PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 43, Fasc. 1 (2023), pp. 83–108
Published online 6.7.2023
doi:10.37190/0208-4147.00118

GROUND-STATE REPRESENTATION FOR THE FRACTIONAL LAPLACIAN ON THE HALF-LINE*

BY

TOMASZ JAKUBOWSKI (WROCŁAW) AND PAWEŁ MACIOCHA (WROCŁAW)

Abstract. We give a ground-state representation for the fractional Laplacian with Dirichlet condition on the half-line.

2020 Mathematics Subject Classification: Primary 60J35; Secondary 60G52, 60J45.

Key words and phrases: fractional Laplacian, Dirichlet heat kernel, Hardy identity.

1. INTRODUCTION

In [10], K. Bogdan et al. proposed a general method of constructing supermedian functions for semigroups. This approach was applied in [11, 25] to study singular Schrödinger perturbations of the fractional Laplacian. In this paper we apply this methodology to the Dirichlet kernel of the half-line $(0, \infty)$ for the fractional Laplacian and we obtain a wide spectrum of ground state representations of the corresponding quadratic form. In doing so we resolve major technical problems related to compensation of kernels and divergent integrals.

We write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We write $f(x) \approx g(x)$ if $f, g \geqslant 0$ and $c^{-1}g(x) \leqslant f(x) \leqslant cg(x)$ for some positive number $c > 0$ and all the arguments x.

1.1. Fractional Laplacian and α-stable Lévy process.

Let $\alpha \in (0, 2)$,

$$\mathcal{A}_\alpha = \frac{\alpha\Gamma(\alpha)\sin(\pi\alpha/2)}{\pi} \quad \text{and} \quad \nu(y) = \mathcal{A}_\alpha |y|^{-1-\alpha}.$$

For (smooth and compactly supported) $\phi \in C_c^\infty(\mathbb{R})$, the *fractional Laplacian* is

$$\Delta^{\alpha/2}\phi(x) = \lim_{\varepsilon \downarrow 0} \int_{B(0,\varepsilon)^c} (\phi(x+y) - \phi(x))\nu(y)\, dy, \qquad x \in \mathbb{R}.$$

* The paper was partially supported by the NCN grant 2015/18/E/ST1/00239.

Let p_t be the smooth real-valued function on $\mathbb{R}$ with Fourier transform

$$\int_{\mathbb{R}} p_t(x)e^{ix\xi}\,dx = e^{-t|\xi|^\alpha}, \quad t>0,\ \xi\in\mathbb{R}. \tag{1.1}$$

According to the Lévy–Khinchin formula, p_t is the density of the probabilistic convolution semigroup with Lévy measure $\nu(y)dy$; see e.g. [7]. Let

$$p(t,x,y) = p_t(y-x).$$

We consider the time-homogeneous transition probability

$$(t,x,A) \mapsto \int_A p(t,x,y)\,dy, \quad t>0,\ x\in\mathbb{R},\ A\subset\mathbb{R}.$$

By the Kolmogorov and Dynkin–Kinney theorems there is a stochastic process $(X_t,\mathbb{P}^x)$ with càdlàg paths and initial distribution $\mathbb{P}^x(X(0)=x)=1$. We denote by $\mathbb{P}^x$ and $\mathbb{E}^x$ the distribution and expectation for the process starting at x. We call X_t the *isotropic α-stable process* with index of stability $\alpha\in(0,2)$. In fact, X_t is a Lévy process with zero Gaussian part and drift, and with Lévy measure $\nu(y)dy$.

From (1.1), we have the scaling property

$$p_t(x) = t^{-1/\alpha}p_1(t^{-1/\alpha}x).$$

It is well known (see e.g. [7]) that

$$p_t(x) \approx t^{-1/\alpha} \wedge \frac{t}{|x|^{1+\alpha}}, \quad t>0,\ x\in\mathbb{R}.$$

Additionally, the function $p(t,x,y)$ satisfies the Chapman–Kolmogorov equation

$$p(t+s,x,y) = \int_{\mathbb{R}} p(t,x,z)p(s,z,y)\,dz, \quad s,t>0,\ x,y\in\mathbb{R}.$$

1.2. Killed process. Throughout the paper we let $D=(0,\infty)\subset\mathbb{R}$. We define the time of the first exit of the process X_t from D by

$$\tau_D = \inf\{t\geqslant 0 : X_t\in D^c\}.$$

The random variable τ_D is an almost surely finite Markov moment (see (3.2)). We denote by P_t^D the semigroup generated by the process X_t killed on exiting D. The semigroup is determined by the transition densities $p_D(t,x,y)$ given by the Hunt formula (see e.g. [16])

$$p_D(t,x,y) = p(t,x,y) - \mathbb{E}^x[p(t-\tau_D, X_{\tau_D}, y)\mathbb{1}_{\{\tau_D<t\}}], \quad t>0,\ x,y\in D. \tag{1.2}$$

It well known that the density $p_D(t,x,y)$ is symmetric, i.e. $p_D(t,x,y) = p_D(t,y,x)$, continuous in (t,x,y) for $t > 0$, $x, y \in D$, satisfies the Chapman–Kolmogorov equation

$$p_D(t+s,x,y) = \int_D p_D(t,x,z)p_D(s,z,y)\,dz,$$

and for any non-negative Borel function, we have

$$P_t^D f(x) := \int_D p_D(t,x,y)f(y)\,dy = \mathbb{E}^x(\mathbb{1}_{\{t<\tau_D\}}f(X_t)), \quad t > 0,\ x \in D. \tag{1.3}$$

We note that p_D admits the same scaling as p,

$$p_D(t,x,y) = t^{-1/\alpha}p_D(1,t^{-1/\alpha}y,t^{-1/\alpha}x). \tag{1.4}$$

It is known (see e.g. [12]) that

$$p_D(t,x,y) \approx p(t,x,y)\left(1 \wedge \frac{x^{\alpha/2}}{t^{1/2}}\right)\left(1 \wedge \frac{y^{\alpha/2}}{t^{1/2}}\right), \quad t > 0,\ x, y \in D, \tag{1.5}$$

while for $x \notin D$ or $y \notin D$, we have $p_D(t,x,y) = 0$ for $t > 0$. By (1.3) and integrating the Hunt formula (1.2), we get

$$\mathbb{P}^x(\tau_D > t) = \int_D p_D(t,x,y)\,dy. \tag{1.6}$$

1.3. Main results. Recall that $D = (0,\infty)$ and $\alpha \in (0,2)$. Consider $\beta \in (0,1)$, $\gamma \in (\beta + \alpha/2, 1 + \alpha/2)$ and define

$$\mathcal{C} = \int_0^\infty \int_D p_D(t,1,y)t^{(-\alpha/2-\beta+\gamma)/\alpha}y^{-\gamma}\,dy\,dt \tag{1.7}$$

and

$$f(t) = \begin{cases} \mathcal{C}^{-1}t^{(-\alpha/2-\beta+\gamma)/\alpha} & \text{for } t > 0, \\ 0 & \text{for } t \leqslant 0. \end{cases}$$

According to [10] we define

$$h_\beta(x) = \int_0^\infty \int_D p_D(t,x,y)f(t)y^{-\gamma}\,dy\,dt, \quad x \in D,$$

$$q_\beta(x) = \frac{1}{h_\beta(x)}\int_0^\infty \int_D p_D(t,x,y)f'(t)y^{-\gamma}\,dy\,dt, \quad x \in D.$$

By the scaling property (1.4), $h_\beta(x) = x^{\alpha/2-\beta}$ (see Lemma 3.4). It turns out that q_β does not depend on γ either. In the previous papers [11, 25] this construction was applied to the convolution semigroups with the Dirac measure δ_0 in place of $\mu(dy) = y^{-\gamma}\,dy$. In our case such a choice is impossible because $p_D(t,x,y) = 0$ for $y \leqslant 0$. Our approach shows how the construction introduced in [10] may be used for more general semigroups than convolution ones, provided one can find the proper measure μ. Our main results are stated in the following theorems.

THEOREM 1.1. *Let* $\beta \in (0,1)$. *Then*

$$q_\beta(x) = \kappa_\beta x^{-\alpha}, \quad x \in D,$$

where

$$\kappa_\beta = \frac{\Gamma(\beta+\alpha/2)\Gamma(1-\beta+\alpha/2)}{\Gamma(\beta)\Gamma(1-\beta)}. \tag{1.8}$$

The main difficulty here is to obtain the exact value of the constant κ_β. In contrast to the case of the free process considered in [10], generally we cannot calculate the constant $\mathcal{C}$ in (1.7) for arbitrary γ. One may try here the approach involving the Mellin transform of the supremum process and calculate $\mathcal{C}$ for $\gamma = 0$ (see Remark 3.1). However, it leads to very complicated formulas, so we choose another method to prove Theorem 1.1 by finding the value of $\mathcal{C}$ for $\gamma = \beta + \alpha/2$. This result is stated in the following theorem.

THEOREM 1.2. *Let* $\beta \in (0,1)$. *Then*

$$\int_0^\infty \int_D p_D(t,x,y) y^{-\beta-\alpha/2}\, dy\, dt = \kappa_\beta^{-1} x^{\alpha/2-\beta}, \quad x \in D, \tag{1.9}$$

where κ_β *is given by* (1.8).

According to (1.3), in probabilistic terms, the statement of Theorem 1.2 reads

$$\mathbb{E}^x\Big(\int_0^{\tau_D} X_t^{-\beta-\alpha/2} dt\Big) = \kappa_\beta^{-1} x^{\alpha/2-\beta}, \quad x \in D.$$

As an application of Theorem 1.1 we prove the Hardy identity for the Dirichlet form

$$\mathcal{E}_D(u,u) = \lim_{t\to 0+} \frac{1}{t}(u - P_t^D u, u), \quad u \in L^2(D).$$

The subject of Hardy identities and inequalities was initiated in 1920, when Hardy [22] discovered that

$$\int_0^\infty [u'(x)]^2\, dx \geqslant \frac{1}{4}\int_0^\infty \frac{u(x)^2}{x^2}\, dx \tag{1.10}$$

for absolutely continuous functions u such that $u(0) = 0$ and $u' \in L^2(0,\infty)$. Later, many generalizations of (1.10) were proven, where the left-hand side of (1.10) was replaced by various symmetric Dirichlet forms $\mathcal{E}$ in the sense of Fukushima, Oshima, and Takeda [20]. In particular, for all $d \geqslant 1$, $0 < \alpha < d \wedge 2$, $0 \leqslant \beta \leqslant d - \alpha$, and $u \in L^2(\mathbb{R}^d)$, the following Hardy-type identity holds (see [17, 10]):

$$\begin{aligned}\mathcal{E}(u,u) = \kappa_\beta^{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha}\, dx \\ + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\frac{u(x)}{|x|^{-\beta}} - \frac{u(y)}{|y|^{-\beta}}\right]^2 |x|^{-\beta}|y|^{-\beta}\nu(x,y)\, dy\, dx,\end{aligned} \tag{1.11}$$

where $\mathcal{E}(u,u) = \lim_{t\to 0+} \frac{1}{t}(u - P_t u, u)$, P_t is the stable semigroup on $\mathbb{R}^d$ and

$$\kappa_\beta^{\mathbb{R}^d} = \frac{2^\alpha \Gamma((\beta+\alpha)/2)\Gamma((d-\beta)/2)}{\Gamma(\beta/2)\Gamma((d-\alpha-\beta)/2)}, \qquad 0<\beta<d-\alpha. \tag{1.12}$$

The identity (1.11) is also called a *ground state representation*. It yields the Hardy-type inequality for $\mathcal{E}$ (see also [23, 4, 30])

$$\mathcal{E}(u,u) \geqslant \kappa_\beta \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha}\, dx. \tag{1.13}$$

We get the analogous result to (1.11) for the form $\mathcal{E}_D$.

THEOREM 1.3. *If* $u \in L^2(D)$, *then*

(1.14)
$$\mathcal{E}_D(u,u) = \kappa_\beta \int_D \frac{u(x)^2}{x^\alpha}\, dx + \frac{1}{2}\int_D\int_D \left(\frac{u(x)}{x^{\alpha/2-\beta}} - \frac{u(y)}{y^{\alpha/2-\beta}}\right)^2 x^{\alpha/2-\beta} y^{\alpha/2-\beta} \nu(x-y)\, dx\, dy.$$

As a corollary, in Proposition 4.1 we give a new proof of the Hardy inequality for $\mathcal{E}_D$ obtained in [9] (see also [26] for its generalization to Sobolev–Bregman forms). For $\beta = 1/2$, the identity (1.14) is a special case of [19, Theorem 1.2] with $p = 2$ and $N = 1$. We note that formula (1.14) may be derived from (1.11) with $d = 1$ by taking u with support in D; see Remark 4.1. However, the range of parameters in this case is limited to $\alpha \in (0,1)$ and $\beta \in (\alpha/2, 1-\alpha/2)$. Note that in our approach this case is the easiest one (see e.g. Fact 2.1) and the generalization to the whole range of parameters $\alpha \in (0,2)$ and $\beta \in (0,1)$ is non-trivial.

REMARK 1.1. By using the relationship between the fractional Laplacian with the Dirichlet condition and the regional fractional Laplacian (see [8, 9]) the ground state representation for the latter operator is an immediate corollary of Theorem 1.3. Namely, for $u \in C_c(D)$ we have

$$\mathcal{E}_R(u,u) = \left(\kappa_\beta - \frac{\mathcal{A}_\alpha}{\alpha}\right) \int_D \frac{u(x)^2}{x^\alpha}\, dx + \frac{1}{2}\int_D\int_D \left(\frac{u(x)}{x^{\alpha/2-\beta}} - \frac{u(y)}{y^{\alpha/2-\beta}}\right)^2 x^{\alpha/2-\beta} y^{\alpha/2-\beta} \nu(x-y)\, dx\, dy,$$

where $\mathcal{E}_R$ is the Dirichlet form related to the regional fractional Laplacian.

It is also worth mentioning the recent paper [14] by Bogdan and Merz, where the authors considered the fractional Laplacian with Hardy potential in angular momentum channels. They obtained a ground state representation for this operator with an explicit constant κ.

1.4. Further discussion. If we compare our result for $\alpha \in (0,2)$ with the case $\alpha = 2$ and $d = 1$, we may observe several similarities. First, note that for $\beta \in (0,1)$ and sufficiently regular u we have the following Hardy identity (see e.g. [29, Theorem 3] for the special case $\beta = 1/2$):

$$\int_0^\infty [u'(x)]^2\,dx = \beta(1-\beta)\int_0^\infty \frac{u(x)^2}{x^2}\,dx + \int_0^\infty [x^{1-\beta}(x^{\beta-1}u(x))']^2\,dx.$$

Hence, (1.14) may be considered as a fractional counterpart of the formula above. Additionally, if we put $\alpha = 2$ in (1.8), we get $\kappa_\beta = \beta(1-\beta)$. For both $\alpha \in (0,2)$ and $\alpha = 2$, κ_β attains its maximum for $\beta = 1/2$ (see Figure 1) and $\kappa_{1/2} = \Gamma((\alpha+1)/2)^2/\pi$ is the best constant in the Hardy inequality.

The subject of this paper is also strictly connected with Schrödinger perturbations of semigroups by Hardy potentials; see e.g. [1, 2, 11, 25, 5, 17, 15, 18]. In particular, in [11] the authors studied the Schrödinger perturbations of the fractional Laplacian in $\mathbb{R}^d$ by $\kappa|x|^{-\alpha}$ and obtained sharp estimates of the perturbed semigroup. It turns out that the critical value of κ for which the perturbed density is finite coincides with the best constant in the Hardy inequality (1.13).

We point out that Theorem 1.1 may be the starting point for a further study of the Schrödinger perturbation of p_D by the potential $\kappa x^{-\alpha}$. Here, we may expect that just as in [3, 11], the best constant $\kappa_{1/2}$ in the Hardy inequality (4.1) is also critical in the sense that for $\kappa > \kappa_{1/2}$ we will have instantaneous blow-up of the perturbed heat kernel $\tilde{p}$. To get this result, probably one has to get appropriate estimates of $\tilde{p}$ (see e.g. [11]), which we postpone to a forthcoming paper.

The last remark concerns the range of parameters α and β. We note that as in (1.11), the authors of [11] assumed that $\alpha < d$, which for $d = 1$ gives the restriction $\alpha < 1$. This condition was imposed to get the integrability of the potential $|x|^{-\alpha}$ at 0. However, in our case by considering the process killed on exiting the half-line, the additional decay at 0 of the kernel p_D permits studying the perturbations by $\kappa x^{-\alpha}$ for the whole range of $\alpha \in (0,2)$. Moreover, since $p_D(t,x,y) < p(t,x,y)$, potentially the bigger perturbation may also be considered for $\alpha < 1$. In Remark 3.2, we show that for $\beta \in (0,(1-\alpha)/2)$, $\kappa_\beta > \kappa_\beta^{\mathbb{R}}$, where $\kappa_\beta^{\mathbb{R}}$ is given by (1.12) (see Figure 2).

The paper is organized as follows. In Section 2 we calculate some auxiliary integrals involving gamma functions and give some definitions from potential theory. In Section 3 we prove Theorems 1.1 and 1.2. In Section 4 we apply Theorem 1.1 to study Hardy identities for the quadratic form connected with the semigroup P_t^D.

2. PRELIMINARIES

2.1. Gamma and Beta functions. For $x > 0$, we define the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

By using the property $x\Gamma(x) = \Gamma(x+1)$ we extend the definition of $\Gamma(x)$ to negative non-integer values of x. The beta function is defined by

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2.1}$$

for $x, y \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$. Recall that

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt, \qquad x, y > 0. \tag{2.2}$$

LEMMA 2.1. *Let $a \in (-1, 0)$ and $b > 0$. We have*

$$\int_0^1 s^{a-1}((1-s)^{b-1} - 1)\,ds = -\frac{1}{a} + B(a,b).$$

Proof. Note that $\lim_{s\to 0} s^{-1}((1-s)^{b-1}-1) = 1-b$. Hence, using integration by parts, (2.2) and (2.1), we get

$$\begin{aligned}
a\int_0^1 & s^{a-1}((1-s)^{b-1} - 1)\,ds \\
&= a\int_0^1 (s^{a-1}(1-s) + s^a)((1-s)^{b-1} - 1)\,ds \\
&= \int_0^1 as^{a-1}((1-s)^b - (1-s))\,ds + a\int_0^1 s^a((1-s)^{b-1} - 1)\,ds \\
&= \int_0^1 s^a(b(1-s)^{b-1} - 1)\,ds + a\int_0^1 s^a((1-s)^{b-1} - 1)\,ds \\
&= (a+b)B(a+1,b) - \frac{1}{a+1} - \frac{a}{a+1} = aB(a,b) - 1. \quad \blacksquare
\end{aligned}$$

For $\beta \in (0,1)$ and $\alpha \in (0,2)$ such that $\beta + \alpha/2 \neq 1$ and $\beta - \alpha/2 \neq 0$ we define

$$\widehat{C}_{\alpha,\beta} = B(1-\beta-\alpha/2, \alpha) + B(\alpha, \beta - \alpha/2), \tag{2.3}$$

$$\widetilde{C}_{\alpha,\beta} = B(1-\beta-\alpha/2, \beta-\alpha/2). \tag{2.4}$$

FACT 2.1. *For $\alpha \in (0,1)$ and $\beta \in (\alpha/2, 1-\alpha/2)$, we have*

$$\int_D |1-w|^{\alpha-1} w^{-\beta-\alpha/2}\,dw = \widehat{C}_{\alpha,\beta}, \tag{2.5}$$

$$\int_D (1+w)^{\alpha-1} w^{-\beta-\alpha/2}\,dw = \widetilde{C}_{\alpha,\beta}. \tag{2.6}$$

Proof. By (2.1) and (2.2), we have

$$\begin{aligned}\int_D |1-w|^{\alpha-1} w^{-\beta-\alpha/2}\,dw &= \int_0^1 (1-w)^{\alpha-1} w^{-\beta-\alpha/2}\,dw + \int_0^\infty w^{\alpha-1}(w+1)^{-\beta-\alpha/2}\,dw \\ &= B(1-\beta-\alpha/2,\alpha) + B(\alpha,\beta-\alpha/2).\end{aligned}$$

The second equality follows directly from (2.2). ∎

We will also need similar equalities for a wider range of parameters α and β. To secure convergence of integrals we compensate the expressions $|1-w|^{\alpha-1}$ and $(1+w)^{\alpha-1}$, which improves the decay near 0 and 1.

FACT 2.2. *In the cases of* $\alpha \in (0,1)$, $\beta \in (0,\alpha/2)$ *and* $\alpha \in (1,2)$, $\beta \in (0,1-\alpha/2)$, *we have*

$$\int_D (|1-w|^{\alpha-1} - w^{\alpha-1}) w^{-\beta-\alpha/2}\,dw = \widehat{C}_{\alpha,\beta}, \tag{2.7}$$

$$\int_D ((1+w)^{\alpha-1} - w^{\alpha-1}) w^{-\beta-\alpha/2}\,dw = \widetilde{C}_{\alpha,\beta}. \tag{2.8}$$

Proof. We have

$$\begin{aligned}\int_D (|1-w|^{\alpha-1} - w^{\alpha-1}) w^{-\beta-\alpha/2}\,dw &= \int_0^1 ((1-w)^{\alpha-1} - w^{\alpha-1}) w^{-\beta-\alpha/2}\,dw \\ &\quad + \int_0^\infty (w^{\alpha-1} - (w+1)^{\alpha-1})(w+1)^{-\beta-\alpha/2}\,dw.\end{aligned}$$

By (2.2), the first integral is equal to

$$\int_0^1 ((1-w)^{\alpha-1} - w^{\alpha-1}) w^{-\beta-\alpha/2}\,dw = B(1-\beta-\alpha/2,\alpha) - \frac{1}{\alpha/2-\beta}.$$

Let $w+1 = s^{-1}$. By the Fubini theorem and Lemma 2.1, we get

$$\begin{aligned}&\int_0^\infty (w^{\alpha-1} - (w+1)^{\alpha-1})(w+1)^{-\beta-\alpha/2}\,dw \\ &\quad = \int_0^1 \left(\left(\frac{s}{1-s}\right)^{1-\alpha} - s^{1-\alpha}\right) s^{\beta+\alpha/2-2}\,ds \\ &\quad = \int_0^1 s^{\beta-1-\alpha/2}((1-s)^{\alpha-1} - 1)\,ds = \frac{1}{\alpha/2-\beta} + B(\alpha,\beta-\alpha/2),\end{aligned}$$

which proves (2.7). Next, by Fubini's theorem and again by (2.2),

$$\begin{aligned}\int_0^\infty ((1+w)^{\alpha-1} - w^{\alpha-1})w^{-\beta-\alpha/2}\,dw &= \int_0^\infty (\alpha-1)\Big(\int_0^1 (w+t)^{\alpha-2}\,dt\Big)w^{-\beta-\alpha/2}\,dw \\ &= (\alpha-1)\int_0^1 t^{\alpha/2-1-\beta}\int_0^\infty (1+z)^{\alpha-2}z^{-\beta-\alpha/2}\,dz\,dt \\ &= (\alpha-1)B(1-\beta-\alpha/2, 1+\beta-\alpha/2)\int_0^1 t^{\alpha/2-1-\beta}\,dt \\ &= B(1-\beta-\alpha/2, \beta-\alpha/2),\end{aligned}$$

which proves (2.8). ■

FACT 2.3. *Let $\alpha \in (1,2)$ and $\beta \in (1-\alpha/2, 1)$. We have*

$$\int_D (|1-w|^{\alpha-1} - 1)w^{-\beta-\alpha/2}\,dw = \widehat{C}_{\alpha,\beta},$$
$$\int_D ((1+w)^{\alpha-1} - 1)w^{-\beta-\alpha/2}\,dw = \widetilde{C}_{\alpha,\beta}.$$

Proof. By Lemma 2.1,

$$\int_0^1 (|1-w|^{\alpha-1} - 1)w^{-\beta-\alpha/2}\,dw = B(1-\beta-\alpha/2, \alpha) + \frac{1}{\beta+\alpha/2-1}.$$

Next, by (2.2),

$$\begin{aligned}\int_1^\infty (|w-1|^{\alpha-1} - 1)w^{-\beta-\alpha/2}\,dw &= \int_0^\infty (w^{\alpha-1} - 1)(w+1)^{-\beta-\alpha/2}\,dw \\ &= \int_0^\infty w^{\alpha-1}(w+1)^{-\beta-\alpha/2}\,dw - \int_0^\infty (w+1)^{-\beta-\alpha/2}\,dw \\ &= B(\alpha, \beta-\alpha/2) - \frac{1}{\beta+\alpha/2-1}.\end{aligned}$$

Hence we get the first equality in the assertion. We also get the second equality by (2.2):

$$\begin{aligned}\int_D ((1+w)^{\alpha-1} - 1)w^{-\beta-\alpha/2}\,dw &= \int_D \Big((1-\alpha)\int_1^{1+w} y^{\alpha-2}\,dy\Big)w^{-\beta-\alpha/2}\,dw \\ &= (1-\alpha)\int_1^\infty \int_{y-1}^\infty w^{-\beta-\alpha/2}y^{\alpha-2}\,dw\,dy\end{aligned}$$

$$= (1-\alpha)\int_1^\infty \frac{1}{1-\beta-\alpha/2}(y-1)^{1-\beta-\alpha/2}y^{\alpha-2}\,dy$$
$$= \frac{1-\alpha}{1-\beta-\alpha/2}\int_0^\infty y^{1-\beta-\alpha/2}(y+1)^{\alpha-2}\,dy$$
$$= \frac{1-\alpha}{1-\beta-\alpha/2}B(2-\beta-\alpha/2,\beta-\alpha/2). \quad \blacksquare$$

The last fact concerns the case $\alpha = 1$.

FACT 2.4. *Let* $\beta \in (0,1/2)$. *We have*

$$\int_0^\infty y^{-\beta-1/2}\ln(|1-y|/y)\,dy = \frac{\pi\sin(\pi\beta)}{(1/2-\beta)\cos(\pi\beta)},$$
$$\int_0^\infty y^{-\beta-1/2}\ln(|1+y|/y)\,dy = \frac{\pi}{(1/2-\beta)\cos(\pi\beta)}.$$

Proof. Substituting $x = 1/y$ and applying [21, Equation 4.293.7], we obtain

$$\int_0^\infty y^{-\beta-1/2}\ln(|1-y|/y)\,dy = \int_0^\infty x^{(\beta-1/2)-1}\ln(|1-x|)\,dx = \frac{\pi\sin(\pi\beta)}{(1/2-\beta)\cos(\pi\beta)}.$$

By making the same substitution and by [21, Equation 4.293.10], we get

$$\int_0^\infty y^{-\beta-1/2}\ln(|1+y|/y)\,dy = \int_0^\infty x^{(\beta-1/2)-1}\ln(|1+x|)\,dx = \frac{\pi}{(1/2-\beta)\cos(\pi\beta)}. \quad \blacksquare$$

2.2. Green function and Poisson kernel. We recall some facts from potential theory of stable processes; see e.g. [6]. For $\alpha < 1$ the process X_t is transient and its potential kernel is given by

$$K_\alpha(x) = \int_0^\infty p_t(x)\,dt.$$

For $\alpha \geqslant 1$, X_t is recurrent and we consider the compensated kernels

$$K_\alpha(x) = \int_0^\infty (p(t,x) - p(t,x_0))\,dt,$$

where $x_0 = 0$ for $\alpha > 1$ and $x_0 = 1$ for $\alpha = 1$. It turns out that K_α is radial and

$$K_\alpha(x) = \begin{cases} C_\alpha |x|^{\alpha-1} & \text{for } \alpha \neq 1, \\ -\frac{1}{\pi}\ln(|x|) & \text{for } \alpha = 1, \end{cases}$$

where

$$C_\alpha = \frac{1}{2\Gamma(\alpha)\cos(\pi\alpha/2)}. \tag{2.9}$$

We note that $C_{-\alpha} = \mathcal{A}_\alpha$. Recall that $D = (0, \infty)$.

DEFINITION 2.1. We define the *Green function* of the set D by

$$G_D(x, y) = \int_0^\infty p_D(t, x, y)\, dt, \qquad x, y \in D. \tag{2.10}$$

$G_D(x, y)$ is symmetric, continuous on $D \times D$ with values in $[0, \infty]$. If $x \in D^c$ or $y \in D^c$, then $G_D(x, y) = 0$. Moreover for $x \neq y \in D$, $G_D(x, y) < \infty$ (see [12]). By integrating the Hunt formula we obtain

$$\begin{aligned} G_D(x, y) &= K_\alpha(x - y) - \mathbb{E}^x K_\alpha(X_{\tau_D} - y) \\ &= K_\alpha(x - y) - \mathbb{E}^y K_\alpha(X_{\tau_D} - x), \end{aligned} \tag{2.11}$$

where the last equality follows from the symmetry of G_D.

Recall that $\nu(x) = \mathcal{A}_\alpha |x|^{-1-\alpha}$ is the density of the Lévy measure. If A is any Borel subset of $(\overline{D})^c$ and $0 \leqslant a < b \leqslant \infty$, then we have the following *Ikeda–Watanabe formula* (see [24]):

$$\mathbb{P}^x(\tau_D \in (a, b), X_{\tau_D} \in A) = \int_A \int_D \int_a^b p_D(t, x, y)\nu(z - y)\, dt\, dy\, dz,$$

for the joint distribution of the random vector (τ_D, X_{τ_D}). Taking $a = 0$ and $b = \infty$, we get

$$\mathbb{P}^x(X_{\tau_D} \in A) = \int_A \int_D G_D(x, y)\nu(z - y)\, dy\, dz. \tag{2.12}$$

Hence, the distribution of X_{τ_D} is absolutely continuous with respect to the Lebesgue measure. Equation (2.12) is also called the Ikeda–Watanabe formula. The density of the measure $\mu(A) = \mathbb{P}^x(X_{\tau_D} \in A)$ with respect to the Lebesgue measure is called the *Poisson kernel* and we denote it by $P_D(x, y)$. For $y \in D$, $P_D(x, y) = 0$. Hence,

$$P_D(x, z) = \int_D G_D(x, y)\nu(z - y)\, dy, \qquad x \in D,\ z \in (\overline{D})^c.$$

LEMMA 2.2. *For* $\alpha \in (0, 2)$ *and* $\beta \in (0, 1)$,

$$\int_D \nu(1 + z) z^{-\beta + \alpha/2}\, dz = \overline{C}_{\alpha,\beta},$$

where

$$\overline{C}_{\alpha,\beta} = \mathcal{A}_\alpha B(1 - \beta + \alpha/2, \alpha/2 + \beta).$$

Proof. We have

$$\int_D \nu(1+z)z^{-\beta+\alpha/2}\,dz = \mathcal{A}_\alpha \int_0^\infty (1+z)^{-1-\alpha} z^{-\beta+\alpha/2}\,dz$$
$$= \mathcal{A}_\alpha B(1-\beta+\alpha/2, \alpha/2+\beta). \quad \blacksquare$$

3. PROOFS OF THEOREMS 1.1 AND 1.2

As mentioned in the Introduction, we cannot directly follow the ideas from [10] to calculate the constant κ_β. Therefore, we first prove the auxiliary result stated in Theorem 1.2.

3.1. Proof of Theorem 1.2. First, we will present the general idea of the proof. By the definition of the Green function (2.10), we need to prove

$$\int_D G_D(x,y)y^{-\beta-\alpha/2}\,dy = \kappa_\beta^{-1} x^{\alpha/2-\beta}. \tag{3.1}$$

One may try to use here the explicit formula for $G_D(x,y)$ (see e.g. [28]), but the resulting double integral seems difficult to calculate. Our method below can also be used when the formula for the Green function is unknown. In equation (3.1) for $G_D(x,y)$ we substitute (2.11) to obtain

$$\int_D G_D(x,y)y^{-\alpha/2-\beta}\,dy$$
$$= \int_D K_\alpha(x-y)y^{-\alpha/2-\beta}\,dy + \int_D \mathbb{E}^x[K_\alpha(X_{\tau_D}-y)]y^{-\alpha/2-\beta}\,dy$$
$$= C_1 x^{\alpha/2-\beta} - C_2 \int_D G_D(x,y)y^{-\beta-\alpha/2}\,dy,$$

where the constants C_1, C_2 can be computed explicitly. Then $\kappa_\beta = (1+C_2)/C_1$.

However, following this approach we encounter several problems with the convergence of the relevant integrals. Therefore, depending on the range of α and β, we use appropriate compensation of the potential K_α. Consequently, in the proof of Theorem 1.2 we will consider separately the cases $\alpha < 1$, $\alpha > 1$, $\alpha = 1$ and different ranges for β.

Note that to prove Theorem 1.2, by the scaling property of p_D it suffices to show (1.9) only for $x = 1$. However, before we pass to the proof of Theorem 1.2, we prove a few auxiliary lemmas.

LEMMA 3.1. *Let $\alpha \in (0,2)$ and $-\alpha < \gamma < 1 + \alpha/2$. Then*

$$\int_D p_D(t,1,y)y^{-\gamma}\,dy \approx 1 \qquad \text{for } t < 2^{-\alpha},$$
$$\int_D p_D(t,1,y)y^{-\gamma}\,dy \approx t^{-1/2-\gamma/\alpha} \qquad \text{for } t > 2.$$

Proof. By (1.5),

$$p_D(t,1,y)y^{-\gamma} \approx \left(1 \wedge \frac{1}{t^{1/2}}\right)\left(1 \wedge \frac{y^{\alpha/2}}{t^{1/2}}\right)\left(t^{-1/\alpha} \wedge \frac{t}{|y-1|^{1+\alpha}}\right)y^{-\gamma}.$$

For $t < 2^{-\alpha}$ we have

$$\begin{aligned}\int_D p_D(t,1,y)y^{-\gamma}\,dy &\leqslant c\int_0^{1/2} y^{\alpha/2-\gamma}t^{1/2}\,dy + c\int_{1/2}^{3/2} p(t,x,y)\,dy + c\int_{3/2}^{\infty} ty^{-1-\alpha-\gamma}\,dy\\ &\leqslant c_1,\\ \int_D p_D(t,1,y)y^{-\gamma}\,dy &\geqslant c_2 \int_{1-t^{1/\alpha}}^{1+t^{1/\alpha}} t^{-1/\alpha}\,dy = 2c_2.\end{aligned}$$

For $t > 2$ we have

$$\begin{aligned}\int_D p_D(t,1,y)y^{-\gamma}\,dy &\approx \int_0^{t^{1/\alpha}} t^{-1-1/\alpha}y^{\alpha/2-\gamma}\,dy + \int_{t^{1/\alpha}}^{\infty} t^{-1/2}\frac{t}{y^{1+\alpha}}y^{-\gamma}\,dy\\ &= \frac{t^{-\gamma/\alpha-1/2}}{\alpha/2-\gamma+1} + \frac{t^{-\gamma/\alpha-1/2}}{\gamma+\alpha} \approx t^{-1/2-\gamma/\alpha}. \quad \blacksquare\end{aligned}$$

COROLLARY 3.1. *For $\alpha \in (0,2)$ and $\beta \in (0,1)$,*

$$\int_0^{\infty}\int_D p_D(t,1,y)y^{-\alpha/2-\beta}\,dy\,dt < \infty.$$

Proof. By Lemma 3.1 we have

$$\begin{aligned}\int_0^{\infty}\int_D p_D(t,1,y)y^{-\alpha/2-\beta}\,dy\,dt &\leqslant c_1\int_0^{2^{-\alpha}} dt + \int_{2^{-\alpha}}^{2}\int_0^{\infty} p_D(t,x,y)y^{-\alpha/2-\beta}\,dy\,dt\\ &\quad + c_2\int_2^{\infty} t^{-1-\beta/\alpha}\,dt\\ &\leqslant c + c_3\int_0^2 y^{-\beta}\,dy + c_4\int_2^{\infty}\frac{y^{-\alpha/2-\beta}}{|y-1|^{1+\alpha}}\,dy < \infty. \quad \blacksquare\end{aligned}$$

We also note that by (1.6) and Lemma 3.1 with $\gamma = 0$ we get

$$\lim_{t\to\infty} \mathbb{P}^x(\tau_D > t) = 0, \qquad x > 0. \tag{3.2}$$

The integral in Corollary 3.1 is symmetric with respect to β, as shown by the following lemma.

LEMMA 3.2. *For $x > 0$, $\alpha \in (0,2)$ and $\beta \in (0,1)$ we have*

(3.3)
$$\int_0^\infty \int_D p_D(t,x,y) y^{-\beta-\alpha/2}\,dy\,dt = x^{\alpha/2-\beta} \int_0^\infty \int_D p_D(t,1,y) y^{-(1-\beta)-\alpha/2}\,dy\,dt.$$

Proof. Since $1-\beta \in (0,1)$, by Corollary 3.1 the integrals in (3.3) are convergent. Using the scaling property of $p_D(t,x,y)$ and letting $s = ty^{-\alpha}$, and then putting $z = x/y$, we have

$$\begin{aligned}\int_0^\infty \int_D p_D(t,x,y) y^{-\beta-\alpha/2}\,dy\,dt &= \int_0^\infty \int_D y^{-1} p_D(ty^{-\alpha}, x/y, 1) y^{-\beta-\alpha/2}\,dy\,dt \\ &= \int_0^\infty \int_D p_D(s, x/y, 1) y^{-\beta+\alpha/2-1}\,dy\,ds \\ &= x^{\alpha/2-\beta} \int_0^\infty \int_D p_D(s,1,z) z^{-(1-\beta)-\alpha/2}\,dz\,ds. \quad \blacksquare\end{aligned}$$

First, we consider the case $\beta = \alpha/2$.

LEMMA 3.3. *For $\alpha \in (0,2)$,*

$$\int_0^\infty \int_D p_D(t,x,y) y^{-\alpha}\,dy\,dt = \frac{\pi}{\Gamma(\alpha)\sin(\pi\alpha/2)}.$$

Proof. From the Ikeda–Watanabe formula, we have

$$\begin{aligned}\mathbb{P}^x(\tau_D < t) &= \mathcal{A}_\alpha \int_0^t \int_D \int_{D^c} p_D(t,x,y)|y-z|^{-1-\alpha}\,dz\,dy\,ds \\ &= \frac{\mathcal{A}_\alpha}{\alpha} \int_0^t \int_D p_D(t,x,y) y^{-\alpha}\,dy\,ds.\end{aligned}$$

By (3.2), $\lim_{t\to\infty} \mathbb{P}^x(\tau_D < t) = 1$. Hence, letting $t \to \infty$, we get

$$\int_0^\infty \int_D p_D(t,x,y) y^{-\alpha}\,dy\,ds = \frac{\pi}{\Gamma(\alpha)\sin(\pi\alpha/2)}. \quad \blacksquare$$

Now, we will prove

PROPOSITION 3.1. *For $\alpha \in (0,1) \cup (1,2)$ and $\beta \in (0,1/2]$, we have*

$$\int_D G_D(x,y) y^{-\beta-\alpha/2}\,dy = \frac{C_\alpha \widehat{C}_{\alpha,\beta}}{1 + C_\alpha \widetilde{C}_{\alpha,\beta} \overline{C}_{\alpha,\beta}} x^{\alpha/2-\beta}, \quad x \in D, \tag{3.4}$$

where C_α is the constant from (2.9), $\widehat{C}_{\alpha,\beta}$, $\widetilde{C}_{\alpha,\beta}$ are given by (2.3), (2.4) respectively, and $\overline{C}_{\alpha,\beta}$ is the constant from Lemma 2.2.

Proof. Let $\beta \in (0, 1-\alpha/2) \setminus \{\alpha/2\}$. For $z \in \mathbb{R}$ and $y \in D$, we let

$$U(z,y) = \begin{cases} C_\alpha |y-z|^{\alpha-1} & \text{if } \alpha < 1, \beta \in (\alpha/2, 1-\alpha/2), \\ C_\alpha(|y-z|^{\alpha-1} - y^{\alpha-1}) & \text{if } \alpha < 1, \beta \in (0, \alpha/2) \text{ or} \\ & \quad \alpha > 1, \beta \in (0, 1-\alpha/2). \end{cases}$$

We will use Facts 2.1 and 2.2 depending on the range of α and β. By (2.11) and (2.5) or (2.7), we have

$$\int_D G_D(x,y) y^{-\beta-\alpha/2}\, dy = C_\alpha \Big(\widehat{C}_{\alpha,\beta} x^{\alpha/2-\beta} - \int_D \mathbb{E}^x[U(X_{\tau_D}, y)] y^{-\beta-\alpha/2}\, dy \Big).$$

Next, by the Ikeda–Watanabe formula, (2.2) and (2.6) or (2.8),

$$\begin{aligned} &\int_D \mathbb{E}^x[U(X_{\tau_D}, y)] y^{-\beta-\alpha/2}\, dy \\ &\qquad = \int_{D^c} \int_D G_D(x,w)\nu(w-z) \Big[\int_D U(z,y) y^{-\beta-\alpha/2}\, dy \Big]\, dw\, dz \\ &\qquad = \int_D \int_D G_D(x,w)\nu(w+z) \Big[\int_D U(-z,y) y^{-\beta-\alpha/2}\, dy \Big]\, dw\, dz \\ &\qquad = \widetilde{C}_{\alpha,\beta} \int_D \int_D G_D(x,w)\nu(w+z) z^{-\beta+\alpha/2}\, dw\, dz, \end{aligned}$$

where in the last equality we substituted $y = z\xi$. By Lemma 2.2,

$$\int_D \nu(w+z) z^{-\beta+\alpha/2}\, dz = w^{-\beta-\alpha/2} \int_D \nu(1+y) y^{-\beta+\alpha/2}\, dy = \overline{C}_{\alpha,\beta} w^{-\beta-\alpha/2}.$$

Hence, we obtain

$$\begin{aligned} &\int_D G_D(x,y) y^{-\beta-\alpha/2}\, dy \\ &\qquad = C_\alpha \widehat{C}_{\alpha,\beta} x^{\alpha/2-\beta} - C_\alpha \widetilde{C}_{\alpha,\beta} \overline{C}_{\alpha,\beta} \int_D G_D(x,w) w^{-\beta-\alpha/2}\, dw, \end{aligned}$$

which gives (3.4).

Next, consider $\alpha > 1$ and $\beta \in (1-\alpha/2, 1/2]$. Here, due to the lack of convergence of certain integrals we need to be more subtle. By scaling and symmetry of the Green function, we have

$$\begin{aligned} G_D(x,y) &= x^{\alpha-1} C_\alpha \big(|1-y/x|^{\alpha-1} - \mathbb{E}^{y/x} |X_{\tau_D} - 1|^{\alpha-1} \big) \\ &= x^{\alpha-1} C_\alpha \big((|1-y/x|^{\alpha-1} - 1) - (\mathbb{E}^{y/x} |X_{\tau_D} - 1|^{\alpha-1} - 1) \big). \end{aligned}$$

Hence, by Fact 2.3,

$$\begin{aligned}
&\int_D G_D(x,y)y^{-\beta-\alpha/2}\,dy\\
&\qquad = C_\alpha x^{\alpha-1}\int_D \big((|1-y/x|^{\alpha-1}-1)-(\mathbb{E}^{y/x}|X_{\tau_D}-1|^{\alpha-1}-1)\big)y^{-\beta-\alpha/2}\,dy\\
&\qquad = C_\alpha x^{\alpha/2-\beta}\int_D \big((|1-w|^{\alpha-1}-1)-(\mathbb{E}^{w}|X_{\tau_D}-1|^{\alpha-1}-1)\big)w^{-\beta-\alpha/2}\,dw\\
&\qquad = C_\alpha x^{\alpha/2-\beta}\Big(\widehat{C}_{\alpha,\beta}-\int_D (\mathbb{E}^{w}|X_{\tau_D}-1|^{\alpha-1}-1)w^{-\beta-\alpha/2}\,dw\Big) = \mathcal{S}x^{\alpha/2-\beta},
\end{aligned}$$

where

$$\mathcal{S} = C_\alpha\Big(\widehat{C}_{\alpha,\beta}-\int_D (\mathbb{E}^{w}|X_{\tau_D}-1|^{\alpha-1}-1)w^{-\beta-\alpha/2}\,dw\Big).$$

Our aim is to calculate $\mathcal{S}$. By the Ikeda–Watanabe formula, Fubini's theorem, symmetry of the Green function, Fact 2.3 and Lemma 2.2, we get

$$\begin{aligned}
&\int_D (\mathbb{E}^{w}|X_{\tau_D}-1|^{\alpha-1}-1)w^{-\beta-\alpha/2}\,dw\\
&\qquad = \int_{D^c}\int_D\Big[\int_D G_D(w,y)w^{-\beta-\alpha/2}\,dw\Big]\nu(y-z)[|1-z|^{\alpha-1}-1]\,dy\,dz\\
&\qquad = \int_{D^c}\int_D \mathcal{S}y^{\alpha/2-\beta}\nu(y-z)[|1-z|^{\alpha-1}-1]\,dy\,dz\\
&\qquad = \mathcal{S}\mathcal{A}_\alpha\int_D\int_D y^{\alpha/2-\beta}|y+w|^{-1-\alpha}[(1+w)^{\alpha-1}-1]\,dy\,dw\\
&\qquad = \mathcal{S}\mathcal{A}_\alpha\int_D\int_D y^{\alpha/2-\beta}(1+y)^{-1-\alpha}[(1+w)^{\alpha-1}-1]w^{-\beta-\alpha/2}\,dy\,dw\\
&\qquad = \mathcal{S}\mathcal{A}_\alpha\widetilde{C}_{\alpha,\beta}B(1-\beta+\alpha/2,\alpha/2+\beta) = \mathcal{S}\widetilde{C}_{\alpha,\beta}\overline{C}_{\alpha,\beta}.
\end{aligned}$$

Hence, $\mathcal{S} = C_\alpha\widehat{C}_{\alpha,\beta} - \mathcal{S}C_\alpha\widetilde{C}_{\alpha,\beta}\overline{C}_{\alpha,\beta}$, which ends the proof. ∎

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. First note that for $\beta=\alpha/2$ the assertion follows by Lemma 3.3. Below we will use the identities

$$\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}, \tag{3.5}$$

$$\Gamma(1-x)\Gamma(1+x) = \frac{\pi x}{\sin(\pi x)}. \tag{3.6}$$

We will separately consider the cases $\alpha\neq 1$ and $\alpha=1$.

Case $\alpha \neq 1$. By Proposition 3.1 and Lemma 3.2 we only need to prove that for $\beta \in (0, 1/2] \setminus \{\alpha/2\}$,

$$\frac{C_\alpha \widehat{C}_{\alpha,\beta}}{1 + C_\alpha \widetilde{C}_{\alpha,\beta} \overline{C}_{\alpha,\beta}} = \frac{\pi}{\Gamma(\beta + \alpha/2)\Gamma(1 - \beta + \alpha/2)\sin(\pi\beta)}.$$

Note that

$$C_\alpha \widehat{C}_{\alpha,\beta} = \frac{1}{2\Gamma(\alpha)\cos(\pi\alpha/2)} \left(\frac{\Gamma(\alpha)\Gamma(1 - \beta - \alpha/2)}{\Gamma(1 - \beta + \alpha/2)} + \frac{\Gamma(\alpha)\Gamma(\beta - \alpha/2)}{\Gamma(\beta + \alpha/2)} \right).$$

Now, applying (3.5) for $x = \beta \pm \alpha/2$ we get

$$C_\alpha \widehat{C}_{\alpha,\beta} = \frac{\Gamma(\beta - \alpha/2)}{\Gamma(\beta + \alpha/2)} \frac{\sin(\pi\beta)}{\sin(\pi(\beta + \alpha/2))}.$$

By (3.5) and (3.6), we have

$$\begin{aligned} C_\alpha \widetilde{C}_{\alpha,\beta} \overline{C}_{\alpha,\beta} &= \frac{1}{2\Gamma(\alpha)\cos(\pi\alpha/2)} B(1 - \beta - \alpha/2, \beta - \alpha/2) \\ &\quad \times \frac{\alpha\Gamma(\alpha)\sin(\pi\alpha/2)}{\pi} B(1 - \beta + \alpha/2, \beta + \alpha/2) \\ &= \frac{\tan(\pi\alpha/2)}{2} \frac{\Gamma(1 - \beta - \alpha/2)\Gamma(\beta + \alpha/2)}{\pi} \\ &\quad \times \frac{\Gamma(1 - \beta + \alpha/2)\Gamma(\beta - \alpha/2)}{\pi} \frac{\alpha\pi}{\Gamma(1 - \alpha)\Gamma(1 + \alpha)} \\ &= \frac{\tan(\pi\alpha/2)\sin(\pi\alpha)}{2\sin(\pi(\beta + \alpha/2))\sin(\pi(\beta - \alpha/2))} = \frac{1 - \cos(\pi\alpha)}{\cos(\pi\alpha) - \cos(2\pi\beta)}. \end{aligned}$$

Therefore,

$$1 + C_\alpha \widetilde{C}_{\alpha,\beta} \overline{C}_{\alpha,\beta} = \frac{1 - \cos(2\pi\beta)}{\cos(\pi\alpha) - \cos(2\pi\beta)} = \frac{\sin^2(\pi\beta)}{\sin(\pi(\beta + \alpha/2))\sin(\pi(\beta - \alpha/2))}.$$

Hence, from the above transformations

$$\frac{C_\alpha \widehat{C}_{\alpha,\beta}}{1 + C_\alpha \widetilde{C}_{\alpha,\beta} \overline{C}_{\alpha,\beta}} = \frac{\pi}{\Gamma(\beta + \alpha/2)\Gamma(1 - \beta + \alpha/2)\sin(\pi\beta)}.$$

Case $\alpha = 1$. We only need to consider $\beta \in (0, 1/2)$. By (2.11), we get

$$G_D(x, y) = \frac{1}{\pi} \mathbb{E}^x [\ln(|X_{\tau_D} - y|/y)] - \frac{1}{\pi} \ln(|x - y|/y).$$

We integrate both sides with respect to $y^{-\beta-1/2}\,dy$. By Fact 2.4, we obtain

$$\frac{1}{\pi}\int_D \ln(|x-y|/y)y^{-\beta-1/2}\,dy = \frac{x^{1/2-\beta}}{\pi}\int_D \ln(|1-y|/y)y^{-\beta-1/2}\,dy$$
$$= \frac{\sin(\pi\beta)}{(1/2-\beta)\cos(\pi\beta)}x^{1/2-\beta}.$$

By the Ikeda–Watanabe formula and Fact 2.4,

$$\begin{aligned}
&\int_D \mathbb{E}^x[\ln|X_{\tau_D}-y|/y]y^{-\beta-1/2}\,dy\\
&\quad= \int_D\int_{D^c} G_D(x,w)\nu(w-z)\Big[\int_D \ln(|y-z|/y)y^{-\beta-1/2}\,dy\Big]\,dz\,dw\\
&\quad= \int_D\int_D G_D(x,w)\nu(w+z)z^{-\beta+1/2}\Big[\int_D \ln(1+u)u^{(\beta-1/2)-1}\,du\Big]dz\,dw\\
&\quad= \frac{\pi}{(1/2-\beta)\cos(\pi\beta)}\int_D\int_D G_D(x,w)\nu(w+z)z^{-\beta+1/2}\,dz\,dw\\
&\quad= \frac{\pi}{(1/2-\beta)\cos(\pi\beta)}\overline{C}_{1,\beta}\int_D G_D(x,w)w^{-\beta-1/2}\,dw.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\int_D G_D(x,y)y^{-\beta-1/2}\,dy &= \frac{\sin(\pi\beta)x^{1/2-\beta}}{(1/2-\beta)\cos(\pi\beta)}\left(-1+\frac{B(3/2-\beta,1/2+\beta)}{\pi(1/2-\beta)\cos(\pi\beta)}\right)^{-1}\\
&= \frac{\sin(\pi\beta)x^{1/2-\beta}}{(1/2-\beta)\cos(\pi\beta)}\left(-1+\frac{1}{\cos^2(\pi\beta)}\right)^{-1}\\
&= \frac{\sin(\pi\beta)x^{1/2-\beta}}{(1/2-\beta)\cos(\pi\beta)}\,\frac{\cos^2(\pi\beta)}{\sin^2(\pi\beta)}\\
&= \frac{\pi x^{1/2-\beta}}{\Gamma(\beta+1/2)\Gamma(1-\beta+1/2)\sin(\pi\beta)}. \quad \blacksquare
\end{aligned}$$

3.2. Proof of Theorem 1.1. Recall that

$$\begin{aligned}
f(t) &= \mathcal{C}^{-1}t^{(-\alpha/2-\beta+\gamma)/\alpha}\mathbb{1}_{(0,\infty)}(t),\\
h_\beta(x) &= \int_0^\infty\int_D p_D(t,x,y)f(t)y^{-\gamma}\,dy\,dt, \qquad x\in D,\\
q_\beta(x) &= \frac{1}{h_\beta(x)}\int_0^\infty\int_D p_D(t,x,y)f'(t)y^{-\gamma}\,dy\,dt, \qquad x\in D.
\end{aligned}$$

where the constant $\mathcal{C}$ is given by (1.7). First, we prove

LEMMA 3.4. *For* $\alpha \in (0,2)$, $\beta \in (0,1)$ *we have*

$$h_\beta(x) = x^{\alpha/2-\beta}, \qquad x \in D.$$

Proof. We use a similar argument to that in Lemma 3.2. By Lemma 3.1 the integral

$$\int_0^\infty \int_D p_D(t,x,y) f(t) y^{-\gamma}\, dy\, dt$$

is finite. By (1.4) and (1.7),

$$\begin{aligned}\int_0^\infty \int_D p_D(t,x,y) f(t) y^{-\gamma}\, dy\, dt &= \int_0^\infty \int_D x^{-1} p_D(tx^{-\alpha},1,y/x) f(t) y^{-\gamma}\, dy\, dt \\ &= \int_0^\infty \int_D x^{\alpha-1} p_D(s,1,y/x) x^{-\alpha/2-\beta+\gamma} f(s) y^{-\gamma}\, dy\, ds \\ &= \int_0^\infty \int_D x^{\alpha/2-1-\beta+\gamma} p_D(s,1,z) f(s) x^{-\gamma+1} z^{-\gamma}\, dz\, ds = x^{\alpha/2-\beta}. \quad \blacksquare\end{aligned}$$

Proof of Theorem 1.1. By Lemma 3.1 the integral

$$\int_0^\infty \int_D p_D(t,x,y) f'(t) y^{-\gamma}\, dy\, dt$$

is finite. By (1.4) and using a similar argument to that in Lemma 3.4 we get

$$\begin{aligned}h_\beta(x) q_\beta(x) &= x^{-\alpha/2-\beta} \int_0^\infty \int_D p_D(s,1,z) f'(s) z^{-\gamma}\, dz\, ds \\ &= x^{-\alpha/2-\beta} q_\beta(1).\end{aligned} \tag{3.7}$$

It remains to show that $q_\beta(1) = \kappa_\beta$. By Lemma 3.4, (3.7) and the Fubini theorem, we have

$$\begin{aligned}1 = h_\beta(1) &= \int_0^\infty \int_D p_D(t,1,y) f(t) y^{-\gamma}\, dy\, dt = \int_0^\infty \int_D \int_0^t p_D(t,1,y) f'(s) y^{-\gamma}\, ds\, dy\, dt \\ &= \int_0^\infty \int_D \int_0^\infty p_D(t+s,1,y) f'(s) y^{-\gamma}\, dt\, dy\, ds \\ &= \int_0^\infty \int_D \int_0^\infty \int_D p_D(t,1,w) p_D(s,w,y) f'(s) y^{-\gamma}\, dy\, ds\, dw\, dt \\ &= \int_0^\infty \int_D p_D(t,1,w) h_\beta(w) q_\beta(w)\, dw\, dt \\ &= q_\beta(1) \int_0^\infty \int_D p_D(t,1,w) w^{-\alpha/2-\beta}\, dw\, dt.\end{aligned}$$

Hence, by Theorem 1.2 we get $q_\beta(1) = \kappa_\beta$. $\blacksquare$

By Lemma 3.2, $\kappa_\beta = \kappa_{1-\beta}$. Now, we will complete this section by showing that κ_β is increasing for $\beta \in (0, 1/2)$ and decreasing when $\beta \in (1/2, 1)$ and reaches its maximum at $\beta = 1/2$ (see Figure 1). We will follow the idea of the proof from [10]. We have

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{k+1}\right),$$

where γ is the Euler–Mascheroni constant. Taking the derivative of $\ln(\kappa_\beta)$ we get

$$\begin{aligned}\frac{\kappa'_\beta}{\kappa_\beta} &= -\sum_{k=0}^{\infty}\left(\frac{1}{\beta+\alpha/2+k} - \frac{1}{1-\beta+\alpha/2+k} - \frac{1}{\beta+k} + \frac{1}{1-\beta+k}\right)\\ &= \sum_{k=0}^{\infty}\left(\frac{2\beta-1}{(\beta+\alpha/2+k)(1-\beta+\alpha/2+k)} - \frac{2\beta-1}{(\beta+k)(1-\beta+k)}\right)\\ &= \sum_{k=0}^{\infty}\frac{(1-2\beta)(\alpha/2)(2k+1+\alpha/2)}{(\beta+\alpha/2+k)(1-\beta+\alpha/2+k)(\beta+k)(1-\beta+k)}.\end{aligned}$$

Hence $\kappa'_\beta > 0$ for $\beta \in (0, 1/2)$, $\kappa'_\beta = 0$ when $\beta = 1/2$, and $\kappa'_\beta < 0$ when $\beta \in (1/2, 1)$.

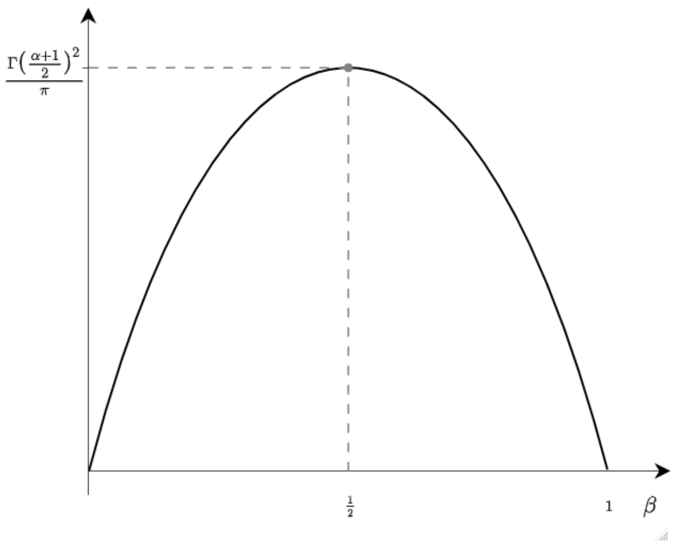

FIGURE 1. The function $\beta \mapsto \kappa_\beta$

REMARK 3.1. One may try to prove Theorem 1.1 by following more directly the idea from [11] and calculate the integral

$$\int_0^\infty \int_D t^\beta p_D(t,x,y)\,dy\,dt = \int_0^\infty t^\beta \mathbb{P}^x(\tau_D > t)\,dt = \frac{1}{\beta+1}\mathbb{E}^x(\tau_D^{\beta+1}).$$

We may proceed as follows. We denote $S_t = \sup_{0\leqslant s\leqslant t} X_s$. Then $\mathbb{P}^x(\tau_D > t) = \mathbb{P}^0(S_t < x)$. Following [27], we denote by $\mathcal{M}(z) = \mathbb{E}^0(S_1^{z-1})$ the Mellin transform of S_1. By [27, Theorem 8], $\mathcal{M}$ may be expressed by a complicated formula involving some special functions (double Gamma functions). Now, using the scaling properties and after some calculations we get $\mathcal{M}(z) = \mathbb{E}^1(\tau_D^{(1-z)/\alpha})$. Hence,

$$\int_0^\infty \int_D t^\beta p_D(t,1,y)\,dy\,dt = \mathcal{M}(1-\alpha(\beta+1)).$$

It seems that by using [27, Theorem 7] it is possible to obtain the assertion of Theorem 1.2 at least for some range of the parameters α and β. However, just as in our approach, one has to struggle with some integrability issues.

REMARK 3.2. Since $p_D(t,x,y) < p(t,x,y)$ and in particular

$$\lim_{x\to 0} p_D(t,x,y)/p(t,x,y) = 0,$$

it seems natural that p_D may be perturbed by a bigger potential than p. We verify below that indeed $\kappa_\beta > \kappa_\beta^{\mathbb{R}}$, where $\kappa_\beta^{\mathbb{R}}$ is given by (1.12) with $d = 1$.

Let $0 < \alpha < 1$ and $0 < \beta \leqslant \frac{1-\alpha}{2}$. The condition $\kappa_\beta < \kappa_\beta^{\mathbb{R}}$ is equivalent to

$$\frac{\Gamma(\beta)\Gamma(1-\beta)\Gamma\big(\frac{1-\beta}{2}\big)}{\Gamma\big(\frac{\beta}{2}\big)} < \frac{\Gamma\big(\beta+\frac{\alpha}{2}\big)\Gamma\big(1-\beta+\frac{\alpha}{2}\big)\Gamma\big(\frac{1-\beta}{2}-\frac{\alpha}{2}\big)}{2^\alpha\Gamma\big(\frac{\beta}{2}+\frac{\alpha}{2}\big)},$$

hence it suffices to show that for fixed $\beta \in (0, \frac{1-\alpha}{2})$,

$$u(t) = \frac{\Gamma(\beta+t)\Gamma(1-\beta+t)\Gamma\big(\frac{1-\beta}{2}-t\big)}{2^\alpha\Gamma\big(\frac{\beta}{2}+t\big)}$$

is increasing on the interval $[0, \frac{1}{2}-\beta)$. We consider

$$[\ln(u(t))]' = \sum_{k=0}^\infty \Big(-\frac{1}{\beta+t+k} - \frac{1}{1-\beta+t+k} + \frac{1}{\frac{1-\beta}{2}-t+k} + \frac{1}{\frac{\beta}{2}+t+k}\Big) \\ -2\ln(2).$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(-\frac{1}{\beta+t+k} + \frac{1}{\frac{\beta}{2}+t+k}\Big) < 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(-\frac{1}{1-\beta+t+k} + \frac{1}{\frac{1-\beta}{2}-t+k}\Big) > 0,$$

so both functions above are strictly monotone. Hence, taking $t = \frac{1}{2} - \beta$ and $t = 0$ respectively, we get

$$[\ln(u(t))]' \geqslant \sum_{k=0}^{\infty} \left(\frac{2}{\frac{1-\beta}{2} + k} - \frac{1}{1-\beta+k} - \frac{1}{\frac{1}{2}+k} \right) - 2\ln(2).$$

Now we investigate monotonicity with respect to β:

$$\frac{\mathrm{d}}{\mathrm{d}\beta} \left(\frac{2}{\frac{1-\beta}{2} + k} - \frac{1}{1-\beta+k} - \frac{1}{\frac{1}{2}+k} \right) = \frac{1}{\left(\frac{1-\beta}{2} + k\right)^2} - \frac{1}{(1-\beta+k)^2} > 0,$$

so it attains its minimum at $\beta = 0$. Hence, (see [21, Equation 8.363.3])

$$[\ln(u(t))]' \geqslant \sum_{k=0}^{\infty} \left(\frac{1}{\frac{1}{2}+k} - \frac{1}{1+k} \right) - 2\ln(2) = 0.$$

In Figure 2 we compare κ_β and $\kappa_\beta^{\mathbb{R}}$ for $\alpha = \frac{1}{2}$.

4. HARDY IDENTITY

In this section we prove the Hardy identity for the transition density $p_D(t, x, y)$. We will use the results from [10]. Recall that P_t^D denotes the semigroup generated by the process X_t killed on exiting D. We first define the quadratic form $\mathcal{E}$ of $\Delta_D^{\alpha/2}$:

$$\mathcal{E}_D(u, u) = \lim_{t \to 0+} \frac{1}{t}(u - P_t^D u, u), \quad u \in L^2(D).$$

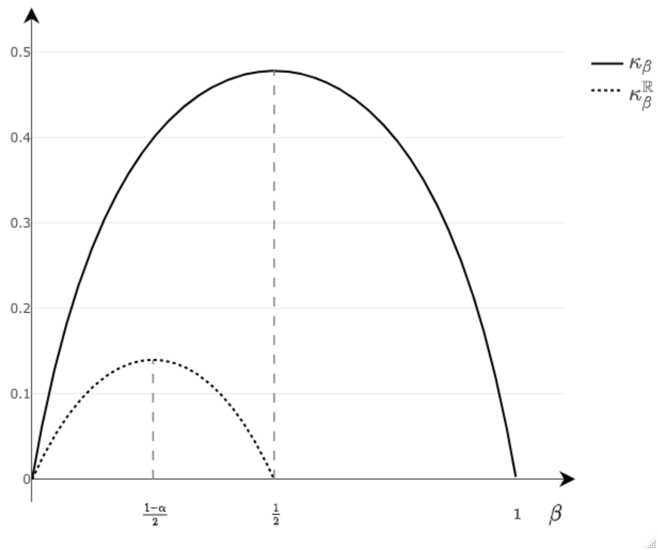

FIGURE 2. Comparison of κ_β and $\kappa_\beta^{\mathbb{R}}$ for $\alpha = \frac{1}{2}$.

Here, as usual, $(u,v) = \int_D u(x)v(x)\,dx$ for $u,v \in L^2(D)$. We let

$$\mathcal{D}(\mathcal{E}_D) = \{u \in L^2(D) : \mathcal{E}_D(u,u) < \infty\}.$$

Now, we will prove an auxiliary lemma:

LEMMA 4.1. *Let $x, y \in D$. We have*

$$\lim_{t\to 0} \frac{p_D(t,x,y)}{t} = \nu(x-y).$$

Proof. By Hunt's formula,

$$\frac{p_D(t,x,y)}{t} = \frac{p(t,x,y)}{t} - \frac{\mathbb{E}^x[p(t-\tau_D, X_{\tau_D}, y)\mathbb{1}_{\{\tau_D<t\}}]}{t}.$$

The limit of the first term is known (see e.g. [13]):

$$\lim_{t\to 0} \frac{p(t,x,y)}{t} = \nu(x-y).$$

Hence it suffices to show that the second term converges to zero. We have

$$\begin{aligned}\frac{\mathbb{E}^x[p(t-\tau_D, X_{\tau_D}, y)\mathbb{1}_{\{\tau_D<t\}}]}{t} &\leqslant c\mathbb{E}^x\left[\frac{t-\tau_D}{|X_{\tau_D}-y|^{1+\alpha}}\mathbb{1}_{\{\tau_D<t\}}\right]t^{-1}\\ &\leqslant \frac{c}{y^{1+\alpha}}\mathbb{P}^x(\tau_D<t).\end{aligned}$$

By the Ikeda–Watanabe formula and the scaling property,

$$\begin{aligned}\mathbb{P}^x(\tau_D<t) &= \int_D\int_{D^c}\int_0^t p_D(s,x,y)\nu(z-y)\,ds\,dy\,dz\\ &= c\int_D\int_0^t p_D(s,x,y)y^{-\alpha}\,ds\,dy = c\int_D\int_0^{tx^{-\alpha}} p_D(s,1,y)y^{-\alpha}\,ds\,dy.\end{aligned}$$

For t such that $tx^{-\alpha} < 1$ and by Lemma 3.1 we get

$$\int_D\int_0^{tx^{-\alpha}} p_D(s,1,y)y^{-\alpha}\,ds\,dy \leqslant ct \xrightarrow{t\to 0} 0. \quad \blacksquare$$

Proof of Theorem 1.3. Since the assumptions of [10, Theorem 2] are satisfied, we have

$$\begin{aligned}\mathcal{E}_D(u,u) &= \kappa_\beta \int_D \frac{u(x)^2}{x^\alpha}\,dx\\ &\quad + \lim_{t\to 0}\int_D\int_D\left(\frac{u(x)}{x^{\alpha/2-\beta}} - \frac{u(y)}{y^{\alpha/2-\beta}}\right)^2 x^{\alpha/2-\beta}y^{\alpha/2-\beta}\frac{p_D(t,x,y)}{2t}\,dx\,dy.\end{aligned}$$

By Lemma 4.1 we get

$$\lim_{t\to 0}\frac{p_D(t,x,y)}{t}=\nu(x-y),$$

so we have only to justify that we can change the order of the limit and the integral. Note that $\frac{p_D(t,x,y)}{t}\leqslant c\nu(x-y)$. Hence if

$$\int_D\int_D\left(\frac{u(x)}{x^{\alpha/2-\beta}}-\frac{u(y)}{y^{\alpha/2-\beta}}\right)^2 x^{\alpha/2-\beta}y^{\alpha/2-\beta}\nu(x-y)\,dx\,dy<\infty,$$

then we apply the dominated convergence theorem. If

$$\int_D\int_D\left(\frac{u(x)}{x^{\alpha/2-\beta}}-\frac{u(y)}{y^{\alpha/2-\beta}}\right)^2 x^{\alpha/2-\beta}y^{\alpha/2-\beta}\nu(x-y)\,dx\,dy=\infty,$$

then we apply Fatou's lemma. ■

Since κ_β is increasing for $\beta\in(0,1/2)$ and decreasing when $\beta\in(1/2,1)$, it attains its maximum at $1/2$. Hence, we obtain the following result first obtained in [9, (1.5)].

PROPOSITION 4.1 (Hardy inequality). *For $u\in L^2(D)$ we have*

$$\mathcal{E}_D(u,u)\geqslant\frac{\Gamma((\alpha+1)/2)^2}{\pi}\int_D\frac{u(x)^2}{x^\alpha}\,dx. \tag{4.1}$$

It turns out that $\kappa_{1/2}=\Gamma((\alpha+1)/2)^2/\pi$ is the best possible constant in the inequality above; see [9] for details.

REMARK 4.1. We note that for $d=1,\alpha<1$ and $\beta<1-\alpha$, (1.14) may be obtained directly from (1.11). For u with support in D we put $\beta-\alpha/2$ for β in (1.11) and we get

$$\begin{aligned}\mathcal{E}(u,u)&=(\kappa^{\mathbb{R}}_{\beta-\alpha/2}+\overline{C}_{\alpha,\beta})\int_D\frac{u(x)^2}{|x|^\alpha}\,dx\\&\quad+\frac{1}{2}\int_D\int_D\left[\frac{u(x)}{x^{\alpha/2-\beta}}-\frac{u(y)}{|y|^{\alpha/2-\beta}}\right]^2|x|^{\alpha/2-\beta}|y|^{\alpha/2-\beta}\nu(x,y)\,dy\,dx.\end{aligned}$$

Since $\mathcal{E}_D$ coincides with $\mathcal{E}$ for functions supported in D, we get (1.14) provided

$$\kappa^{\mathbb{R}}_{\beta-\alpha/2}+\overline{C}_{\alpha,\beta}=\kappa_\beta.$$

Indeed, the equality above holds for $\alpha\in(0,2)$ and $\beta\in(0,1)$. By (3.5) and the formula

$$\frac{\Gamma(x)}{\Gamma(1/2-x)}=\frac{2^{1-2x}\cos(\pi x)\Gamma(2x)}{\pi^{1/2}},$$

we get

$$\kappa^{\mathbb{R}}_{\beta-\alpha/2} = \frac{1}{\pi}\Gamma(1-\beta+\alpha/2)\Gamma(\beta+\alpha/2) 2\cos(\pi(\beta/2+\alpha/4))\sin(\pi(\beta/2-\alpha/4)).$$

Now, applying the identity $\cos(b+a)\sin(b-a) = \frac{1}{2}(\sin(2b) - \sin(2a))$, we get

$$\kappa^{\mathbb{R}}_{\beta-\alpha/2} + \overline{C}_{\alpha,\beta} = \Gamma(1-\beta+\alpha/2)\Gamma(\beta+\alpha/2)\frac{\sin(\pi\beta)}{\pi} = \kappa_\beta.$$

Acknowledgments. We thank Krzysztof Bogdan for many helpful discussions and comments on the paper.

REFERENCES

[1] B. Abdellaoui, M. Medina, I. Peral, and A. Primo, *The effect of the Hardy potential in some Calderón–Zygmund properties for the fractional Laplacian*, J. Differential Equations 260 (2016), 8160–8206.

[2] B. Abdellaoui, M. Medina, I. Peral, and A. Primo, *Optimal results for the fractional heat equation involving the Hardy potential*, Nonlinear Anal. 140 (2016), 166–207.

[3] P. Baras and J. A. Goldstein, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. 284 (1984), 121–139.

[4] W. Beckner, *Pitt's inequality and the uncertainty principle*, Proc. Amer. Math. Soc. 123 (1995), 1897–1905.

[5] A. BenAmor, *The heat equation for the Dirichlet fractional Laplacian with Hardy's potentials: properties of minimal solutions and blow-up*, Rev. Roumaine Math. Pures Appl. 66 (2021), 43–66.

[6] K. Bogdan and T. Byczkowski, *Potential theory of Schrödinger operator based on fractional Laplacian*, Probab. Math. Statist. 20 (2000), 293–335.

[7] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, and Z. Vondraček, *Potential Analysis of Stable Processes and its Extensions*, Lecture Notes in Math. 1980, Springer, Berlin, 2009.

[8] K. Bogdan, K. Burdzy, and Z. Q. Chen, *Censored stable processes*, Probab. Theory Related Fields 127 (2003), 89–152.

[9] K. Bogdan and B. Dyda, *The best constant in a fractional Hardy inequality*, Math. Nachr. 284 (2011), 629–638.

[10] K. Bogdan, B. Dyda, and P. Kim, *Hardy inequalities and non-explosion results for semigroups*, Potential Anal. 44 (2016), 229–247.

[11] K. Bogdan, T. Grzywny, T. Jakubowski, and D. Pilarczyk, *Fractional Laplacian with Hardy potential*, Comm. Partial Differential Equations 44 (2019), 20–50.

[12] K. Bogdan, T. Grzywny, and M. Ryznar, *Heat kernel estimates for the fractional Laplacian with Dirichlet conditions*, Ann. Probab. 38 (2010), 1901–1923.

[13] K. Bogdan, T. Grzywny, and M. Ryznar, *Density and tails of unimodal convolution semigroups*, J. Funct. Anal. 266 (2014), 3543–3571.

[14] K. Bogdan and K. Merz, *Ground state representation for the fractional Laplacian with Hardy potential in angular momentum channels*, arXiv:2305.00881v1 (2023).

[15] S. Cho, P. Kim, R. Song, and Z. Vondraček, *Factorization and estimates of Dirichlet heat kernels for non-local operators with critical killings*, J. Math. Pures Appl. (9) 143, 208–256 (2020).

[16] K. L. Chung and Z. X. Zhao, *From Brownian Motion to Schrödinger's Equation*, Grundlehren Math. Wiss. 312, Springer, Berlin, 1995.

[17] R. L. Frank, E. H. Lieb, and R. Seiringer, *Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators*, J. Amer. Math. Soc. 21 (2008), 925–950.

[18] R. L. Frank, K. Merz, and H. Siedentop, *Equivalence of Sobolev norms involving generalized Hardy operators*, Int. Math. Res. Notices 2021, 2284–2303.

[19] R. L. Frank and R. Seiringer, *Sharp fractional Hardy inequalities in half-spaces*, in: Around the Research of Vladimir Maz'ya. I, Int. Math. Ser. (N.Y.) 11, Springer, New York, 2010, 161–167.

[20] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Stud. Math. 19, Walter de Gruyter, Berlin, 2011.

[21] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed., Elsevier/Academic Press, Amsterdam, 7 edition, 2007.

[22] G. H. Hardy, *Note on a theorem of Hilbert*, Math. Z. 6 (1920), 314–317.

[23] I. W. Herbst, *Spectral theory of the operator* $(p^2 + m^2)^{1/2} - Ze^2/r$, Comm. Math. Phys. 53 (1977), 285–294.

[24] N. Ikeda and S. Watanabe, *On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes*, J. Math. Kyoto Univ. 2 (1962), 79–95.

[25] T. Jakubowski and J. Wang, *Heat kernel estimates of fractional Schrödinger operators with negative Hardy potential*, Potential Anal. 53 (2020), 997–1024.

[26] M. Kijaczko and J. Lenczewska, *Sharp Hardy inequalities for Sobolev–Bregman forms*, arXiv:2109.01704v2 (2023).

[27] A. Kuznetsov, *On extrema of stable processes*, Ann. Probab. 39 (2011), 1027–1060.

[28] M. Kwaśnicki, *Fractional Laplace operator and its properties*, in: Handbook of Fractional Calculus with Applications, Vol. 1, De Gruyter, Berlin, 2019, 159–193.

[29] S. Machihara, T. Ozawa, and H. Wadade, *Remarks on the Hardy type inequalities with remainder terms in the framework of equalities*, in: Asymptotic Analysis for Nonlinear Dispersive and Wave Equations, Adv. Stud. Pure Math. 81, Math. Soc. Japan, Tokyo, 2019, 247–258.

[30] D. Yafaev, *Sharp constants in the Hardy–Rellich inequalities*, J. Funct. Anal. 168 (1999), 121–144.

Tomasz Jakubowski, Paweł Maciocha
Wrocław University of Science and Technology
27 Wybrzeże Wyspiańskiego St.
50-370 Wrocław, Poland
E-mail: tomasz.jakubowski@pwr.edu.pl
pawel.maciocha@pwr.edu.pl

Received 18.2.2023;
accepted 12.6.2023