s,X_s^{n-1})]\, dB_s\Big|^2\Big] \\ &\quad \leqslant \frac{3D(T+TC_1+C_2)T^{2\alpha-2}}{\Gamma(\alpha)^2} \int_0^T \widehat{\mathbb{E}}\Big(\sup_{0\leqslant t_2\leqslant t_1} |X^n(t_2) - X^{n-1}(t_2)|^2\Big)\, dt_1 \\ &\quad \leqslant \frac{3D(T+TC_1+C_2)T^{2\alpha-2}}{\Gamma(\alpha)^2} \int_0^T \widehat{\mathbb{E}}\Big(\sup_{0\leqslant t_2\leqslant t_1} |X_{t_2}^n - X_{t_2}^{n-1}|^2\Big)\, dt_1. \end{aligned}$$

Then we can write, by setting $\gamma = \frac{3D(T+TC_1+C_2)T^{2\alpha-1}}{(2\alpha-1)\Gamma(\alpha)^2}$,

$$\begin{aligned} \widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |X^{n+1}(s) - X^n(s)|^2\Big] &\leqslant \gamma \int_0^T \widehat{\mathbb{E}}\Big(\sup_{0\leqslant t_2\leqslant t_1} |X^n(t_2) - X^{n-1}(t_2)|^2\Big)\, dt_1 \\ &\leqslant \gamma^2 \int_0^T\int_0^{t_1} \widehat{\mathbb{E}}\Big(\sup_{0\leqslant t_3\leqslant t_2} |X^{n-1}(t_3) - X^{n-2}(t_3)|^2\Big)\, dt_1\, dt_2 \\ &\leqslant \gamma^n K \int_0^T\int_0^{t_1} \dots \int_0^{t_n} dt_1 \dots dt_n \leqslant K\frac{(\gamma T)^n}{n!}. \end{aligned}$$

It follows that

$$N(X^{n+1}-X^n)\leqslant\left(K\frac{(\gamma T)^n}{n!}\right)^{1/2}. \tag{3.5}$$

For $m>n$,

$$\begin{aligned}N(X^m-X^n)&=N\Big(\sum_{i=n+1}^{m}(X^i-X^{i-1})\Big)\leqslant\sum_{i=n+1}^{m}N(X^i-X^{i-1})\\&\leqslant\sum_{i>n}N(X^i-X^{i-1})\leqslant\sum_{i>n}\sqrt{K\frac{(\gamma T)^i}{i!}}.\end{aligned}$$

This implies that $(X^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H_T and also in $\mathcal{M}_G^2([0,T])$. Let X be the limit of this sequence.

Step 3: We prove that for all $t\in[0,T]$, X_t is the solution of the G-FSDE (1.1). By using the continuity of the norm N, we deduce from (3.3) that

$$\widehat{\mathbb{E}}\Big(\sup_{s\in[0,T]}|X(s)|^2\Big)\leqslant r_4E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}), \tag{3.6}$$

which implies that

$$\begin{aligned}\widehat{\mathbb{E}}\Big(\int_{-\tau}^{T}|X(s)|^2\,ds\Big)&=\widehat{\mathbb{E}}\Big(\int_{-\tau}^{0}|X(s)|^2\,ds\Big)+\widehat{\mathbb{E}}\Big(\int_{0}^{T}|X(s)|^2\,ds\Big)\\&\leqslant\tau\|\Psi\|^2+r_4TE_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1})<\infty.\end{aligned}$$

Therefore, $X\in\mathcal{M}_G^2([-\tau,T])$.

By the uniqueness of the limit, it suffices to prove that for each $t\in[0,T]$, the sequence of random variables $(K_n(t))_n$ (resp. $(T_n(t))_n,(V_n(t))_n$) converges in $L_G^2(\Omega)$ to the random variable $\int_0^t(t-s)^{\alpha-1}b(s,X_s)\,ds$ (resp. $\int_0^t(t-s)^{\alpha-1}h(s,X_s)\,d\langle B\rangle_s,\int_0^t(t-s)^{\alpha-1}\sigma(s,X_s)\,dB_s$), where

$$\begin{aligned}K_n(t)&=\int_0^t(t-s)^{\alpha-1}b(s,X_s^{n-1})\,ds,\\T_n(t)&=\int_0^t(t-s)^{\alpha-1}h(s,X_s^{n-1})\,d\langle B\rangle_s,\\V_n(t)&=\int_0^t(t-s)^{\alpha-1}\sigma(s,X_s^{n-1})\,dB_s.\end{aligned}$$

Indeed, by Hölder's inequality we have

$$\begin{aligned}\widehat{\mathbb{E}}\Big[\int\limits_0^t (t-s)^{\alpha-1} b(s,X_s^n)\,ds - \int\limits_0^t (t-s)^{\alpha-1} b(s,X_s)\,ds\Big]^2 \\ = \widehat{\mathbb{E}}\Big[\int\limits_0^t (t-s)^{\alpha-1}[b(s,X_s^n)-b(s,X_s)]\,ds\Big]^2 \\ \leqslant T\int\limits_0^T (t-s)^{2\alpha-2}\widehat{\mathbb{E}}|b(s,X_s^n)-b(s,X_s)|^2\,ds \\ \leqslant DT^{2\alpha-1}\int\limits_0^t \widehat{\mathbb{E}}\Big(\sup_{r\in[0,T]}|X_r^n-X_r|^2\Big)\,ds.\end{aligned}$$

Then

$$\begin{aligned}\widehat{\mathbb{E}}\Big[\frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} b(s,X_s^n)\,ds - \frac{1}{\Gamma(\alpha)}\int\limits_0^t (t-s)^{\alpha-1} b(s,X_s)\,ds\Big]^2 \\ \leqslant \frac{DT^{2\alpha-2}}{\Gamma(\alpha)^2}(N(X^n-X))^2,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\int\limits_0^t (t-s)^{\alpha-1} b(s,X_s^n)\,ds = \int\limits_0^t (t-s)^{\alpha-1} b(s,X_s)\,ds \quad \text{in } L_G^2(\Omega).$$

Similarly, by using G-BDG inequalities, we get

$$\begin{aligned}\widehat{\mathbb{E}}\Big[\int\limits_0^t (t-s)^{\alpha-1} h(s,X_s^n)\,d\langle B\rangle_s - \int\limits_0^t (t-s)^{\alpha-1} h(s,X_s)\,d\langle B\rangle_s\Big]^2 \\ \leqslant DC_1T^{2\alpha-2}(N(X^n-X))^2\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathbb{E}}\Big[\int\limits_0^t (t-s)^{\alpha-1} \sigma(s,X_s^n)\,dB_s - \int\limits_0^t (t-s)^{\alpha-1} \sigma(s,X_s)\,dB_s\Big]^2 \\ \leqslant DC_2T^{2\alpha-1}(N(X^n-X))^2,\end{aligned}$$

so that

$$\lim_{n\to\infty}\int\limits_0^t (t-s)^{\alpha-1} h(s,X_s^n)\,d\langle B\rangle_s = \int\limits_0^t (t-s)^{\alpha-1} h(s,X_s)\,d\langle B\rangle_s \quad \text{in } L_G^2(\Omega)$$

and

$$\lim_{n\to\infty}\int\limits_0^t (t-s)^{\alpha-1} \sigma(s,X_s^n)\,dB_s = \int\limits_0^t (t-s)^{\alpha-1} \sigma(s,X_s)\,dB_s \quad \text{in } L_G^2(\Omega).$$

The proof is complete. ■

COROLLARY 3.1. *Let X be the solution of* (1.1). *Then*

$$\widehat{\mathbb{E}}\Big[\sup_{t\in[-\tau,T]}|X(t)|^2\Big] \leqslant \|\Psi\|^2 + r_4 E_{2\alpha-1}(r_3\Gamma(2\alpha-1)T^{2\alpha-1}).$$

Proof. Follows from (3.6) and the fact that $\sup_{t\in[-\tau,0]}|X(t)| = \|\Psi\|$. ∎

4. THE AVERAGING PRINCIPLE

In this section, we study the averaging principle for (1.1). Let us consider the standard form of (1.1):

(4.1)
$$\begin{cases} D_t^\alpha X^\varepsilon(t) = \varepsilon b(t, X_t^\varepsilon)dt + \sqrt{\varepsilon}\, h(t, X_t^\varepsilon)d\langle B\rangle_t + \sqrt{\varepsilon}\,\sigma(t, X_t^\varepsilon)\, dB_t, \quad t\in[0,T], \\ X_0 = \Psi, \end{cases}$$

where $\varepsilon\in(0,\varepsilon_0]$ is a small parameter with ε_0 fixed. Before turning to the averaging principle, we introduce some Lipschitz and linear growing coefficients $\overline{b}(\cdot), \overline{h}(\cdot)$ and $\overline{\sigma}(\cdot) : BC([-\tau,0];\mathbb{R})\to\mathbb{R}$ satisfying the following hypothesis:

(H_3) For any $(T_1, x)\in[0,T]\times BC([-\tau,0];\mathbb{R})$, there exist bounded positive functions $\alpha_i(T_1)$, $i = 1,2,3$, such that

$$\frac{1}{T_1}\int_0^{T_1}|b(s,x)-\overline{b}(x)|\,ds \leqslant \alpha_1(T_1)(1+\|x\|) \quad \text{q.s.},$$
$$\frac{1}{T_1}\int_0^{T_1}|h(s,x)-\overline{h}(x)|^2\,ds \leqslant \alpha_2(T_1)(1+\|x\|^2) \quad \text{q.s.},$$
$$\frac{1}{T_1}\int_0^{T_1}|\sigma(s,x)-\overline{\sigma}(x)|^2\,ds \leqslant \alpha_3(T_1)(1+\|x\|^2) \quad \text{q.s.}$$

where $\lim_{T_1\to\infty}\alpha_i(T_1) = 0$, $i = 1,2,3$.

REMARK 4.1. Typical examples for b, h, σ satisfying hypothesis (H_3) are as follows:

(1) Let $T = \pi$ and $b(s,x) = h(s,x) = \sigma(s,x) = \cos^2 s\sum_{k\geqslant 1}\frac{\sin(kx(0))}{k^3}$. Then

$$\overline{b}(x) = \overline{h}(x) = \overline{\sigma}(x) = \frac{1}{\pi}\sum_{k\geqslant 1}\frac{\sin(kx(0))}{k^3}\int_0^\pi \cos^2 s\, ds$$
$$= \frac{1}{2}\sum_{k\geqslant 1}\frac{\sin(kx(0))}{k^3}.$$

It is easy to check that for $T_1 \in [0, \pi]$,

$$\frac{1}{T_1}\int_0^{T_1} |b(s,x) - \overline{b}(x)|\, ds = \frac{1}{2T_1}\sum_{k\geqslant 1}\frac{|\sin(kx(0))|}{k^3}\int_0^{T_1} |\cos 2s|\, ds$$
$$\leqslant \frac{\|x\|}{2T_1}\sum_{k\geqslant 1}\frac{1}{k^2}\int_0^{\pi} |\cos 2s|\, ds \leqslant \frac{\pi^2}{12T_1}(1+\|x\|)$$

and for $g = h, \sigma$, we have

$$\frac{1}{T_1}\int_0^{T_1} |g(s,x) - \overline{g}(x)|^2\, ds = \frac{1}{4T_1}\left(\sum_{k\geqslant 1}\frac{\sin(kx(0))}{k^3}\right)^2\int_0^{T_1} \cos^2 2s\, ds$$
$$\leqslant \frac{\|x\|^2}{4T_1}\left(\sum_{k\geqslant 1}\frac{1}{k^2}\right)^2\int_0^{\pi} \cos^2 2s\, ds$$
$$\leqslant \frac{\pi}{8T_1}\left(\sum_{k\geqslant 1}\frac{1}{k^2}\right)^2(1+\|x\|^2)$$
$$\leqslant \frac{\pi^5}{288T_1}(1+\|x\|^2),$$

which means that hypothesis (H_3) is satisfied.

(2) Let $T = \pi/2$, $b(s,x) = 2\|x\|\sin^2 s$ and $h(s,x) = \sigma(s,x) = 1$. Then

$$\overline{b}(x) = 2\frac{\|x\|}{\pi}\int_0^{\pi/2} (1-\cos 2s)\, ds = \|x\|,$$

so that for all $T_1 \in [0, \pi/2]$,

$$\frac{1}{T_1}\int_0^{T_1} |b(s,x) - \overline{b}(x)|\, ds = \frac{\|x\|}{T_1}\int_0^{T_1} |2\sin^2 s - 1|\, ds$$
$$\leqslant \frac{\|x\|}{T_1}\int_0^{\pi/2} |\cos 2s|\, ds \leqslant \frac{1}{T_1}(1+\|x\|).$$

On the other hand, if

$$\overline{h}(x) = \overline{\sigma}(x) = 1,$$

then

$$\frac{1}{T_1}\int_0^{T_1} |h(s,x) - \overline{h}(x)|^2 ds = \frac{1}{T_1}\int_0^{T_1} |\sigma(s,x) - \overline{\sigma}(x)|^2 ds = 0,$$

which means that hypothesis (H_3) is satisfied.

Then we have the averaging form of (4.1):

(4.2)
$$\begin{cases} D_t^\alpha Y^\varepsilon(t) = \varepsilon\overline{b}(Y_t^\varepsilon)dt + \sqrt{\varepsilon}\,\overline{h}(Y_t^\varepsilon)d\langle B\rangle_t + \sqrt{\varepsilon}\,\overline{\sigma}(Y_t^\varepsilon)\,dB_t, \quad t \in [0,T], \\ Y_0 = \Psi. \end{cases}$$

Note that, following Corollary 3.1,

$$\widehat{\mathbb{E}}\Big[\sup_{\eta\in[0,T]} \|Y_\eta^\varepsilon\|^2\Big] \leqslant \widehat{\mathbb{E}}\Big[\sup_{s\in[-\tau,T]} |Y^\varepsilon(s)|^2\Big] \leqslant \|\Psi\|^2 + \widehat{\mathbb{E}}\Big[\sup_{s\in[0,T]} |Y^\varepsilon(s)|^2\Big] < \infty.$$

Now we come to the averaging principle result: we will prove that the solution to (4.1) will converge to the solution of (4.2) in the mean square sense as $\varepsilon \to 0$.

THEOREM 4.1. *Assume that* (H_1)–(H_3) *are satisfied. Then for a given arbitrarily small number* $\delta > 0$*, there exist* $L > 0, \varepsilon_1 \in (0, \varepsilon_0]$ *and* $\theta \in (0, 1/2)$ *such that for all* $\varepsilon \in (0, \varepsilon_1]$*,*

$$\widehat{\mathbb{E}}\Big(\sup_{t\in[-\tau,L\varepsilon^{-\theta}]} |X^\varepsilon(t) - Y^\varepsilon(t)|^2\Big) \leqslant \delta.$$

Proof. For any $t \in [0, u] \subset [0, T]$, we have

$$\begin{aligned} X^\varepsilon(t) - Y^\varepsilon(t) &= \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[b(s, X_s^\varepsilon) - \overline{b}(Y_s^\varepsilon)]\,ds \\ &\quad + \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[h(s, X_s^\varepsilon) - \overline{h}(Y_s^\varepsilon)]\,d\langle B\rangle_s \\ &\quad + \frac{\sqrt{\varepsilon}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[\sigma(s, X_s^\varepsilon) - \overline{\sigma}(Y_s^\varepsilon)]\,dB_s. \end{aligned}$$

Firstly, we have

(4.3)
$$\begin{aligned} &\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]} |X^\varepsilon(t) - Y^\varepsilon(t)|^2\Big] \\ &\leqslant \frac{3\varepsilon^2}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]}\Big|\int_0^t (t-s)^{\alpha-1}(b(s, X_s^\varepsilon) - \overline{b}(Y_s^\varepsilon))\,ds\Big|^2\Big] \\ &\quad + \frac{3\varepsilon}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]}\Big|\int_0^t (t-s)^{\alpha-1}(h(s, X_s^\varepsilon) - \overline{h}(Y_s^\varepsilon))\,d\langle B\rangle_s\Big|^2\Big] \\ &\quad + \frac{3\varepsilon}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]}\Big|\int_0^t (t-s)^{\alpha-1}(\sigma(s, X_s^\varepsilon) - \overline{\sigma}(Y_s^\varepsilon))\,dB_s\Big|^2\Big] \\ &= I_1 + I_2 + I_3, \end{aligned}$$

so that

$$I_1 \leqslant \frac{6\varepsilon^2}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]}\Big|\int_0^t (t-s)^{\alpha-1}(b(s,X_s^\varepsilon)-b(s,Y_s^\varepsilon))\,ds\Big|^2\Big] + \frac{6\varepsilon^2}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]}\Big|\int_0^t (t-s)^{\alpha-1}(b(s,Y_s^\varepsilon)-\overline{b}(Y_s^\varepsilon))\,ds\Big|^2\Big] = I_{1.1}+I_{1.2}. \tag{4.4}$$

Thanks to the Cauchy–Schwarz inequality and assumption (H_1), we get

$$\begin{aligned} I_{1.1} &\leqslant \frac{6D\varepsilon^2 u}{\Gamma(\alpha)^2}\int_0^u (t-s)^{2\alpha-2}\widehat{\mathbb{E}}\Big[\sup_{0\leqslant s_1\leqslant s}\|X_{s_1}^\varepsilon - Y_{s_1}^\varepsilon\|^2\Big]\,ds \\ &\leqslant \frac{6D\varepsilon^2 u}{\Gamma(\alpha)^2}\int_0^u (t-s)^{2\alpha-2}\widehat{\mathbb{E}}\Big[\sup_{0\leqslant s_1\leqslant s}|X^\varepsilon(s_1)-Y^\varepsilon(s_1)|^2\Big]\,ds. \end{aligned} \tag{4.5}$$

Applying the Lipschitz condition and integration by parts, we obtain

$$\begin{aligned} I_{1.2} &\leqslant \frac{6\varepsilon^2}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]}\Big|\int_0^t (t-s)^{\alpha-1} d\Big[\int_0^s (b(\eta,Y_\eta^\varepsilon)-\overline{b}(Y_\eta^\varepsilon))\,d\eta\Big]\,ds\Big|^2\Big] \\ &\leqslant \frac{6\varepsilon^2(\alpha-1)^2}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]}\Big|\int_0^t\Big(\int_0^s (b(\eta,Y_\eta^\varepsilon)-\overline{b}(Y_\eta^\varepsilon))\,d\eta\Big)(t-s)^{\alpha-2}\,ds\Big|^2\Big]. \end{aligned}$$

Then together with the Cauchy–Schwarz inequality we get

$$I_{1.2} \leqslant \frac{6\varepsilon^2(\alpha-1)^2u^{2\alpha-3}}{(2\alpha-3)\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\int_0^u\Big|\int_0^s (b(\eta,Y_\eta^\varepsilon)-\overline{b}(Y_\eta^\varepsilon))\,d\eta\Big|^2 ds\Big].$$

Set $D^\varepsilon(\omega) := \big\{x(\omega) := Y_\eta^\varepsilon(\omega) : \eta \in [0,T]\big\}$ for $\omega \in \Omega$. For all $\eta \in [0,T] \subset [0,T]$ and all $\omega \in \Omega$, we have

$$\begin{aligned} \Big|\int_0^s (b(\eta,Y_\eta^\varepsilon(\omega))-\overline{b}(Y_\eta^\varepsilon)(\omega))\,d\eta\Big| &\leqslant \int_0^s |b(\eta,Y_\eta^\varepsilon(\omega))-\overline{b}(Y_\eta^\varepsilon(\omega))|\,d\eta \\ &\leqslant \int_0^s \sup_{x\in D^\varepsilon(\omega)}|b(\eta,x)-\overline{b}(x)|\,d\eta. \end{aligned}$$

On the other hand, for each $\delta > 0$, there exists $x^\delta \in D^\varepsilon(\omega)$ such that

$$\sup_{x\in D^\varepsilon(\omega)}|b(\eta,x)-\overline{b}(x)| \leqslant |b(\eta,x^\delta)-\overline{b}(x^\delta)|+\delta,$$

so that by using hypothesis (H_3) and the last two inequalities, we get

$$\begin{aligned} \Big|\int_0^s (b(\eta,Y_\eta^\varepsilon(\omega))-\overline{b}(Y_\eta^\varepsilon)(\omega))\,d\eta\Big| &\leqslant \int_0^s |b(\eta,x^\delta)-\overline{b}(x^\delta)|d\eta+\delta s \\ &\leqslant s\alpha_1(\eta)(1+|x^\delta|)+\delta s \leqslant s\sup_{\eta\in[0,T]}\alpha_1(\eta)\Big(1+\sup_{x\in D^\varepsilon(\omega)}|x|\Big)+\delta s \quad \text{q.s.} \end{aligned}$$

This implies that

$$\Big|\int_0^s (b(\eta, Y^\varepsilon_\eta) - \overline{b}(Y^\varepsilon_\eta))\, d\eta\Big| \leqslant s \sup_{\eta\in[0,T]} \alpha_1(\eta)\Big(1 + \sup_{\eta\in[0,T]} |Y^\varepsilon_\eta|\Big) \quad \text{q.s.}$$

and

$$\begin{aligned}
I_{1.2} &\leqslant \frac{6\varepsilon^2(\alpha-1)^2u^{2\alpha-3}}{(2\alpha-3)\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\int_0^u \Big(s \sup_{\eta\in[0,T]} \alpha_1(\eta)\Big(1 + \sup_{\eta\in[0,T]} |Y^\varepsilon_\eta|\Big)\Big)^2 ds\Big] \\
&\leqslant \frac{2\varepsilon^2(\alpha-1)^2u^{2\alpha}}{(2\alpha-3)\Gamma(\alpha)^2} \sup_{\eta\in[0,T]} \alpha_1(\eta)^2\Big(1 + \widehat{\mathbb{E}}\Big[\sup_{\eta\in[0,T]} \|Y^\varepsilon_\eta\|^2\Big]\Big).
\end{aligned} \tag{4.6}$$

It follows from (4.5) and (4.6) that

$$I_1 \leqslant r_{1.1}\varepsilon^2 u \int_0^u (u-s)^{2\alpha-2}\widehat{\mathbb{E}}\Big(\sup_{0\leqslant s_1\leqslant s} |X^\varepsilon(s_1) - Y^\varepsilon(s_1)|^2\Big)\, ds + r_{1.2}\varepsilon^2 u^{2\alpha}, \tag{4.7}$$

where

$$r_{1.1} = \frac{6D}{\Gamma(\alpha)^2}, \quad r_{1.2} = \frac{2(\alpha-1)^2}{\Gamma(\alpha)^2(2\alpha-3)} \sup_{\eta\in[0,T]} \alpha_1(\eta)^2\Big(1 + \widehat{\mathbb{E}}\Big[\sup_{\eta\in[0,T]} \|Y^\varepsilon_\eta\|^2\Big]\Big).$$

For I_2, we get in the same way

$$\begin{aligned}
I_2 &\leqslant \frac{6\varepsilon^2}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]} \Big|\int_0^t (t-s)^{\alpha-1}(h(s, X^\varepsilon_s) - h(s, Y^\varepsilon_s))\, d\langle B\rangle_s\Big|^2\Big] \\
&\quad + \frac{6\varepsilon^2}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]} \Big|\int_0^t (t-s)^{\alpha-1}(h(s, Y^\varepsilon_s) - \overline{h}(Y^\varepsilon_s))\, d\langle B\rangle_s\Big|^2\Big] \\
&= I_{2.1} + I_{2.2}.
\end{aligned}$$

Thanks to the Cauchy–Schwarz inequality, assumption (H_1) and G-BDG inequalities we get

$$I_{2.1} \leqslant \frac{6DC_1\varepsilon u}{\Gamma(\alpha)^2} \int_0^u (u-s)^{2\alpha-2}\widehat{\mathbb{E}}\Big(\sup_{0\leqslant s_1\leqslant s} |X^\varepsilon(s_1) - Y^\varepsilon(s_1)|^2\Big)\, ds. \tag{4.8}$$

Applying the Lipschitz condition and integration by parts, we obtain

$$\begin{aligned}
I_{2.2} &\leqslant \frac{6C_1u\varepsilon}{\Gamma(\alpha)^2}\widehat{\mathbb{E}}\Big[\int_0^u (u-s)^{2\alpha-2} d\Big[\int_0^s |(h(\eta, Y^\varepsilon_\eta) - \overline{h}(Y^\varepsilon_\eta))|^2\, d\eta\Big]\, ds\Big] \\
&\leqslant \frac{6C_1u\varepsilon(2\alpha-2)}{\Gamma(\alpha)^2}\widehat{\mathbb{E}} \int_0^u (u-s)^{2\alpha-3}\Big(\int_0^s |(h(\eta, Y^\varepsilon_\eta) - \overline{h}(Y^\varepsilon_\eta))|^2\, d\eta\Big)\, ds.
\end{aligned}$$

By assumption (H_3) we obtain

$$I_{2.2} \leqslant \frac{6C_1\varepsilon u^{2\alpha}}{\Gamma(\alpha)^2} \sup_{\eta\in[0,T]} \alpha_2(\eta)\Big(1+\widehat{\mathbb{E}}\Big[\sup_{\eta\in[0,T]} \|Y^\varepsilon_\eta\|^2\Big]\Big). \tag{4.9}$$

It follows from (4.8) and (4.9) that

$$I_2 \leqslant r_{2.1}u\varepsilon \int_0^u (u-s)^{2\alpha-2}\widehat{\mathbb{E}}\Big(\sup_{0\leqslant s_1\leqslant s} |X^\varepsilon(s_1)-Y^\varepsilon(s_1)|^2\Big)\, ds + r_{2.2}u^{2\alpha}\varepsilon, \tag{4.10}$$

where

$$r_{2.1} = C_1 r_{1.1}, \quad r_{2.2} = \frac{6C_1}{\Gamma(\alpha)^2(2\alpha-1)} \sup_{\eta\in[0,T]} \alpha_2(\eta)\Big(1+\widehat{\mathbb{E}}\Big[\sup_{\eta\in[0,T]} \|Y^\varepsilon_\eta\|^2\Big]\Big).$$

For the last term I_3, we get in the same manner

$$I_3 \leqslant r_{3.1}\varepsilon \int_0^u (u-s)^{2\alpha-2}\widehat{\mathbb{E}}\Big(\sup_{0\leqslant s_1\leqslant s} |X^\varepsilon(s_1)-Y^\varepsilon(s_1)|^2\Big)\, ds + r_{3.2}u^{2\alpha-1}\varepsilon, \tag{4.11}$$

where

$$r_{3.1} = C_2 r_{1.1}, \quad r_{3.2} = \frac{6C_2}{\Gamma(\alpha)^2(2\alpha-1)} \sup_{\eta\in[0,T]} \alpha_3(\eta)\Big(1+\widehat{\mathbb{E}}\Big[\sup_{\eta\in[0,T]} \|Y^\varepsilon_\eta\|^2\Big]\Big)$$

Now, inserting (4.7), (4.10) and (4.11) into (4.3), we get, for any $u \in [0,T]$,

$$\begin{aligned}
&\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]} |X^\varepsilon(t)-Y^\varepsilon(t)|^2\Big] \\
&\qquad\leqslant a + b\int_0^u (u-s)^{(2\alpha-1)-1}\widehat{\mathbb{E}}\Big[\sup_{0\leqslant s_1\leqslant s} |X^\varepsilon(s_1)-Y^\varepsilon(s_1)|^2\Big]\, ds,
\end{aligned}$$

where

$$\begin{aligned}
a &= r_{1.2}\varepsilon^2 u^{2\alpha} + r_{2.2}u^{2\alpha}\varepsilon + r_{3.2}u^{2\alpha-1}\varepsilon, \\
b &= r_{1.1}\varepsilon^2 u + r_{2.1}u\varepsilon + r_{3.1}\varepsilon.
\end{aligned}$$

By Corollary 2.1 we have

$$\widehat{\mathbb{E}}\Big[\sup_{t\in[0,u]} |X^\varepsilon(t)-Y^\varepsilon(t)|^2\Big] \leqslant aE_{2\alpha-1}(b\Gamma(2\alpha-1)u^{2\alpha-1}).$$

Let $\theta \in (0,1/2)$ and $L > 0$. Then

$$\widehat{\mathbb{E}}\Big[\sup_{t\in[0,L\epsilon^{-\theta}]} |X^\varepsilon(t)-Y^\varepsilon(t)|^2\Big] \leqslant Q(\varepsilon), \tag{4.12}$$

where

$$Q(\varepsilon) = [r_{1.2}\varepsilon^{-2\theta\alpha+2}L^{2\alpha} + r_{2.2}L^{2\alpha}\varepsilon^{-2\theta\alpha+1} + r_{3.2}L^{2\alpha-1}\varepsilon^{-2\theta\alpha+\theta+1}]$$
$$\times E_{2\alpha-1}\big((r_{1.1}L^{1+\beta}\varepsilon^{2-\theta-\theta\beta} + r_{2.1}L^{1+\beta}\varepsilon^{1-\theta-\theta\beta} + r_{3.1}L^{\beta}\varepsilon^{1-\theta\beta})\Gamma(\beta)\big),$$

where $\beta = 2\alpha - 1$. Since all the powers of ε, which appear in $Q(\varepsilon)$ are positive, we have $\lim_{\varepsilon\to 0} Q(\varepsilon) = 0$. It follows that, for any given δ, there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that for each $\varepsilon \in (0, \varepsilon_1]$,

$$\widehat{\mathbb{E}}\Big[\sup_{t\in[-\tau, L\varepsilon^{-\theta}]} |X^\varepsilon(t) - Y^\varepsilon(t)|^2\Big] \leqslant \widehat{\mathbb{E}}\Big[\sup_{t\in[0, L\varepsilon^{-\theta}]} |X^\varepsilon(t) - Y^\varepsilon(t)|^2\Big] \leqslant \delta.$$

The proof is complete. ■

COROLLARY 4.1. *Suppose that both the original G-FSDE* (4.1) *and the averaged G-FSDE* (4.2) *satisfy hypotheses* (H_1)–(H_3). *Then for any* $\xi > 0$,

$$\lim_{\varepsilon\to 0} \mathbb{C}\Big(\sup_{t\in[-\tau, L\varepsilon^{-\theta}]} |X^\varepsilon(t) - Y^\varepsilon(t)| > \xi\Big) = 0,$$

where $\mathbb{C}$ *is the capacity associated with* $\widehat{\mathbb{E}}$.

Proof. Follows from by Lemma 2.1 with $p = 2$ and formula (4.12). ■

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REFERENCES

[1] H. M. Ahmed and Q. Zhu, *The averaging principle of Hilfer fractional stochastic delay differential equations with Poisson jumps*, Appl. Math. Lett. 112 (2021), art. 106755, 7 pp.

[2] H. Bao and J. Cao, *Existence of solutions for fractional stochastic impulsive neutral functional differential equations with infinite delay*, Adv. Difference Equations 2017, art. 66, 14 pp.

[3] A. Chadha and D. N. Pandey, *Existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay*, Nonlinear Anal. 128 (2015), 149–175.

[4] J. Cui and L. Yan, *Existence result for fractional neutral stochastic integro-differential equations with infinite delay*, J. Phys. 44 (2011), art. 335201, 16 pp.

[5] L. Denis and C. Martini, *A theoretical framework for the pricing of contingent claims in the presence of model uncertainty*, Ann. Appl. Probab. 16 (2006), 827–852.

[6] M. Hu and S. Peng, *On representation theorem of G-expectations and paths of G-Brownian motion*, Acta Math. Appl. Sin. Engl. Ser. 25, (2009), 539–546.

[7] W. Mao, B. Chen and S. You, *On averaging principle for SDEs driven by G-Brownian motion with non-Lipschitz coefficients*, Adv. Difference Equations 2021, art. 71, 16 pp.

[8] B. P. Moghaddam, L. Zhang, A. M. Lopes, J. A. Tenreiro Machado and Z. S. Mostaghim. *Sufficient conditions for existence and uniqueness of fractional stochastic delay differential equations*, Stochastics 92 (2020), 379–396.

[9] S. Peng, *G-expectation, G-Brownian motion and related stochastic calculus of Itô type*, in: Stochastic Analysis and Applications, Proc. 2nd Abel Symposium, Springer, Berlin, 2006, 541–567.

[10] S. Peng, *G-Brownian motion and dynamic risk measure under volatility uncertainty*, arXiv:0711.2834) (2007).

[11] S. Peng. *Nonlinear Expectations and Stochastic Calculus under Uncertainty – with Robust CLT and G-Brownian Motion*, Springer, 2019.

[12] P. Umamaheswaria, K. Balachandrana and N. Annapoorani, *Existence and stability results for caputo fractional stochastic differential equations with Lévy noise*, Filomat 34 (2020), 1739–1751.

[13] Wenjing Xu, J. Duan and Wei Xu, *An averaging principle for fractional stochastic differential equation with Lévy noise*, Chaos 30 (2020), art. 083126, 7 pp.

[14] Wenjing Xu, Wei Xu and S. Zhang, *The averaging principle for stochastic differential equation with Caputo fractional derivative*, Appl. Math. Lett. 93 (2019), 79–84.

[15] H. Ye, J. Gao and Y. Ding, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl., 328 (2007), 1075–1081.

[16] X. Zhang, P. Agarwal, Z. Liu, H. Peng, F. You and Y. Zhu, *Existence and uniqueness of solutions for stochastic differential equations of fractional-order* $q > 1$*with finite delays*, Adv. Difference Equations 2017, art. 123, 18 pp.

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