PROBABILITY
AND
MATHEMATICAL STATISTICS

Vol. 43, Fasc. 1 (2023), pp. 41–61
Published online 5.9.2023
doi:10.37190/0208-4147.00106

MOMENT INEQUALITIES FOR NONNEGATIVE RANDOM VARIABLES

BY

NETZAHUALCÓYOTL CASTAÑEDA-LEYVA (AGUASCALIENTES),
SILVIA RODRÍGUEZ-NARCISO (AGUASCALIENTES),
ANGÉLICA HERNÁNDEZ-QUINTERO (AGUASCALIENTES), AND
AROLDO PÉREZ (CUNDUACÁN)

Abstract. We give reciprocal versions of the Sclove et al. and Feller inequalities for moments of nonnegative random variables. Our results apply to any nonnegative random variable. The strongest assumption is that the moments involved must be finite. Thus, the results obtained also hold for any empirical distribution with nonnegative data. These facts allow potential applications in numerical analysis, probability, and statistical inference, among other disciplines. Moreover, the proposed methodology offers an alternative approach to obtain new inequalities and even to improve some known inequalities. For instance, we give new inequalities for the ratio of gamma functions. In this context, we also improve an inequality by Bustoz and Ismail and some cases of inequalities due to Gurland and Dragomir et al. Additionally, we present a new inequality for finite sums of nonnegative or nonpositive numbers. For some cases, this relation improves even the Cauchy–Bunyakovsky–Schwarz inequality.

2020 Mathematics Subject Classification: Primary 60E15; Secondary 26D15, 33B15, 11L07.

Key words and phrases: moment inequalities, ratio of gamma functions, alternative expectation formula, alternative covariance formula.

1. INTRODUCTION

The moments of a probability distribution determine many population properties. Therefore, identities and inequalities for moments are a topic of scientific interest. The purpose of this paper is to provide new moment inequalities for nonnegative random variables. The potential applications span disciplines such as numerical analysis, probability, statistical inference, longevity, survival, insurance, reliability, and queuing theory.

For a nonnegative random variable X, the Sclove et al. inequality [38, (5)] is

$$\mathrm{E}\,X^{r+1}\,\mathrm{E}\,X^{s-1} \leqslant \mathrm{E}\,X^{r}\,\mathrm{E}\,X^{s}, \quad 0 \leqslant r \leqslant s-1.$$

A generalization of this inequality follows from Olkin and Shepp [32]:

$$\mathrm{E}\,X^{r+\alpha}\,\mathrm{E}\,X^{s-\alpha} \leqslant \mathrm{E}\,X^{r}\,\mathrm{E}\,X^{s}, \quad r \geqslant 0,\ 0 \leqslant \alpha \leqslant s-r. \tag{1.1}$$

On the other hand, the inequality from Feller [11, (V.8.10)] states

$$(\mathrm{E}\,X^{s})^2 \leqslant \mathrm{E}\,X^{s+\alpha}\,\mathrm{E}\,X^{s-\alpha}, \quad 0 \leqslant \alpha \leqslant s. \tag{1.2}$$

For extensions of both inequalities, see Yeh et al. [41, Theorem 2.2(C3)]. We prove in Proposition 2.1 that inequality (1.2) follows from (1.1). Both inequalities apply to any nonnegative random variable such that the moments involved are finite. In particular, these inequalities apply to absolutely continuous, discrete, or mixed random variables. They also apply empirically:

$$\overline{x^{r+\alpha}}\;\overline{x^{s-\alpha}} \leqslant \overline{x^{r}}\;\overline{x^{s}}, \tag{1.3}$$

$$(\overline{x^{s}})^2 \leqslant \overline{x^{s+\alpha}}\;\overline{x^{s-\alpha}}. \tag{1.4}$$

Here $n\overline{x^r} = \sum_{i=1}^{n} x_i^r$, where $x_1, \dots, x_n \geqslant 0$ is a random sample of size $n \geqslant 1$ of a certain target distribution.

Inequalities (1.3) and (1.4) are useful to obtain existence and uniqueness conditions for parameter estimates based on numerical methods that optimize a utility or loss function, such as maximum likelihood or least squares. For instance, in Rockette et al. [36, Lemma 1], an inequality equivalent to (1.3) gives conditions for existence and uniqueness of maximum likelihood estimates for a three-parameter Weibull distribution.

There are other moment inequalities for specific families of absolutely continuous distributions that, in contrast to (1.1)–(1.2), do not immediately extend to the discrete case or even to the corresponding empirical case. For example, Ahmad [2, Theorem 2.1] obtained inequalities with scaled moments $\mathrm{E}\,X^r/r!$; for $r \geqslant 0$, where the nonnegative random variable X belongs to the class of absolutely continuous distributions with increasing failure rate (IFR). This is a family of aging distributions in which the target population suffers degradation or wear over time. However, such moment inequalities do not immediately extend to the analogous discrete case; see Hu et al. [16, Theorem 3.3]. In particular, the coefficient of variation of an IFR random variable X is less than or equal to 1: $\mathrm{E}\,X^2 \leqslant 2(\mathrm{E}\,X)^2$. However, this inequality is not necessarily satisfied for the analogous IFR discrete family; see Kemp [20, Section 3.7]. Additionally, the corresponding empirical distribution presents challenges in the same sense. On the other hand, specific empirical moment inequalities are required to obtain existence and uniqueness parameter estimates. See, for instance, Panjer [33, Theorem 2.2.2], who characterizes the solution of the maximum likelihood estimator from the left-truncated extreme value distribution. See also (5) and its explanation in Domínguez-Molina et al. [8], for the moment method estimate for the generalized Gaussian distribution.

In this work, inequalities for moments of nonnegative random variables are obtained. Our methodology offers a wide family of moment inequalities for nonnegative random variables, where the strongest assumption is that the moments involved must be finite. As a consequence, the results obtained can be applied to any empirical distribution with nonnegative data. Our main contributions are the reciprocal versions of the inequalities of Sclove et al. (1.1) and of Feller (1.2). As a corollary, new inequalities are obtained for the ratio of gamma functions. Moreover, we improve the Bustoz and Ismail inequality [4, (4.4)]. For some cases, we also improve the inequalities of Gurland [14, (1)] and of Dragomir et al. [9, (3.16)].

The second section presents the main results of this work. Before giving the reciprocal versions of inequalities (1.1)–(1.2), we state a lemma with the Feller alternative expectation formula [11, Lemma V.6.1], and the Hoeffding alternative covariance formula [15, (5.6)] and [26, Section 4.]. Also, new extensions for the discrete case are presented. Next, as a corollary, new inequalities for the ratio of gamma functions are obtained. We additionally extend (1.1), together with the corresponding reciprocal inequality, to moments of power and analytic functions of nonnegative random variables.

In the third section, a new inequality for finite sums of nonnegative or nonpositive numbers is shown. In some cases, this result improves the Cauchy–Bunyakovsky–Schwarz inequality.

2. POPULATION MOMENTS

In this section, we present our main contributions. In Proposition 2.1, we give the reciprocal inequalities to those of Sclove et al. (1.1) and of Feller (1.2). The utility of the proposed methodology for creating inequalities is represented by Corollaries 2.1–2.3; see also Remark 2.1. In particular, we give new inequalities for the ratio of gamma functions and improve a Bustoz and Ismail inequality. The Gurland and Dragomir et al. inequalities are also improved in some cases. In Corollary 2.3 we conclude with a generalization of inequalities of Sclove et al.; the result involves power and analytic functions of nonnegative random variables.

Let (X, Y) be a vector of nonnegative random variables, with marginal survival functions $S(x) = P(X > x)$, $S(y) = P(Y > y)$, and joint survival function $S(x, y) = P(X > x, Y > y)$, for $x, y \geqslant 0$.

If X and Y are discrete random variables, with respective possible values $\{x_i\}_{i \geqslant 0}$ and $\{y_j\}_{j \geqslant 0}$, define the *auxiliary discrete survival* functions

$$\dot{S}(x_i) = P(X \in \{x_{i+1}, x_{i+2}, \dots\}) = \sum_{u=i+1}^{\infty} f(x_u), \tag{2.1}$$

$$\dot{S}(y_j) = P(Y \in \{y_{j+1}, y_{j+2}, \dots\}) = \sum_{v=j+1}^{\infty} f(y_v),$$

$$\dot{S}(x_i, y_j) = P(X \in \{x_{i+1}, x_{i+2}, \dots\},\, Y \in \{y_{j+1}, y_{j+2}, \dots\}) \tag{2.2}$$
$$= \sum_{u=i+1}^{\infty} \sum_{v=j+1}^{\infty} f(x_u, y_v) \quad \text{for } i, j = 0, 1, 2, \dots,$$

where $f(x_i)$, $f(y_j)$, and $f(x_i, y_j)$ denote the respective probability functions for X, Y, and (X, Y). By convention, $x_0 = y_0 = 0$. If zero is not a possible value (has zero probability), consider it as a dummy value; see [31]. The decreasing sequence (2.1) determines the distribution of the random variable X, since $f(x_i) = P(X = x_i) = \dot{S}(x_{i-1}) - \dot{S}(x_i)$. Similarly, (2.2) determines the joint distribution of (X, Y). In particular, if the possible values for X and Y are increasing, then $\dot{S}(x_i)$, $\dot{S}(y_j)$, and $\dot{S}(x_i, y_j)$ are the classical discrete survival functions.

LEMMA 2.1. *Let (X, Y) be a vector of nonnegative random variables and $r, s > 0$, such that* $\mathrm{E}\, X^r, \mathrm{E}\, Y^s, \mathrm{E}[X^r Y^s] < \infty$. *Then:*

(1) *The following identities hold:*

$$\mathrm{E}\, X^r = r \int_0^{\infty} x^{r-1} S(x)\, dx, \tag{2.3}$$

$$\mathrm{C}(X^r, Y^s) = rs \int_0^{\infty}\int_0^{\infty} x^{r-1} y^{s-1} [S(x, y) - S(x)S(y)]\, dy\, dx. \tag{2.4}$$

(2) *Moreover, if X and Y are discrete random variables, with possible values eventually monotonic, then*

$$\mathrm{E}\, X^r = \sum_{i=0}^{\infty} [x_{i+1}^r - x_i^r] \dot{S}(x_i), \tag{2.5}$$

$$\mathrm{C}(X^r, Y^s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} [x_{i+1}^r - x_i^r][y_{j+1}^s - y_j^s][\dot{S}(x_i, y_j) - \dot{S}(x_i)\dot{S}(y_j)]. \tag{2.6}$$

(3) *In particular, if the possible values for X and Y are increasing:* $0 = x_0 < x_1 < x_2 < \cdots$ *and* $0 = y_0 < y_1 < y_2 < \cdots$ *respectively, then*

$$\mathrm{E}\, X^r = \sum_{i=0}^{\infty} [x_{i+1}^r - x_i^r] S(x_i), \tag{2.7}$$

$$\mathrm{C}(X^r, Y^s) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} [x_{i+1}^r - x_i^r][y_{j+1}^s - y_j^s][S(x_i, y_j) - S(x_i)S(y_j)]. \tag{2.8}$$

The Feller alternative expectation formula (2.3) is deduced in [11, Lemma V.6.1]. The alternative covariance formula (2.4) is obtained in [26, Section 4], and it is a generalization of the Hoeffding covariance formula [15, (5.6)]:

$$\mathrm{C}(X, Y) = \int_0^{\infty}\int_0^{\infty} [S(x, y) - S(x)S(y)]\, dy\, dx.$$

See also [39, (2.4)], [7, Theorem 1], and [25, Corollary 2.7(e) and Theorem 3.1]. Both (2.3)–(2.4) apply to general nonnegative random variables whenever the moments involved are finite.

Formulas (2.5)–(2.6) are innovations for the discrete case. In turn, these formulas imply (2.7)–(2.8). Formula (2.5) generalizes (2.7), since, unlike [6, (3)], [31, Theorem 2 and Corollary 1], and [12, proof of Theorem 1], we do not assume ordered possible values for the random variable X. For the deduction of (2.7) via (2.3), follow [12, proof of Theorem 1] with $g(x) = x^r$. Similarly, the covariance formula for the discrete case (2.6) extends (2.8); see [23, Theorem 1] and [31, (3.17)].

Proof of Lemma 2.1. (1) Formula (2.3) is deduced in [11, Lemma V.6.1] by integration by parts and the Fubini theorem. With similar technique, (2.4) is obtained in [26, Section 4]; it is also a consequence of [25, Theorem 3.1] with $f(x) = x^r$ and $g(y) = y^s$.

(2) As in the proof of (1), the underlying idea is the careful use of the Fubini theorem. For $i = 1, 2, \ldots$, note that

$$x_i^r = \sum_{j=0}^{i-1}(x_{j+1}^r - x_j^r) = \sum_{j=0}^{i-1}(x_{j+1}^r - x_j^r)^+ - \sum_{j=0}^{i-1}(x_{j+1}^r - x_j^r)^-. \tag{2.9}$$

The terms of the two sums on the right-hand side are nonnegative. By the eventual monotonicity of $\{x_i\}_{i\geqslant 0}$, one of these sums has at most n positive terms, for some $n \geqslant 0$ and all $i \geqslant 1$. Then the Fubini theorem can be applied:

$$\begin{aligned}
\mathrm{E}\, X^r &= \sum_{i=0}^{\infty} x_i^r f(x_i) = \sum_{i=1}^{\infty}\Big[\sum_{j=0}^{i-1}(x_{j+1}^r - x_j^r)^+ - \sum_{j=0}^{i-1}(x_{j+1}^r - x_j^r)^-\Big] f(x_i) \\
&= \sum_{j=0}^{\infty}(x_{j+1}^r - x_j^r)^+ \sum_{i=j+1}^{\infty} f(x_i) - \sum_{j=0}^{\infty}(x_{j+1}^r - x_j^r)^- \sum_{i=j+1}^{\infty} f(x_i) \\
&= \sum_{j=0}^{\infty}(x_{j+1}^r - x_j^r)^+ \dot{S}(x_j) - \sum_{j=0}^{\infty}(x_{j+1}^r - x_j^r)^- \dot{S}(x_j) \\
&= \sum_{j=0}^{\infty}(x_{j+1}^r - x_j^r)\dot{S}(x_j).
\end{aligned}$$

Similar to (2.9), for $i, j = 1, 2, \ldots$, we have

$$\begin{aligned}
x_i^r y_j^s &= \sum_{u=0}^{i-1}(x_{u+1}^r - x_u^r)\sum_{v=0}^{j-1}(y_{v+1}^s - y_v^s) \\
&= \Big[\sum_{u=0}^{i-1}(x_{u+1}^r - x_u^r)^+ - \sum_{u=0}^{i-1}(x_{u+1}^r - x_u^r)^-\Big] \\
&\quad \times \Big[\sum_{v=0}^{j-1}(y_{v+1}^s - y_v^s)^+ - \sum_{v=0}^{j-1}(y_{v+1}^s - y_v^s)^-\Big]
\end{aligned}$$

$$= \sum_{u=0}^{i-1}\sum_{v=0}^{j-1}(x_{u+1}^r - x_u^r)^+(y_{v+1}^s - y_v^s)^+ - \sum_{u=0}^{i-1}\sum_{v=0}^{j-1}(x_{u+1}^r - x_u^r)^+(y_{v+1}^s - y_v^s)^-$$
$$- \sum_{u=0}^{i-1}\sum_{v=0}^{j-1}(x_{u+1}^r - x_u^r)^-(y_{v+1}^s - y_v^s)^+ + \sum_{u=0}^{i-1}\sum_{v=0}^{j-1}(x_{u+1}^r - x_u^r)^-(y_{v+1}^s - y_v^s)^-.$$

The terms of the four double sums on the right-hand side are nonnegative. By the eventual monotonicity of $\{x_i\}_{i\geqslant 0}$ and $\{y_j\}_{j\geqslant 0}$, three of these four sums have at most n positive terms, for some $n \geqslant 0$ and all $i, j \geqslant 1$. Then the Fubini theorem can be applied:

$$\begin{aligned}
\mathrm{E}[X^r Y^s] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_i^r y_j^s f(x_i, y_j) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{u=0}^{i-1}\sum_{v=0}^{j-1}[x_{u+1}^r - x_u^r][y_{v+1}^s - y_v^s] f(x_i, y_j) \\
&= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{i=u+1}^{\infty}\sum_{j=v+1}^{\infty}[x_{u+1}^r - x_u^r][y_{v+1}^s - y_v^s] f(x_i, y_j) \\
&= \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}[x_{u+1}^r - x_u^r][y_{v+1}^s - y_v^s]\dot{S}(x_u, y_v).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{C}(X^r, Y^s) &= \mathrm{E}[X^r Y^s] - \mathrm{E}\, X^r\, \mathrm{E}\, Y^s \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(x_{i+1}^r - x_i^r)(y_{j+1}^s - y_j^s)\dot{S}(x_i, y_j) \\
&\quad - \sum_{i=0}^{\infty}(x_{i+1}^r - x_i^r)\dot{S}(x_i)\sum_{j=0}^{\infty}(y_{j+1}^s - y_j^s)\dot{S}(y_j) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(x_{i+1}^r - x_i^r)(y_{j+1}^s - y_j^s)[\dot{S}(x_i, y_j) - \dot{S}(x_i)\dot{S}(y_j)].
\end{aligned}$$

(3) This part is a particular case of (2), where the survival functions are the respective auxiliary ones: $S(x_i) = \dot{S}(x_i)$, $S(y_j) = \dot{S}(y_j)$, and $S(x_i, y_j) = \dot{S}(x_i, y_j)$. ■

Now, we describe two examples that illustrate the potential of Lemma 2.1(2).

• Contrary to (2.7), formula (2.5) applies for possible values that eventually decrease. The following example is due to the anonymous reviewer. Consider $x_0 = 0$, $x_1 = 8$, $x_2 = 2$, $x_3 = 4$ with probability $1/8$, whereas $f(x_i) = f(1/2^{i-3}) = 1/2^{i-2}$ for $i \geqslant 4$. Then

$$\mathrm{E}\, X = \sum_{i=0}^{\infty} x_i f(x_i) = \frac{0+8+2+4}{8} + \sum_{i=4}^{\infty}\frac{1}{2^{i-3}}\,\frac{1}{2^{i-2}} = \frac{7}{4} + \frac{1}{6} = \frac{23}{12}$$

and

$$\sum_{i=0}^{\infty}(x_{i+1}-x_i)\dot{S}(x_i) = 8\frac{7}{8} - 6\frac{6}{8} + 2\frac{5}{8} - \frac{7}{2}\frac{4}{8} + \sum_{i=4}^{\infty}\left(\frac{1}{2^{i+1-3}} - \frac{1}{2^{i-3}}\right)\frac{1}{2^{i-2}}$$
$$= 2 - \frac{1}{4^2}\sum_{u=0}^{\infty}\frac{1}{4^u} = \frac{23}{12} = \mathrm{E}\,X.$$

Here $\dot{S}(x_0) = \dot{S}(0) = 7/8$, $\dot{S}(x_1) = \dot{S}(8) = 6/8$, $\dot{S}(x_2) = \dot{S}(2) = 5/8$, and

$$\dot{S}(x_i) = \sum_{u=i+1}^{\infty}\frac{1}{2^{u-2}} = \frac{1}{2^{i-2}} \quad \text{for } i \geqslant 3.$$

• The eventual monotonicity of the possible values is a sufficient condition for the convergence of the series involved in (2.5)–(2.6). These results can be extended by a careful application of the Fubini theorem. For instance, consider a random variable without eventual monotonic possible values:

$$X = 0, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots.$$

Note that

$$\sum_{i=0}^{\infty}(x_{i+1}-x_i)^+\dot{S}(x_i) - \sum_{i=0}^{\infty}(x_{i+1}-x_i)^-\dot{S}(x_i)$$
$$= \frac{1}{2}\dot{S}(0) + \sum_{i=1}^{\infty}\left(i+1-\frac{1}{i+1}\right)\dot{S}(x_{2i-1}) - \sum_{i=1}^{\infty}\left(i+1-\frac{1}{i+2}\right)\dot{S}(x_{2i}).$$

The last two series are finite if eventually $\dot{S}(x_i) \leqslant 1/i^{2+\varepsilon}$ with $\varepsilon > 0$. In this case, (2.5) holds for $r = 1$. In contrast, the series in (2.7) does not converge, since the sequence $\{(x_{i+1}-x_i)S(x_i)\}_{i\geqslant 0}$ has a divergent subsequence: when $i \to \infty$,

$$(x_{2(i+1)} - x_{2i+1})S(x_{2i+1}) = \left(i+1-\frac{1}{i+1}\right)S\left(\frac{1}{i+1}\right) \to \infty S(0) = \infty.$$

PROPOSITION 2.1. (1) *Let X be a nonnegative random variable such that*

(2.10) $$\mathrm{E}\,X^{r+s} < \infty \quad \textit{with } r < s \textit{ and } rs > 0.$$

If

(2.11) $$0 \leqslant \alpha \leqslant s - r \quad \textit{and} \quad \beta \leqslant \frac{rs}{(r+\alpha)(s-\alpha)} \quad (\leqslant 1),$$

then

(2.12) $$\mathrm{E}\,X^{r+\alpha}\,\mathrm{E}\,X^{s-\alpha} \leqslant \mathrm{E}\,X^r\,\mathrm{E}\,X^s \leqslant \beta\,\mathrm{E}\,X^{r+\alpha}\,\mathrm{E}\,X^{s-\alpha} + (1-\beta)\,\mathrm{E}\,X^{r+s},$$

or equivalently

(2.13) $$\beta\,\mathrm{C}(X^{r+\alpha}, X^{s-\alpha}) \leqslant \mathrm{C}(X^r, X^s) \leqslant \mathrm{C}(X^{r+\alpha}, X^{s-\alpha}).$$

(2) *Let X be a nonnegative random variable such that* $\operatorname{E} X^{2s} < \infty$ *with* $s \neq 0$. *If*

$$0 \leqslant \alpha \leqslant |s| \quad \textit{and} \quad \beta \leqslant \frac{s^2 - \alpha^2}{s^2} \quad (\leqslant 1), \tag{2.14}$$

then

$$(\operatorname{E} X^s)^2 \leqslant \operatorname{E} X^{s+\alpha} \operatorname{E} X^{s-\alpha} \leqslant \beta(\operatorname{E} X^s)^2 + (1-\beta) \operatorname{E} X^{2s}, \tag{2.15}$$

or equivalently

$$\beta \operatorname{V} X^s \leqslant \operatorname{C}(X^{s+\alpha}, X^{s-\alpha}) \leqslant \operatorname{V} X^s. \tag{2.16}$$

Proof. (1) Let X be a nonnegative random variable that satisfies (2.10). Consider α and β as in (2.11). Note that, from (2.10) and the Lyapunov inequality, all moments of the random variable X in (2.12) are finite. In fact, for the case $0 < r < s$, $r + s$ is the greatest positive power involved: $0 < r, s, r + \alpha, s - \alpha < r + s$. On the other hand, if $r < s < 0$, hypothesis (2.10) implies $P(X = 0) = 0$. Thus, we can define

$$Y = \frac{1}{X} > 0, \quad u = -s, \quad \text{and} \quad v = -r. \tag{2.17}$$

Then $0 < u < v$ and $\operatorname{E} Y^{u+v} = \operatorname{E} X^{r+s} < \infty$. From the Lyapunov inequality, the following random variables also have finite moments: $Y^u = X^s$, $Y^v = X^r$, $Y^{u+\alpha} = X^{s-\alpha}$, and $Y^{v-\alpha} = X^{r+\alpha}$. Here, all powers of Y are positive, where $u + v$ is the greatest. Note now that, by the definition of covariance, the first and second inequalities of (2.12) are equivalent to the second and first of (2.13), respectively. Hence, it is sufficient to prove the first inequality of (2.12) and the first of (2.13).

CASE $0 < r < s$. The first inequality in (2.12) follows from the method of cloning [32], which assumes $0 \leqslant 2\alpha \leqslant s - r$. As the first condition in (2.11) shows, we have improved this constraint. In fact, considering Y as an independent replica of the random variable X, we have

$$\begin{aligned}
&2(\operatorname{E} X^r \operatorname{E} X^s - \operatorname{E} X^{r+\alpha} \operatorname{E} X^{s-\alpha}) \\
&\quad = \operatorname{E} X^r \operatorname{E} Y^s + \operatorname{E} X^s \operatorname{E} Y^r - \operatorname{E} X^{r+\alpha} \operatorname{E} Y^{s-\alpha} - \operatorname{E} X^{s-\alpha} \operatorname{E} Y^{r+\alpha} \\
&\quad = \operatorname{E}[X^r Y^s + X^s Y^r - X^{r+\alpha} Y^{s-\alpha} - X^{s-\alpha} Y^{r+\alpha}] \\
&\quad = \operatorname{E}[X^r Y^r (Y^\alpha - X^\alpha)(Y^{s-r-\alpha} - X^{s-r-\alpha})] \\
&\quad \geqslant 0 \quad \text{for } 0 \leqslant \alpha \leqslant s - r.
\end{aligned}$$

Therefore, the first inequality in (2.12) holds. On the other hand, Lemma 2.1(1) applies even for the case $X = Y$, where $S(x)$ is the marginal survival function of X and Y while the joint survival function is

$$S(x, y) = \min(S(x), S(y)), \quad x, y \geqslant 0.$$

Then the arguments x and y of (2.4) are interchangeable:

$$\begin{aligned}\mathrm{C}(X^r, X^s) &= rs \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} [S(x,y) - S(x)S(y)] \, dy \, dx \\ &= rs \int_0^\infty \int_0^\infty x^{s-1} y^{r-1} [S(x,y) - S(x)S(y)] \, dy \, dx \qquad \text{for } r, s > 0.\end{aligned}$$

Thus, considering (2.11), we have the first inequality in (2.13):

$$\begin{aligned}&\mathrm{C}(X^r, X^s) - \beta\, \mathrm{C}(X^{r+\alpha}, X^{s-\alpha}) \\ &\quad = rs \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} [S(x,y) - S(x)S(y)] \, dy \, dx \\ &\qquad - \beta(r+\alpha)(s-\alpha) \int_0^\infty \int_0^\infty x^{r+\alpha-1} y^{s-\alpha-1} [S(x,y) - S(x)S(y)] \, dy \, dx \\ &\quad \geqslant rs \int_0^\infty \int_0^\infty [x^{r-1} y^{s-1} - x^{r+\alpha-1} y^{s-\alpha-1}][S(x,y) - S(x)S(y)] \, dy \, dx \\ &\quad = \frac{rs}{2} \int_0^\infty \int_0^\infty [x^{r-1} y^{s-1} + x^{s-1} y^{r-1} - x^{r+\alpha-1} y^{s-\alpha-1} - x^{s-\alpha-1} y^{r+\alpha-1}] \\ &\qquad \times [S(x,y) - S(x)S(y)] \, dy \, dx \\ &\quad = \frac{rs}{2} \int_0^\infty \int_0^\infty x^{r-1} y^{r-1} (y^\alpha - x^\alpha)(y^{s-r-\alpha} - x^{s-r-\alpha})[S(x,y) - S(x)S(y)] \, dy \, dx \\ &\quad \geqslant 0.\end{aligned}$$

CASE $r < s < 0$. Define Y, u, and v as in (2.17). Then

$$0 \leqslant \alpha \leqslant s - r = v - u, \quad \beta \leqslant \frac{rs}{(r+\alpha)(s-\alpha)} = \frac{uv}{(u+\alpha)(v-\alpha)}.$$

With r and s replaced by u and v, the random variable Y satisfies (2.12):

$$\mathrm{E}\, Y^{u+\alpha}\, \mathrm{E}\, Y^{v-\alpha} \leqslant \mathrm{E}\, Y^u\, \mathrm{E}\, Y^v \leqslant \beta\, \mathrm{E}\, Y^{u+\alpha}\, \mathrm{E}\, Y^{v-\alpha} + (1-\beta)\, \mathrm{E}\, Y^{u+v}. \tag{2.18}$$

Remember that, from (2.10), it follows that all the moments involved of the random variable Y are finite. Finally, notice that (2.18) is just (2.12).

(2) Consider α and β as in (2.14).

CASE $0 \leqslant \alpha < s$. By the definition of variance and covariance, the first and second inequalities of (2.15) are equivalent to the second and first of (2.16), respectively:

$$(\mathrm{E}\, X^s)^2 \leqslant \mathrm{E}\, X^{s+\alpha}\, \mathrm{E}\, X^{s-\alpha} \leqslant \beta (\mathrm{E}\, X^s)^2 + (1-\beta)\, \mathrm{E}\, X^{2s},$$

$$(\mathrm{E}\,X^s)^2 - \mathrm{E}\,X^{2s} \leqslant \mathrm{E}\,X^{s+\alpha}\,\mathrm{E}\,X^{s-\alpha} - \mathrm{E}\,X^{2s} \leqslant \beta[(\mathrm{E}\,X^s)^2 - \mathrm{E}\,X^{2s}],$$
$$\beta\,\mathrm{V}\,X^s \leqslant \mathrm{C}(X^{s+\alpha}, X^{s-\alpha}) \leqslant \mathrm{V}\,X^s.$$

Define $u = s-\alpha$ and $v = s+\alpha$. Then $s = u+\alpha = v-\alpha$ and $\beta \leqslant (s^2-\alpha^2)/s^2 = uv/[(u+\alpha)(v-\alpha)]$. Therefore, the last two inequalities are a particular case of (2.13), with

$$\mathrm{V}\,X^s = \mathrm{C}(X^{u+\alpha}, X^{v-\alpha}) \quad \text{and} \quad \mathrm{C}(X^{s+\alpha}, X^{s-\alpha}) = \mathrm{C}(X^u, X^v).$$

CASE $0 \leqslant \alpha < -s$. Since $\mathrm{E}\,X^{2s} < \infty$, we have $P(X = 0) = 0$ and all moments involved for the random variable X in (2.15) are finite. Therefore, the conclusions of the above case also hold here.

Finally, the case $0 < \alpha = |s|$ is trivial. ∎

Inequalities (2.12) also hold for $r = 0 < s$, which is the Chebyshev inequality for sums and integrals; see [22, p. 601], [28, Theorems 2.5.1 and 2.5.10], and [19, (3)].

The potential application of Proposition 2.1 is illustrated in the following three corollaries and a remark. In particular, we give new or improve some known inequalities for the ratio of gamma functions.

COROLLARY 2.1. (1) *Let* $0 < x \leqslant y$, $x+r+s, rs > 0$, $0 \leqslant \alpha \leqslant s-r$, *and* $0 \leqslant \rho \leqslant 1$. *Then*

$$\begin{aligned}(2.19)\quad 1 &\leqslant \frac{\Gamma(x+r)\,\Gamma(x+s)}{\Gamma(x+r+\alpha)\,\Gamma(x+s-\alpha)}\\ &\leqslant \frac{rs}{(r+\alpha)(s-\alpha)} + \frac{\alpha(s-r-\alpha)\,\Gamma(x)\,\Gamma(x+r+s)}{(r+\alpha)(s-\alpha)\,\Gamma(x+r+\alpha)\,\Gamma(x+s-\alpha)},\end{aligned}$$

$$\begin{aligned}(2.20)\quad 1 &\leqslant \frac{\Gamma(x+r)\,\Gamma(x+s)\,\Gamma(y+r+\alpha)\,\Gamma(y+s-\alpha)}{\Gamma(y+r)\,\Gamma(y+s)\,\Gamma(x+r+\alpha)\,\Gamma(x+s-\alpha)}\\ &\leqslant \frac{rs}{(r+\alpha)(s-\alpha)}\\ &\quad + \frac{\alpha(s-r-\alpha)\,\Gamma(x)\,\Gamma(x+r+s)\,\Gamma(y+r+\alpha)\,\Gamma(y+s-\alpha)}{(r+\alpha)(s-\alpha)\,\Gamma(y)\,\Gamma(y+r+s)\,\Gamma(x+r+\alpha)\,\Gamma(x+s-\alpha)}.\end{aligned}$$

Moreover, if $r+s+1 > 0$, *then*

$$\begin{aligned}(2.21)\quad 1 &\leqslant \frac{\Gamma(r+1)\,\Gamma(s+1)\,\Gamma(\rho(r+\alpha)+1)\,\Gamma(\rho(s-\alpha)+1)}{\Gamma(\rho r+1)\,\Gamma(\rho s+1)\,\Gamma(r+\alpha+1)\,\Gamma(s-\alpha+1)}\\ &\leqslant \frac{rs}{(r+\alpha)(s-\alpha)}\\ &\quad + \frac{\alpha(s-r-\alpha)\,\Gamma(r+s+1)\,\Gamma(\rho(r+\alpha)+1)\,\Gamma(\rho(s-\alpha)+1)}{(r+\alpha)(s-\alpha)\,\Gamma(\rho(r+s)+1)\,\Gamma(r+\alpha+1)\,\Gamma(s-\alpha+1)}.\end{aligned}$$

(2) *Let* $0 < x \leqslant y$, $x + 2s > 0$, $0 \leqslant \alpha \leqslant |s|$, *and* $0 \leqslant \rho \leqslant 1$. *Then*

$$1 \leqslant \frac{\Gamma(x+s+\alpha)\,\Gamma(x+s-\alpha)}{\Gamma^2(x+s)} \leqslant \frac{s^2-\alpha^2}{s^2} + \frac{\alpha^2\,\Gamma(x)\,\Gamma(x+2s)}{s^2\,\Gamma^2(x+s)}, \tag{2.22}$$

$$\begin{aligned} 1 &\leqslant \frac{\Gamma(x+s+\alpha)\,\Gamma(x+s-\alpha)\,\Gamma^2(y+s)}{\Gamma(y+s+\alpha)\,\Gamma(y+s-\alpha)\,\Gamma^2(x+s)} \\ &\leqslant \frac{s^2-\alpha^2}{s^2} + \frac{\alpha^2\,\Gamma(x)\,\Gamma(x+2s)\,\Gamma^2(y+s)}{s^2\,\Gamma(y)\,\Gamma(y+2s)\,\Gamma^2(x+s)}. \end{aligned} \tag{2.23}$$

Moreover, if $2s + 1 > 0$, *then*

$$\begin{aligned} 1 &\leqslant \frac{\Gamma(s+\alpha+1)\,\Gamma(s-\alpha+1)\,\Gamma^2(\rho s+1)}{\Gamma(\rho(s+\alpha)+1)\,\Gamma(\rho(s-\alpha)+1)\,\Gamma^2(s+1)} \\ &\leqslant \frac{s^2-\alpha^2}{s^2} + \frac{\alpha^2\,\Gamma(2s+1)\,\Gamma^2(\rho s+1)}{s^2\,\Gamma(2\rho s+1)\,\Gamma^2(s+1)}. \end{aligned} \tag{2.24}$$

Proof. Let X be a positive random variable, $0 < x \leqslant y$, and $0 \leqslant \rho \leqslant 1$.
(1) For $rs > 0$ and $0 \leqslant \alpha \leqslant s - r$, (2.12) is equivalent to

$$\begin{aligned} 1 &\leqslant \frac{\mathrm{E}\,X^r\,\mathrm{E}\,X^s}{\mathrm{E}\,X^{r+\alpha}\,\mathrm{E}\,X^{s-\alpha}} \\ &\leqslant \frac{rs}{(r+\alpha)(s-\alpha)} + \frac{\alpha(s-r-\alpha)}{(r+\alpha)(s-\alpha)}\,\frac{\mathrm{E}\,X^{r+s}}{\mathrm{E}\,X^{r+\alpha}\,\mathrm{E}\,X^{s-\alpha}} \end{aligned} \tag{2.25}$$

whenever all these moments are finite. In particular, if the distribution for X is gamma with parameters of shape $x > 0$ and of scale 1, then

$$\mathrm{E}\,X^t = \frac{\Gamma(x+t)}{\Gamma(x)}, \quad x + t > 0. \tag{2.26}$$

Therefore, (2.19) holds. Similarly, (2.20) is given by (2.25) for the particular case of a beta distribution, with shape parameters $x, y - x > 0$. Here

$$\mathrm{E}\,X^t = \frac{\Gamma(y)\,\Gamma(x+t)}{\Gamma(x)\,\Gamma(y+t)}, \quad x + t > 0. \tag{2.27}$$

Now, for a random variable X with a stable distribution, with characteristic exponent $0 < \rho < 1$, its moments are

$$\mathrm{E}\,X^t = \frac{\Gamma(1-t/\rho)}{\Gamma(1-t)}, \quad t < \rho. \tag{2.28}$$

See [24, Lemma 5] and [30, (3.50)]. From (2.25) and (2.28), we have

$$1 \leqslant \frac{\Gamma(1-r/\rho)\,\Gamma(1-s/\rho)\,\Gamma(1-r-\alpha)\,\Gamma(1-s+\alpha)}{\Gamma(1-r)\,\Gamma(1-s)\,\Gamma(1-(r+\alpha)/\rho)\,\Gamma(1-(s-\alpha)/\rho)} \leqslant \frac{rs}{(r+\alpha)(s-\alpha)} + \frac{\alpha(s-r-\alpha)\,\Gamma(1-(r+s)/\rho)\,\Gamma(1-r-\alpha)\,\Gamma(1-s+\alpha)}{(r+\alpha)(s-\alpha)\,\Gamma(1-r-s)\,\Gamma(1-(r+\alpha)/\rho)\,\Gamma(1-(s-\alpha)/\rho)}$$

for $r+s<\rho<1$. Let us consider the next reparameterization for r, s, and α:

$$u=-\frac{s}{\rho}, \quad v=-\frac{r}{\rho}, \quad \text{and} \quad \gamma=\frac{\alpha}{\rho}.$$

Then $uv, u+v+1>0$, $0\leqslant\gamma\leqslant v-u$, and

$$\begin{aligned}1 &\leqslant \frac{\Gamma(1+v)\,\Gamma(1+u)\,\Gamma(1+\rho(v-\gamma))\,\Gamma(1+\rho(u+\gamma))}{\Gamma(1+\rho v)\,\Gamma(1+\rho u)\,\Gamma(1+v-\gamma)\,\Gamma(1+u+\gamma)} \\ &\leqslant \frac{uv}{(v-\gamma)(u+\gamma)} \\ &\quad + \frac{\gamma(v-u-\gamma)\,\Gamma(1+u+v)\,\Gamma(1+\rho(v-\gamma))\,\Gamma(1+\rho(u+\gamma))}{(v-\gamma)(u+\gamma)\,\Gamma(1+\rho(u+v))\,\Gamma(1+v-\gamma)\,\Gamma(1+u+\gamma)}.\end{aligned}$$

Thus, (2.21) is obtained by letting $r=u$, $s=v$, and $\alpha=\gamma$. Here $rs, r+s+1>0$, $0\leqslant\alpha\leqslant s-r$, together with $0\leqslant\rho\leqslant 1$.

(2) The inequalities in (2.15) are equivalent to

$$1 \leqslant \frac{\operatorname{E} X^{s+\alpha}X^{s-\alpha}}{(\operatorname{E} X^s)^2} \leqslant \frac{s^2-\alpha^2}{s^2} + \frac{\alpha^2 \operatorname{E} X^{2s}}{s^2(\operatorname{E} X^s)^2}, \quad 0\leqslant\alpha\leqslant|s|.$$

Therefore, (2.22), (2.23), and (2.24) follow from (2.26), (2.27), and (2.28), respectively. ■

COROLLARY 2.2. (1) *Let $x, y, x+r+s>0$ and $rs\geqslant 0$. Then*

$$1+\frac{rs}{x+r+s} \leqslant \frac{\Gamma(x)\,\Gamma(x+r+s)}{\Gamma(x+r)\,\Gamma(x+s)}, \tag{2.29}$$

$$1+\frac{rs}{x+s-1} \leqslant \frac{\Gamma(x)\,\Gamma(x+r+s)}{\Gamma(x+r)\,\Gamma(x+s)} \quad \textit{for } s-r\geqslant 1, \tag{2.30}$$

$$\begin{aligned}1+\frac{rs}{x(x+r+s)} &\leqslant \frac{\Gamma(x)\,\Gamma(x+r+s)}{\Gamma(x+r)\,\Gamma(x+s)} \\ &\leqslant 1+\frac{rs(2x+r+s)\,\Gamma(2x+r+s)}{x(x+r)(x+s)(x+r+s)\,\Gamma(x+r)\,\Gamma(x+s)} \\ &\qquad \textit{for } r,s\geqslant 0,\end{aligned} \tag{2.31}$$

$$1+\frac{s^2}{x+2s} \leqslant \frac{\Gamma(x)\,\Gamma(x+2s)}{\Gamma^2(x+s)} \quad \textit{for } x+2s>0, \tag{2.32}$$

$$(2.33)\qquad 1+\frac{s^2}{x(x+2s)} \leqslant \frac{\Gamma(x)\,\Gamma(x+2s)}{\Gamma^2(x+s)} \leqslant 1+\frac{2s^2\,\Gamma(2(x+s))}{x(x+s)(x+2s)\,\Gamma^2(x+s)} \quad \textit{for } s \geqslant 0,$$

$$(2.34)\qquad \frac{(x+y)(x+y+1)(1-xy)}{xy(x+1)(y+1)} \leqslant \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \leqslant \frac{(x+y)(x+y+1)}{xy(x+1)(y+1)},$$

$$(2.35)\qquad \frac{x(x+1)^2}{2(2x+1)} \leqslant \frac{\Gamma(2x)}{\Gamma^2(x)} \leqslant \frac{x(x+1)}{2(2x+1)(1-x)} \quad (0<x<1 \textit{ for the last inequality}),$$

$$(2.36)\qquad \frac{x+1/2}{\sqrt{x+1}} \leqslant \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leqslant \sqrt{x+\frac{x\,\Gamma(2x)}{(x+1)(x+1/2)\,\Gamma^2(x+1/2)}},$$

$$(2.37)\qquad 1+\frac{(y-x)^2}{4xy} \leqslant \frac{\Gamma(x)\,\Gamma(y)}{\Gamma^2((x+y)/2)} \leqslant 1+\frac{(y-x)^2\,\Gamma(x+y)}{xy(x+y)\,\Gamma^2((x+y)/2)},$$

$$(2.38)\qquad 1+\frac{(y-x)^2}{4\min(x,y)} \leqslant \frac{\Gamma(x)\,\Gamma(y)}{\Gamma^2((x+y)/2)} \leqslant \frac{x+y}{2\min(x,y)} \quad (|y-x| \leqslant 2 \textit{ for the last inequality}).$$

(2) *Let* $0<x\leqslant y$, $x+r+s>0$, $rs\geqslant 0$, *and* $0\leqslant\rho\leqslant 1$. *Then*

$$(2.39)\qquad 1+\frac{rs(y-x)}{(x+s-1)(y+r)} \leqslant \frac{\Gamma(x)\,\Gamma(y+r)\,\Gamma(y+s)\,\Gamma(x+r+s)}{\Gamma(y)\,\Gamma(x+r)\,\Gamma(x+s)\,\Gamma(y+r+s)} \quad \textit{for } s-r\geqslant 1,$$

$$(2.40)\qquad 1+\frac{s^2(y-x)}{(x+s-1)(y+s)} \leqslant \frac{\Gamma(x)\,\Gamma(x+2s)\,\Gamma^2(y+s)}{\Gamma(y)\,\Gamma(y+2s)\,\Gamma^2(x+s)} \quad \textit{for } x+2s>0,\ |s|\geqslant 1,$$

$$(2.41)\qquad 1 \leqslant \frac{\Gamma(r+s+1)\,\Gamma(\rho r+1)\,\Gamma(\rho s+1)}{\Gamma(\rho(r+s)+1)\,\Gamma(r+1)\,\Gamma(s+1)} \quad \textit{for } r+s+1>0,$$

$$(2.42)\qquad \frac{\Gamma(\rho x)\,\Gamma(\rho y)}{\Gamma^2(\rho(x+y)/2)} \leqslant \frac{\Gamma(x)\,\Gamma(y)}{\Gamma^2((x+y)/2)} \leqslant \frac{4xy\,\Gamma(\rho x)\,\Gamma(\rho y)}{(x+y)^2\,\Gamma^2(\rho(x+y)/2)} + \frac{\rho(y-x)^2\,\Gamma(x+y)\,\Gamma(\rho x)\,\Gamma(\rho y)}{(x+y)^2\,\Gamma^2((x+y)/2)\,\Gamma(\rho(x+y))} \quad \textit{for } 0<\rho\leqslant 1.$$

Proof. (1) Inequalities (2.29)–(2.31) follow from the second inequality in (2.19). With $\alpha=1<s-r$, we have

$$\frac{x+s-1}{x+r} \leqslant \frac{rs}{(r+1)(s-1)} + \frac{(s-r-1)(x+s-1)\,\Gamma(x)\,\Gamma(x+r+s)}{(r+1)(s-1)(x+r)\,\Gamma(x+r)\,\Gamma(x+s)},$$

$$\frac{\Gamma(x)\,\Gamma(x+r+s)}{\Gamma(x+r)\,\Gamma(x+s)} \geqslant \left(\frac{x+s-1}{x+r} - \frac{rs}{(r+1)(s-1)}\right)\frac{(r+1)(s-1)(x+r)}{(s-r-1)(x+s-1)} = 1+\frac{rs}{x+s-1}.$$

Thus, (2.30) holds. Moreover, (2.29) is given by (2.30) with the reparameterization $v = s - 1 \geqslant r$, together with symmetry arguments between r and v. Similarly, for $x = 1$, $r > 0$, and $0 \leqslant \alpha \leqslant s - r$, (2.19) gives

$$\begin{aligned} 1 &\leqslant \frac{rs\,\Gamma(r)\,\Gamma(s)}{(r+\alpha)(s-\alpha)\,\Gamma(r+\alpha)\,\Gamma(s-\alpha)} \\ &\leqslant \frac{rs}{(r+\alpha)(s-\alpha)} + \frac{\alpha(s-r-\alpha)(r+s)\,\Gamma(r+s)}{(r+\alpha)^2(s-\alpha)^2\,\Gamma(r+\alpha)\,\Gamma(s-\alpha)}, \\ 1 + \frac{\alpha(s-r-\alpha)}{rs} &\leqslant \frac{\Gamma(r)\,\Gamma(s)}{\Gamma(r+\alpha)\,\Gamma(s-\alpha)} \\ &\leqslant 1 + \frac{\alpha(r+s)(s-r-\alpha)\,\Gamma(r+s)}{rs(r+\alpha)(s-\alpha)\,\Gamma(r+\alpha)\,\Gamma(s-\alpha)}. \end{aligned}$$

Then (2.31) is obtained by a reparameterization such that $x > 0$ and $r, s \geqslant 0$.

On the other hand, with $r = s$, inequalities (2.32) and (2.33) are particular cases of (2.29) and (2.31), respectively.

For $x = 1$ in (2.31), we have

$$\begin{aligned} \frac{(r+1)(s+1)}{r+s+1} &\leqslant \frac{\Gamma(r+s+1)}{\Gamma(r+1)\,\Gamma(s+1)} \\ &\leqslant 1 + \frac{rs(r+s+2)\,\Gamma(r+s+2)}{(r+1)(s+1)(r+s+1)\,\Gamma(r+1)\,\Gamma(s+1)}, \\ \frac{rs(r+1)(s+1)}{(r+s)(r+s+1)} &\leqslant \frac{\Gamma(r+s)}{\Gamma(r)\,\Gamma(s)} \leqslant \frac{rs(r+1)(s+1)}{(r+s)(r+s+1)(1-rs)}. \end{aligned}$$

The last inequality applies only for $0 < rs < 1$. Therefore, we obtain (2.34). The particular case $x = y$ gives (2.35).

On the other hand, (2.36) is implied by the particular case $s = 1/2$ of (2.33). Inequalities (2.37) are a symmetrical form of (2.33).

The denominator in the lower bound of inequality (2.32) can be replaced by $\min(x, x+2s)$; look at the case $0 < x + 2s < x$. Thus, the first inequality of (2.38) holds. On the other hand, the second inequality follows from the first one of (2.24). In fact, for $\alpha = 1 < s$, we get

$$\frac{\Gamma(\rho(s+1))\,\Gamma(\rho(s-1))}{\Gamma^2(\rho s)} \leqslant \frac{s}{s-1} \quad \text{for } 0 < \rho \leqslant 1 < s.$$

With a reparameterization, this gives

$$\frac{\Gamma(x)\,\Gamma(y)}{\Gamma^2((x+y)/2)} \leqslant \frac{x+y}{2x} \quad \text{for } 0 < x < y \leqslant x + 2,$$

from which the second inequality of (2.38) follows by arguments of symmetry between x and y, for $x, y > 0$ and $|y - x| \leqslant 2$.

(2) For the particular case $\alpha = 1$, the second inequalities in (2.20) and (2.23) represent (2.39) and (2.40), respectively.

On the other hand, the first inequality of (2.21), with $r = 0 \leqslant \alpha \leqslant s$, implies

$$1 \leqslant \frac{\Gamma(s+1)\,\Gamma(\rho\alpha+1)\,\Gamma(\rho(s-\alpha)+1)}{\Gamma(\rho s+1)\,\Gamma(\alpha+1)\,\Gamma(s-\alpha+1)}.$$

Similarly, for $s = 0 \leqslant \alpha \leqslant -r < 1$, (2.21) gives

$$1 \leqslant \frac{\Gamma(r+1)\,\Gamma(\rho(r+\alpha)+1)\,\Gamma(1-\rho\alpha)}{\Gamma(\rho r+1)\,\Gamma(r+\alpha+1)\,\Gamma(1-\alpha)}.$$

Through a reparameterization, both of these inequalities are written as (2.41). Another implication from (2.21) is (2.42), with $\alpha = (s-r)/2$ and $0 < r \leqslant s$. ■

REMARK 2.1. (1) In (2.20) we give a double inequality, where the first generalizes the inequality in [3, Theorem 2.5]. Indeed, for $x = 1$ and through the reparameterization $u = s$, $v = y - 1 + s$, $a = \alpha$, and $b = s - r - \alpha$, (2.20) can be written as

$$\begin{aligned} 1 &\leqslant \frac{\Gamma(u+1)\,\Gamma(u-a-b+1)\,\Gamma(v-a+1)\,\Gamma(v-b+1)}{\Gamma(u-a+1)\,\Gamma(u-b+1)\,\Gamma(v+1)\,\Gamma(v-a-b+1)} \\ &\leqslant \frac{u(u-a-b)}{(u-a)(u-b)} \\ &\quad + \frac{ab\,\Gamma(2u-a-b+1)\,\Gamma(v-a+1)\,\Gamma(v-b+1)}{(u-a)(u-b)\,\Gamma(v-u+1)\,\Gamma(v+u-a-b+1)\,\Gamma(u-a+1)\,\Gamma(u-b+1)}, \end{aligned}$$

where $a, b, v-u \geqslant 0$ and $2u-a-b+1, u(u-a-b) > 0$.

(2) The Bustoz and Ismail inequality [4, (4.4)] is

$$1 \leqslant \frac{\Gamma(x)\,\Gamma(x+r+s)}{\Gamma(x+r)\,\Gamma(x+s)}, \qquad x > 0,\ r, s \geqslant 0. \tag{2.43}$$

See also [9, Theorem 4]. Inequalities (2.29), (2.30), and the first in (2.31) improve (2.43). The improvements additionally include an extended range for the parameters together with the new double inequality (2.31). On the other hand, with the additional constraint $s - r \geqslant 1$, (2.30) is better than (2.29). Additionally, the first inequality in (2.31) is better than (2.29) for $0 < x < 1$, and than (2.30) for $0 < r \leqslant s - 1$ and small x:

$$0 < x < ([(r+s-1)^2 + 4(s-1)]^{1/2} - (r+s-1))/2.$$

(3) The Gurland inequality [14, (1)] states

$$1 + \frac{s^2}{x} \leqslant \frac{\Gamma(x)\,\Gamma(x+2s)}{\Gamma^2(x+s)}, \quad x, x+2s > 0.$$

This inequality is improved by (2.32) and the first inequality of (2.33), for the cases $0 < x + 2s < x$ and $x + 2s < 1$ respectively. Moreover, (2.33) is a double inequality. The Gurland ratio has been widely studied; see, for instance, [13, 37, 5, 35, 10, 21, 27, 34, 40].

(4) From [17, (3.2)], it follows that

$$\frac{x+y-xy}{xy} \leqslant \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \leqslant \frac{x+y}{xy(xy+1)}, \quad 0 < x, y \leqslant 1. \tag{2.44}$$

The second inequality also holds for $x, y > 1$; see [18, (3.1)]. For x and y near zero, the first inequality in (2.34) improves the one of (2.44). In fact,

$$\frac{x+y-xy}{xy} \leqslant \frac{(x+y)(x+y+1)(1-xy)}{xy(x+1)(y+1)} \iff (x+y)^2 \leqslant (1-x)(1-y).$$

Our contribution in the second inequality of (2.34) is a range without constraints of arguments. Moreover, this inequality improves [29, (13)] or [3, (3.3)] and [9, (3.16)] for x, y near zero: $x^2 + xy + y^2 < 1$.

(5) From [3, (3.11)] and [17, (2.8)], it follows that

$$\frac{x}{2} \leqslant \frac{\Gamma(2x)}{\Gamma^2(x)} \leqslant \frac{x}{2-x}, \qquad x > 0 \; (0 < x \leqslant 1 \text{ for the last inequality}). \tag{2.45}$$

Our first inequality in (2.35) improves the one of (2.45), whereas the second one is also better than the respective inequality for $0 < x < 1/3$.

(6) For $s = 1/2$, [34, (2.8)] states

$$\sqrt{x} \leqslant \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \leqslant \sqrt{x+1/2}, \quad x > 0. \tag{2.46}$$

The first inequality in (2.36) improves the lower bound of (2.46). Moreover, from the second inequality in (2.45), the second inequality in (2.36) is also better than the respective upper bound of (2.46), for at least $0 < x \leqslant 1/2$.

(7) From [17, (1.7) and (3.1)], it follows that

$$1 + \frac{(y-x)^2}{4xy} \leqslant \frac{\Gamma(x)\,\Gamma(y)}{\Gamma^2((x+y)/2)} \leqslant \frac{x^2+y^2}{2xy}, \quad x, y > 0 \tag{2.47}$$

($0 < x, y \leqslant 1$ for the last inequality). In (2.37), we give an alternative method to obtain the first inequality in (2.47). Moreover, the first inequality in (2.38) improves the one of (2.47) for $\max(x, y) > 1$. Also the range of values for the second inequality is improved and widened. In fact,

$$1 + \frac{(y-x)^2\,\Gamma(x+y)}{xy(x+y)\,\Gamma^2((x+y)/2)} \leqslant \frac{x^2+y^2}{2xy},$$

$$\frac{\Gamma(x+y)}{\Gamma^2((x+y)/2)} \leqslant \frac{x+y}{2}.$$

The last relationship is deduced from (2.45):

$$\frac{\Gamma(x+y)}{\Gamma^2((x+y)/2)} \leqslant \frac{(x+y)/2}{2-(x+y)/2} \leqslant \frac{x+y}{2}, \quad x, y > 0,\, x+y < 2.$$

For $|y-x| > 1$, the lower bound of (2.38) improves the one of [10, (6)]:

$$1 + \frac{(y-x)^2}{4} \sum_{k=0}^{\infty} \frac{1}{(x+k)(y+k)} \leqslant \frac{\Gamma(x)\,\Gamma(y)}{\Gamma^2((x+y)/2)}, \quad x, y > 0.$$

See also [27, (6) and (40)]. In fact, for $0 < x < y-1$,

$$\begin{aligned} 1 + \sum_{k=0}^{\infty} \frac{(y-x)^2/4}{(x+k)(y+k)} &< 1 + \sum_{k=0}^{\infty} \frac{(y-x)^2/4}{(x+k)(x+k+1)} \\ &= 1 + \frac{(y-x)^2}{4} \lim_{n\to\infty} \frac{n+1}{x(x+n+1)} = 1 + \frac{(y-x)^2}{4x}. \end{aligned}$$

(8) Finally, we have not found a reference for inequalities (2.39)–(2.42).

COROLLARY 2.3. *Let X be a nonnegative random variable such that $\mathrm{E}\,X^{r+s+k} < \infty$ for $0 < r < s$ and all $k \geqslant 0$. Let $h(x)$ be an analytic function with $h^{(k)}(0) \geqslant 0$ for $k \geqslant 0$. If $0 \leqslant \alpha \leqslant s-r$ and $\beta \leqslant r/(r+\alpha)$, then*

$$\begin{aligned} \mathrm{E}\,X^{r+\alpha}\,\mathrm{E}[X^{s-\alpha}h(X)] &\leqslant \mathrm{E}\,X^{r}\,\mathrm{E}[X^{s}h(X)] \\ &\leqslant \beta\,\mathrm{E}\,X^{r+\alpha}\,\mathrm{E}[X^{s-\alpha}h(X)] + (1-\beta)\,\mathrm{E}[X^{r+s}h(X)]. \end{aligned} \tag{2.48}$$

In particular, if $h(x) = \exp(x)$ for $x \geqslant 0$, then

$$\begin{aligned} \mathrm{E}\,X^{r+\alpha}\,\mathrm{E}[X^{s-\alpha}e^{X}] &\leqslant \mathrm{E}\,X^{r}\,\mathrm{E}[X^{s}e^{X}] \\ &\leqslant \beta\,\mathrm{E}\,X^{r+\alpha}\,\mathrm{E}[X^{s-\alpha}e^{X}] + (1-\beta)\,\mathrm{E}[X^{r+s}e^{X}]. \end{aligned}$$

Proof. Inequalities (2.48) follow immediately by applying (2.12) to each of the terms of the Taylor expansion around $x=0$ of $h(x)$. In addition to the conditions in (2.11), we need $\beta \leqslant r/(r+\alpha)$. ∎

Other known inequalities involving the quotients of functions can be recovered from Proposition 2.1. For instance, the inequality

$$1 \leqslant \frac{(e^y-1)/y}{(e^x-1)/x} \quad \text{for } x \leqslant y$$

follows from (2.12) or (2.15), where the underlying distribution of the random variable X is log normal, with parameters $\mu = 0$ and $\sigma = 1$.

3. FINITE SUMS

This section gives a new inequality for finite sums of nonnegative or nonpositive numbers involving the exponential function. For some cases, this inequality even improves the Cauchy–Bunyakovsky–Schwarz inequality. Its extension to the case of nonnegative or nonpositive random variables is an open problem. This section is self-contained.

PROPOSITION 3.1. *For* $x_1, \ldots, x_n \geqslant 0$ *or* $x_1, \ldots, x_n \leqslant 0$,

$$\Big(\sum_{i=1}^n x_i e^{x_i}\Big)^2 \leqslant \sum_{i=1}^n x_i^2 e^{x_i} \sum_{i=1}^n (e^{x_i}-1) + \Big(\sum_{i=1}^n (e^{x_i}-1)\Big)^2. \tag{3.1}$$

Proof. For $n=1$, we have

$$(e^x-1)^2 + x^2 e^x (e^x-1) - (xe^x)^2 = (e^x + xe^{x/2} - 1)(e^x - xe^{x/2} - 1), \quad x \in \mathbb{R}.$$

The first factor of the right-hand side is positive [negative] if $x>0$ [$x<0$]. The sign of the second factor is the same, since the auxiliary function

$$g(x) = e^x - xe^{x/2} - 1 \quad \text{for } x \in \mathbb{R}$$

is increasing, with $g(0)=0$:

$$g'(x) = e^x - \Big(1+\frac{x}{2}\Big)e^{x/2} = e^{x/2}\Big(e^{x/2} - 1 - \frac{x}{2}\Big) > 0, \quad x \neq 0.$$

For $n \geqslant 2$, we have

$$\begin{aligned}
&\Big(\sum_{i=1}^n (e^{x_i}-1)\Big)^2 + \sum_{i=1}^n x_i^2 e^{x_i} \sum_{i=1}^n (e^{x_i}-1) - \Big(\sum_{i=1}^n x_i e^{x_i}\Big)^2 \\
&= \sum_{i=1}^n (e^{x_i}-1)^2 + 2\sum_{i<j}(e^{x_i}-1)(e^{x_j}-1) + \sum_{i=1}^n x_i^2 e^{x_i}(e^{x_i}-1) \\
&\quad + \sum_{i\neq j} x_i^2 e^{x_i}(e^{x_j}-1) - \Big(\sum_{i=1}^n x_i^2 e^{2x_i} + 2\sum_{i<j} x_i x_j e^{x_i+x_j}\Big) \\
&= \sum_{i=1}^n [(e^{x_i}-1)^2 + x_i^2 e^{x_i}(e^{x_i}-1) - x_i^2 e^{2x_i}] + \sum_{i<j}[2(e^{x_i}-1)(e^{x_j}-1) \\
&\quad + x_i^2 e^{x_i}(e^{x_j}-1) + x_j^2 e^{x_j}(e^{x_i}-1) - 2x_i x_j e^{x_i+x_j}] \\
&= \sum_{i=1}^n [(e^{x_i}-1)^2 + x_i^2 e^{x_i}(e^{x_i}-1) - x_i^2 e^{2x_i}] + \sum_{i<j} g(x_i, x_j),
\end{aligned} \tag{3.2}$$

where $g(x,y)$ is the auxiliary function

$$g(x,y) = 2(e^x-1)(e^y-1) + x^2 e^x (e^y-1) + y^2 e^y (e^x-1) - 2xye^{x+y}$$

for $xy \geqslant 0$. From the case $n = 1$, it follows that the terms of the first sum on the right-hand side in (3.2) are nonnegative. It remains to verify that $g(x, y) \geqslant 0$. We note that $g(0, y) = 0$. We prove

$$g(0, y) < g(x, y) \quad \text{for } xy > 0.$$

It is enough to show that $g_x(x, y) > [<]\ 0$ for $x, y > [<]\ 0$. First, we consider the case $x, y > 0$:

$$\begin{aligned} g(x,y) &= (2 + (x-y)^2)e^{x+y} - (2+x^2)e^x - (2+y^2)e^y + 2, \\ g_x(x,y) &= (2 + (x-y)^2 + 2(x-y))e^{x+y} - (2 + x^2 + 2x)e^x \\ &= [(1 + (1+x-y)^2)e^y - (1 + (1+x)^2)]e^x > 0, \end{aligned}$$

because the function

$$h(x, y) = (1 + (1 + x - y)^2)e^y, \quad x, y \geqslant 0,$$

is increasing with respect to y:

$$h_y(x, y) = (1 + (1 + x - y)^2 - 2(1 + x - y))e^y = (x - y)^2 e^y \geqslant 0.$$

Otherwise, for $x, y < 0$, the function $h(x, y)$ also increases with respect to y. In this case, $g_x(x, y) < 0$ and $g(x, y) > g(0, y) = 0$. ∎

Inequality (3.1) improves the Cauchy–Bunyakovsky–Schwarz inequality when the values x_i are close to zero. To see this, we verify the second inequality of

(3.3) $$a^2\overline{xy}^2 \leqslant a^2\overline{x^2y}(\bar{y} - 1) + (\bar{y} - 1)^2 \leqslant a^2\overline{x^2}\,\overline{y^2} \quad \text{for } 0 \leqslant a \approx 0,$$

with $y_i = y_i(a) = e^{ax_i}$, $i = 1, \dots, n$, and $n\bar{y} = \sum_{i=1}^n y_i$. Similarly, we define $\overline{x^2}$, $\overline{y^2}$, $\overline{xy}$, and $\overline{x^2y}$. We consider the auxiliary function

$$h(a) = a^2(\overline{x^2}\,\overline{y^2} - \overline{x^2y}(\bar{y} - 1)) - (\bar{y} - 1)^2, \quad a \geqslant 0.$$

Then

$$h''(0) = 2(\overline{x^2} - \bar{x}^2) > 0 = h'(0) = h(0),$$

with

$$\begin{aligned} h'(a) &= a^2(2\overline{x^2}\,\overline{xy^2} - \overline{xy}\,\overline{x^2y} - \overline{x^3y}(\bar{y} - 1)) \\ &\quad + 2a(\overline{x^2}\,\overline{y^2} - \overline{x^2y}(\bar{y} - 1)) - 2\overline{xy}(\bar{y} - 1), \\ h''(a) &= a^2(4\overline{x^2}\,\overline{x^2y^2} - \overline{x^2y}^2 - 2\overline{xy}\,\overline{x^3y} - \overline{x^4y}(\bar{y} - 1)) \\ &\quad + 4a(2\overline{x^2}\,\overline{xy^2} - \overline{xy}\,\overline{x^2y} - \overline{x^3y}(\bar{y} - 1)) \end{aligned}$$

$$+\,2(\overline{x^2}\,\overline{y^2} - 2\overline{x^2y}(\bar{y}-1) - \overline{xy}^2).$$

Here, we have assumed a positive sample variance. Therefore, there exists $a_0 > 0$ such that the second inequality in (3.3) holds for $0 \leqslant a \leqslant a_0$. However, the inequality is reversed when $a \to \infty$. This is clear for the case of nonpositive data, since $\lim_{a\to\infty} h(a) = -1$. For the case of nonnegative data, the dominant term of the auxiliary function $h(a)$ is not bounded below: $-[a^2(x_{(n)}^2 - \overline{x^2}) + 1]e^{2ax_{(n)}} < 0$, where $x_{(n)} > 0$ denotes the sample maximum.

Acknowledgments. The authors thank the anonymous reviewer for her/his valuable comments, which substantially improved this paper.

REFERENCES

[1] S. E. Abu-Youssef, *Moment inequality on new renewal better than used class of life distributions with hypothesis testing application*, Appl. Math. Comput. 149 (2004), 651–659.

[2] I. A. Ahmad, *Moments inequalities of aging families of distributions with hypotheses testing applications*, J. Statist. Plan. Inference 92 (2001), 121–132.

[3] M. M. Al-Jararha and J. M. Al-Jararha, *Inequalities of gamma function appearing in generalizing probability sampling design*, Austral. J. Math. Anal. Appl. 17 (2020), art. 17, 9 pp.

[4] J. Bustoz and M. E. H. Ismail, *On gamma function inequalities*, Math. Comput. 47 (1986), 659–667.

[5] P. P. Chakrabarty, *On certain inequalities connected with gamma function*, Scand. Actuar. J. 1969, 20–23.

[6] S. Chakraborti, F. Jardim and E. Epprecht, *Higher-order moments using the survival function: The alternative expectation formula*, Amer. Statist. 73 (2019), 191–194.

[7] C. M. Cuadras, *On the covariance between functions*, J. Multivariate Anal. 81 (2002), 19–27.

[8] J. A. Domínguez-Molina, G. González-Farías and R. Rodríguez-Dagnino, *A practical procedure to estimate the shape parameter in the generalized Gaussian distribution*, Tech. Rep. CIMAT I-01-18 (2001), www.researchgate.net/publication/241152541.

[9] S. S. Dragomir, R. P. Agarwal and N. S. Barnett, *Inequalities for beta and gamma functions via some classical and new integral inequalities*, J. Inequal. Appl. 5 (2000), 103–165.

[10] C. J. Everett, *Inequalities for the Wallis product*, Math. Mag. 43 (1970), 30–33.

[11] W. Feller, *An Introduction to Probability Theory and its Applications*, Vol. 2, 2nd ed., Wiley, New York, 1971.

[12] J. Glasser and R. G. Regis, *A generalized survival function method for the expectation of functions of nonnegative random variables*, Amer. J. Math. Management Sci. 40-4 (2021), 378–390.

[13] D. V. Gokhale, *On an inequality for gamma functions*, Scand. Actuar. J. 1962, 213–215.

[14] J. Gurland, *An inequality satisfied by the gamma function*, Scand. Actuar. J. 1956, 171–172.

[15] W. Hoeffding, *Massstabinvariante Korrelationtheorie*, Schrift. Math. Inst. & Inst. Angew. Math. Univ. Berlin 5 (1940), 181–233; English transl.: *Scale-invariant correlation theory*, in: The Collected Works of Wassily Hoeffding, Springer, New York, 1994, 56–107.

[16] T. Hu, M. Ma and A. K. Nanda, *Moment inequalities for discrete ageing families*, Comm. Statist. Theory Methods 32 (2003), 61–90.

[17] P. Ivády, *On a beta function inequality*, J. Math. Inequal. 6 (2012), 333–341. dx.

[18] P. Ivády, *Extension of a beta function inequality*, J. Math. Sci. Adv. Appl. 47 (2017), 1–8. dx.

[19] A. Jakubowski, *A complement to the Chebyshev integral inequality*, Statist. Probab. Lett. 168 (2021), art. 108934, 4 pp.

[20] A. W. Kemp, *Classes of discrete lifetime distributions*, Comm. Statist. Theory Methods 33 (2004), 3069–3093.

[21] A. W. Kemp, *On gamma function inequalities*, Scand. Actuar. J. 1973, 65–69.
[22] A. W. Kimball, *On dependent tests of significance in the analysis of variance*, Ann. Math. Statist. 22 (1951), 600–602,
[23] H. S. Kwong and S. Nadarajah, *Expectation formulas for integer valued multivariate random variables*, Comm. Statist. Theory Methods 47 (2018), 5514–5518.
[24] G. D. Lin, *On the Mittag-Leffler distributions*, J. Statist. Plan. Inference 74 (1998), 1–9.
[25] A. Lo, *Functional generalizations of Hoeffding's covariance lemma and a formula for Kendall's tau*, Statist. Probab. Lett. 122 (2017), 218–226.
[26] K. V. Mardia and J. W. Thompson, *Unified treatment of moment-formulae*, Sankhyā Ser. A 34 (1972), 121–132.
[27] M. Merkle, *Gurland's ratio for the gamma function*, Comput. Math. Appl. 49 (2005), 389–406.
[28] D. S. Mitrinović, *Analytic Inequalities*, Springer, Berlin, 1970.
[29] K. Nantomah, *Inequalities concerning the (p,k)-gamma and (p,k)-polygamma functions*, Note Mat. 38 (2019), 93–104.
[30] J. P. Nolan, *Univariate Stable Distributions*, Springer, Cham, 2020.
[31] H. Ogasawara, *Alternative expectation formulas for real-valued random vectors*, Comm. Statist. Theory Methods 49 (2020), 454–470.
[32] I. Olkin and L. Shepp, *Several colorful inequalities*, Amer. Math. Monthly 113 (2006), 817–822.
[33] H. H. Panjer, *Maximum likelihood estimation from distributions useful in actuarial applications*, PhD thesis, Univ. of Western Ontario, 1975.
[34] F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. 2010, art. 493058, 84 pp.
[35] B. R. Rao, *An improved inequality satisfied by the gamma function*, Scand. Actuar. J. 1969, 78–83.
[36] H. Rockette, C. Antle and L. A. Klimko, *Maximum likelihood estimation with the Weibull model*, J. Amer. Statist. Assoc. 69 (1974), 246–249.
[37] H. Ruben, *Variance bounds and orthogonal expansions in Hilbert space with an application to inequalities for gamma functions and π*, J. Reine Angew. Math. 225 (1967), 147–153.
[38] S. L. Sclove, G. Simons and J. V. Ryzin, *Further remarks on the expectation of the reciprocal of a positive random variable*, Amer. Statist. 21 (1967), 33–34.
[39] P. K. Sen, *The impact of Wassily Hoeffding's research on nonparametrics*, in: N. I. Fisher and P. K. Sen (eds.), *The Collected Works of Wassily Hoeffding*, Springer, New York, 1994, 29–55.
[40] J. F. Tian and Z. Yang, *Asymptotic expansions of Gurland's ratio and sharp bounds for their remainders*, J. Math. Anal. Appl. 493 (2021), no. 2, art. 124545, 19 pp.
[41] C. C. Yeh, H. W. Yeh, and W. Chan, *Some equivalent forms of the arithmetic-geometric mean inequality in probability: A survey*, J. Inequal. Appl. 2008, art. 386715, 9 pp.

Netzahualcóyotl Castañeda-Leyva,
Silvia Rodríguez-Narciso,
Angélica Hernández-Quintero
Departamento de Estadística
Universidad Autónoma de Aguascalientes
Aguascalientes, Ags., México
E-mail: netza.castaneda@edu.uaa.mx
silvia.rodriguezn@edu.uaa.mx
angelica.hernandezq@edu.uaa.mx

Aroldo Pérez
División Académica de Ciencias Básicas
Universidad Juárez Autónoma de Tabasco
Cunduacán, Tab., México
E-mail: aroldopz2@gmail.com

Received 6.12.2022;
accepted 14.5.2023