PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 45, Fasc. 1 (2025), pp. 71–99
Published online 7.8.2025
doi:10.37190/0208-4147.00214

ASYMPTOTICS OF ESTIMATORS IN A HETEROSCEDASTIC REGRESSION MODEL WITH STRONG MIXING ERRORS*

BY

YAN WANG (CHIZHOU AND HEFEI), XUEJUN WANG (HEFEI), AND AITING SHEN (HEFEI)

Abstract. Consider the model $Y_{ni} = g(x_{ni}) + \sigma_{ni}\varepsilon_{ni},\ i = 1, \ldots, n$, where $\sigma_{ni}^2 = f(u_{ni})$, the design points (x_{ni}, u_{ni}) are known and nonrandom, $g(\cdot)$ and $f(\cdot)$ are unknown functions defined on $[0, 1]$, and the random errors $\{\varepsilon_{ni},\ 1 \leqslant i \leqslant n\}$ are assumed to have the same distribution as $\{\vartheta_i,\ 1 \leqslant i \leqslant n\}$, which is a sequence of identically distributed α-mixing random variables with mean zero. Estimators of $f(\cdot)$ and $g(\cdot)$ are constructed by the G-M method and their rth ($r > 2$) mean consistency and strong consistency are obtained under appropriate conditions. To demonstrate the validity of theoretical results, finite sample behaviors of the estimators are considered via a simulation study.

2020 Mathematics Subject Classification: Primary 60G05; Secondary 62G20.

Key words and phrases: heteroscedastic regression model, strong mixing random variables, G-M estimator, consistency.

1. INTRODUCTION

Consider the following heteroscedastic regression model:

$$Y_{ni} = g(x_{ni}) + \sigma_{ni}\varepsilon_{ni}, \quad i = 1, \ldots, n, \tag{1.1}$$

where $\sigma_{ni}^2 = f(u_{ni})$, the design points (x_{ni}, u_{ni}) are known and nonrandom with $0 = x_{n0} < x_{n1} \leqslant x_{n2} \leqslant \cdots \leqslant x_{n,n-1} < x_{nn} = 1$ and $0 = u_{n0} < u_{n1} \leqslant \cdots \leqslant u_{n,n-1} < u_{nn} = 1$, Y_{ni} are the response variables, ε_{ni} are random errors with $E\varepsilon_{ni}^2 = 1$, and $g(\cdot)$ and $f(\cdot)$ are unknown functions defined on $[0, 1]$.

The heteroscedastic regression model solves the problem caused by the constant variance in traditional econometrics and has attracted the attention of many

* Supported by the National Natural Science Foundation of China (12471248, 12201600, 12201079) and the Natural Science Foundation of Anhui Province (2308085MA07), the Chizhou University High Level Talent Research Start-up Fund (CZ2024YJRC05), and the Provincial Natural Science Research Project of Anhui Colleges (2024AH051399).

scholars due to its applications in practical problems such as economic analysis and time series (see Fan and Yao [7] for the latter applications). Concerning the properties of this model, the interested readers can refer to Liang and Lu [13] for the asymptotic normality of wavelet estimators of $g(\cdot)$ and $f(\cdot)$ being known or unknown functions in the model (1.1) under the condition that the errors are assumed to be martingale differences, to Liang [10] for the asymptotic normality of wavelet estimators of $g(\cdot)$ and $f(\cdot)$ when $f(\cdot)$ is known or unknown in (1.1) with α-mixing errors, revealing the limit distribution of the wavelet estimator, to Liang and Liu [12] for the asymptotic normality of an estimator of $f(\cdot)$ in (1.1) under the condition that the error sequence is a stationary and α-mixing time series, etc.

It is worth pointing out that the model (1.1) boils down to the usual nonparametric regression model if $\sigma_{ni}^2 = \sigma^2$, and the latter model has been extensively studied under both independent and dependent noises. For the details, one can refer to Gasser and Müller [8], Georgiev and Greblicki [9], Müller [15], Priestley and Chao [16] for the independent case. In the dependent cases, Liang and Jing [11] and Chen [3] considered a weighted function estimator of the unknown function in the model with negatively associated errors and martingale difference errors, respectively, Wu et al. [23] and Ding and Li [5] studied the Priestley–Chao estimator in the model with extended negatively dependent errors and linear process errors generated by LNQD random errors, respectively, etc.

Now let us recall the definition of α-mixing random variables.

DEFINITION 1.1. Let $\{X_n,\, n \geqslant 1\}$ be a sequence of random variables, and let $\mathcal{F}_1^k = \sigma(X_j; 1 \leqslant j \leqslant k)$ and $\mathcal{F}_{k+n}^{\infty} = \sigma(X_j; j \geqslant k+n)$ be two σ-algebras. If

$$\alpha(n) = \sup_{k \geqslant 1} \sup_{A \in \mathcal{F}_1^k,\, B \in \mathcal{F}_{k+n}^{\infty}} |P(AB) - P(A)P(B)| \to 0 \quad \text{as } n \to \infty,$$

then $\{X_n,\, n \geqslant 1\}$ is called *α-mixing*.

Rosenblatt [17] introduced the concept of α-mixing and many related results have been obtained recently. For instance, Ekström [6] gave a central limit theorem for α-mixing sequences, Thanh and Yin [19] proved complete convergence for weighted sums of α-mixing random variables, Asghari and Fakoor [1] derived the asymptotic normality as well as a Berry–Esseen type bound for the kernel density estimator of left truncated and α-mixing data, Tabacu [18] studied the weak convergence of the linear rank statistics under α-mixing conditions, Yang et al. [24] investigated the change point detection for nonparametric regression under α-mixing process, Dedecker and Merlevède [4] proved a strong invariance principle for the Kantorovich distance between the empirical distribution and the marginal distribution of stationary α-mixing sequences, Wang and Shen [20] established some asymptotic results such as weak consistency, strong consistency, uniformly strong consistency and the corresponding rates for the nearest neighbour estimator of the density function under α-mixing samples, Wu et al. [23] established a general form of Berry–Esseen bound for α-mixing random variables, Wang et al. [21]

investigated the Bahadur type representation of a nonparametric expected shortfall estimator for α-mixing financial losses, Wang et al. [22] investigated weak consistency, uniformly weak consistency and the rate of uniformly weak consistency for a generalized edge frequency polygon of the density function under α-mixing samples, etc.

The Gasser–Müller (G-M, for short) estimator was originally proposed by Gasser and Müller [8], and has been widely discussed by many scholars; see e.g. [8] and Müller [15] for more details. Recently, Benhenni et al. [2] analyzed the effect of regularity of the error process on the performance of the estimator with correlated errors, and Yang and Yang [26] proved strong consistency and uniformly strong consistency of the estimator under weaker conditions. Now let us introduce the estimators.

According to [8], the G-M estimator of $g(\cdot)$ for the model (1.1) is defined by

$$\widehat{g}_n(x) = h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\, Y_{ni}. \tag{1.2}$$

Hence, the G-M estimator of $f(\cdot)$ for the model (1.1) is given by

$$\widehat{f}_n(u) = h_{2n}^{-1} \sum_{i=1}^{n} (Y_{ni} - \widehat{g}_n(x_{ni}))^2 \int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds, \tag{1.3}$$

where the kernel function $K(\cdot)$ is measurable and bounded on $\mathbb{R}$ and we have $0 < h_{1n}, h_{2n} \to 0$ as $n \to \infty$.

The G-M estimator is based on kernel functions and it is easy to compute and implement. Furthermore, it can adapt to different bandwidth choices and performs stably under various errors. To date, the consistency properties of G-M estimators for the nonparametric component in the model (1.1) remain unestablished. Inspired by the literature, we aim to investigate the rth ($r > 2$) mean consistency and strong consistency for the estimators (1.3) and (1.2) of $f(\cdot)$ and $g(\cdot)$ in the model (1.1) under α-mixing error structures. Since α-mixing is weaker than φ-mixing or ρ-mixing, the study of α-mixing is reasonable. Additionally, in order to illustrate the validity of the theoretical results established, we also make a simulation study of finite samples.

Throughout this paper, the random errors $\{\varepsilon_{ni},\ 1 \leqslant i \leqslant n\}$ are assumed to have the same distribution as $\{\vartheta_i,\ 1 \leqslant i \leqslant n\}$, which is a sequence of identically distributed α-mixing random variables with mean zero. The work is organized as follows. In the next section, we state the assumptions and main results. In Section 3, a simulation study including numerical simulation and real data example is presented. Preliminary lemmas used to prove our main results are given in Section 4. The proofs of our main results can be found in Section 5.

Let $\lfloor x \rfloor$ stand for the integer part of x, $\|X\|_r = (E|X|^r)^{1/r}$, and let C be a positive constant, whose value may be different in different places. We write

$a_n = O(b_n)$ instead of $a_n \leqslant Cb_n$, and $a_n = o(b_n)$ instead of $a_n/b_n \to 0$ as $n \to \infty$, where $\{a_n,\ n \geqslant 1\}$ and $\{b_n,\ n \geqslant 1\}$ are sequences of positive numbers.

2. ASSUMPTIONS AND MAIN RESULTS

2.1. Assumptions. Firstly, we introduce some assumptions, which play an important role in this paper.

(A1) (i) $g(\cdot)$ and $f(\cdot)$ satisfy the Lipschitz condition of order 1 on $[0,1]$;

(ii) $0 < m_0 \leqslant \min_{1\leqslant i\leqslant n} f(u_{ni}) \leqslant \max_{1\leqslant i\leqslant n} f(u_{ni}) \leqslant M_0 < \infty$ and $\sup_{x\in[0,1]} g(x) := G_1\ (< \infty)$.

(A2) (i) $K(\cdot)$ is bounded on $\mathbb{R}$;

(ii) $\int_{-\infty}^{\infty} K(\nu)\,d\nu = 1$.

(A3) $h_{jn} \to 0,\ \delta_{jn} \to 0,\ \delta_{jn}/h_{jn} \to 0,\ j = 1, 2$, where

$$\delta_{1n} := \max_{1\leqslant i\leqslant n} |x_{ni} - x_{n,i-1}| \quad \text{and} \quad \delta_{2n} := \max_{1\leqslant i\leqslant n} |u_{ni} - u_{n,i-1}|,$$

(A4) For some $0 < \theta_1, \theta_2 < 1$ and any $x, u \in (0,1)$,

(i) $\max_{1\leqslant i\leqslant n} \left|h_{1n}^{-1} \int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\right| = O(n^{-\theta_1})$;

(ii) $\max_{1\leqslant i\leqslant n} \left|h_{2n}^{-1} \int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\right| = O(n^{-\theta_2})$.

(A4)′ (A4) holds for some $1/2 < \theta_1, \theta_2 < 1$ and any $x, u \in (0,1)$.

REMARK 2.1. In fact, (A1)–(A3) are quite common. Specifically, (A1) is a mild regularity condition for a heteroscedastic regression model, like (B1) in Liang and Liu [12]. It is well known that (A2) and (A3) are mild conditions for kernel smoothing, like (A2) and (A3) in Ding and Li [5].

REMARK 2.2. From (A2)(i) and (A3) we can deduce that as $n \to \infty$,

$$\max_{1\leqslant i\leqslant n} \left|h_{1n}^{-1} \int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\right| \leqslant \frac{C\delta_{1n}}{h_{1n}} \to 0$$

and

$$\max_{1\leqslant i\leqslant n} \left|h_{2n}^{-1} \int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\right| \leqslant \frac{C\delta_{2n}}{h_{2n}} \to 0,$$

thus (A4) is reasonable.

Next, our main results are given, which are divided into two parts: rth mean consistency and strong consistency.

2.2. rth mean consistency

THEOREM 2.1. *In the model* (1.1)*, let* (A1)–(A4) *hold, and* $\{\vartheta_i,\ i \geqslant 1\}$ *be a sequence of identically distributed* α*-mixing random variables with mean zero and* $E|\vartheta_1|^{r+\delta} < \infty$ *for some* $r > 2$ *and* $\delta > 0$*. Suppose that* $\alpha(n) = O(n^{-\lambda})$ *for some* $\lambda > r(r+\delta)/(2\delta)$*. Then for each* $x \in (0,1)$*,*

$$E|\widehat{g}_n(x) - g(x)|^r \xrightarrow[n\to\infty]{} 0. \tag{2.1}$$

THEOREM 2.2. *In the model* (1.1)*, let* (A1)–(A3) *and* (A4)′ *hold, and* $\{\vartheta_i,\ i \geqslant 1\}$ *be a sequence of identically distributed* α*-mixing random variables with mean zero and* $E|\vartheta_1|^{2(r+\delta)} < \infty$ *for some* $r > 2$ *and* $\delta > 0$*. Suppose that* $\alpha(n) = O(n^{-\lambda})$ *for some* $\lambda > r(r+\delta)/(2\delta)$*. Then for each* $u \in (0,1)$*,*

$$E|\widehat{f}_n(u) - f(u)|^r \xrightarrow[n\to\infty]{} 0. \tag{2.2}$$

2.3. Almost sure consistency

THEOREM 2.3. *In the model* (1.1)*, let* (A1)–(A3) *and* (A4)′ *hold, and* $\{\vartheta_i,\ i \geqslant 1\}$ *be a sequence of identically distributed* α*-mixing random variables with mean zero and* $E|\vartheta_1|^{2+\delta} < \infty$ *for some* $\delta > 0$*. Suppose that* $\alpha(n) = O(n^{-\lambda})$ *for some* $\lambda > \max\left\{\frac{2+\delta}{\delta}, \frac{(2-\theta_1)(2+\delta)+1}{\theta_1(2+\delta)-1}, \frac{(2-\theta_2)(2+\delta)+1}{\theta_2(2+\delta)-1}\right\}$*. Then for each* $x \in (0,1)$*,*

$$\widehat{g}_n(x) \xrightarrow[n\to\infty]{a.s.} g(x). \tag{2.3}$$

THEOREM 2.4. *In the model* (1.1)*, let* (A1)–(A3) *and* (A4)′ *hold and* $\{\vartheta_i,\ i \geqslant 1\}$ *be a sequence of identically distributed* α*-mixing random variables with mean zero and* $E|\vartheta_1|^{4+2\delta} < \infty$ *for some* $\delta > 0$*. Suppose that* $\alpha(n) = O(n^{-\lambda})$ *for some* $\lambda > \max\left\{\frac{2+\delta}{\delta}, \frac{(3-\theta_1)(2+\delta)+1}{\theta_1(2+\delta)-1}, \frac{(3-\theta_2)(2+\delta)+1}{\theta_2(2+\delta)-1}\right\}$*. Then for each* $u \in (0,1)$*,*

$$\widehat{f}_n(u) \xrightarrow[n\to\infty]{a.s.} f(u). \tag{2.4}$$

REMARK 2.3. Theorems 2.3 and 2.4 state the almost sure convergence for the estimators (1.2) and (1.3) in the model (1.1) based on identically distributed α-mixing errors, which can help readers to solve some statistical inference problems more generally.

3. SIMULATION STUDY

3.1. Numerical simulation. In this subsection, we investigate the finite sample performance of the estimators (1.2) and (1.3). In particular, we test the consistency by drawing the figures of the estimations $\widehat{g}_n(x)$ and $\widehat{f}_n(u)$. Taking $g(x) = \sin(2\pi x)$, $f(u) = [1 + 0.5\cos(2\pi u)]^2$, the observations are generated from

$$Y_{ni} = \sin(2\pi x_{ni}) + [1 + 0.5\cos(2\pi u_{ni})]\epsilon_i, \tag{3.1}$$

where $x_{ni} = (i - 0.5)/n$, $u_{ni} = (i - 0.99)/n$. ϵ_i are errors, which are considered under three different scenarios: (i) an AR(1) process $\epsilon_i = 0.3\epsilon_{i-1} + e_i$, where e_i are i.i.d. $N(0, 1)$ random variables; (ii) an MA(∞) process with a_k coefficients decaying as $k^{-\tau}$ with $\tau = 4$; (iii) a Student's law $t(4)$. Here we take the Gaussian kernel function $K(u) = \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)$, bandwidths $h_{1n} = n_1$ or n_3 and $h_{2n} = n_1$ or n_2 in the estimators (1.2) and (1.3), where $n_1 = n^{-1/3}, n_2 = n^{-5/12}$ and $n_3 = n^{-1/5}$. Thus the conditions of Theorems 2.3 and 2.4 can be verified easily.

From the model (3.1), we generate the observed data with sample sizes $n = 500,\ 1000,\ 2000$ and 5000 respectively, and calculate the estimated values for each x_{ni} and u_{ni} with $M = 5000$ replications. In particular, $\widehat{g}_n(x_{nl}, k)$ and $\widehat{f}_n(u_{nl}, k)$ are the kth estimates of $g(x_{nl})$ and $f(u_{nl})$ in M calculations respectively. Then the GMSE of $\widehat{g}_n(\cdot)$ based on M replications is defined as

$$\text{GMSE}(\widehat{g}_n) = \frac{1}{nM} \sum_{k=1}^{M} \sum_{l=1}^{n} [\widehat{g}_n(x_{nl}, k) - g(x_{nl})]^2 =: D_n^2(\widehat{g}_n). \tag{3.2}$$

The GMSE of $\widehat{f}_n(\cdot)$ based on M replications is defined as

$$\text{GMSE}(\widehat{f}_n) = \frac{1}{nM} \sum_{k=1}^{M} \sum_{l=1}^{n} [\widehat{f}_n(u_{nl}, k) - f(u_{nl})]^2 =: D_n^2(\widehat{f}_n). \tag{3.3}$$

Obviously, $0 \leqslant D_n(\cdot) := \sqrt{\text{GMSE}(\cdot)}$. Furthermore, the estimated values $\widehat{g}_n(\cdot)$ and $\widehat{f}_n(\cdot)$ with the sample sizes $n = 500,\ 1000,\ 2000$ and 5000 are drawn in Figure 1, and only one of them is shown due to similar features.

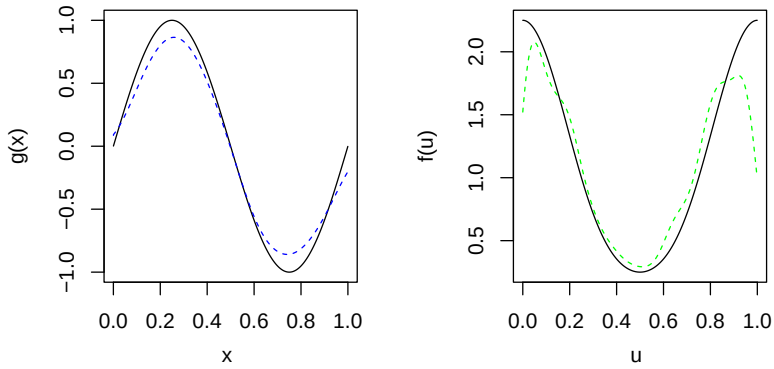

FIGURE 1. The estimated values $\widehat{g}_n(\cdot)$ (left plot) and $\widehat{f}_n(\cdot)$ (right plot); the true values (solid line) and the estimated values (dotted line) with $n = 2000$ and $h_{1n} = n_1$ and $h_{2n} = n_2$

Due to the possible lack of graphical explanation, we also calculated the GMSEs of $\widehat{g}_n(\cdot)$ and $\widehat{f}_n(\cdot)$ based on M replications, using the expressions (3.2) and (3.3). The calculation results are shown in Tables 1 and 2.

TABLE 1. $D_n(\widehat{g}_n)$ under different scenarios

Bandwidths		Sequences	Sample sizes (n)			
h_{1n}	h_{2n}		500	1000	2000	5000
n_1	n_2	AR(1)	0.20009	0.15435	0.10720	0.06702
		MA(∞)	0.19489	0.13761	0.09523	0.06111
		$t(4)$	0.21691	0.14633	0.10670	0.06969
n_3	n_2	AR(1)	0.48492	0.42877	0.37360	0.30163
		MA(∞)	0.48562	0.42912	0.36974	0.30548
		$t(4)$	0.48242	0.42501	0.37431	0.30408
n_1	n_1	AR(1)	0.20990	0.15174	0.10393	0.06827
		MA(∞)	0.21074	0.14041	0.09606	0.06422
		$t(4)$	0.20486	0.14649	0.10251	0.06937
n_3	n_1	AR(1)	0.48492	0.42877	0.37360	0.30163
		MA(∞)	0.48562	0.42912	0.36974	0.30548
		$t(4)$	0.48242	0.42501	0.37431	0.30408

TABLE 2. $D_n(\widehat{f}_n)$ under different scenarios

Bandwidths		Sequences	Sample sizes (n)			
h_{1n}	h_{2n}		500	1000	2000	5000
n_1	n_2	AR(1)	0.36309	0.29549	0.25136	0.20880
		MA(∞)	0.38757	0.32183	0.26145	0.20883
		$t(4)$	1.15736	1.22723	1.26396	1.25455
n_3	n_2	AR(1)	0.48344	0.42187	0.35096	0.27700
		MA(∞)	0.44840	0.39908	0.31131	0.23510
		$t(4)$	1.58030	1.43573	1.31758	1.31559
n_1	n_1	AR(1)	0.49445	0.39779	0.32680	0.26466
		MA(∞)	0.50483	0.44439	0.35745	0.29045
		$t(4)$	1.12825	1.03574	0.99298	1.05035
n_3	n_1	AR(1)	0.54191	0.46483	0.39072	0.31289
		MA(∞)	0.52883	0.47416	0.38772	0.30973
		$t(4)$	1.29740	1.19314	1.12361	1.13306

Concluding remarks. Tables 1–2 show the following:

(i) The values of $D_n(\widehat{g}_n)$ and $D_n(\widehat{f}_n)$ decrease as the sample size increases under different bandwidths. This indicates that the precision of the estimators improves with larger sample sizes. There are some differences in accuracy between the two estimators, mainly because the estimated values $\widehat{f}_n(\cdot)$ depend on the estimated values $\widehat{g}_n(\cdot)$.

(ii) The performance of the G-M estimators varies under different error structures (AR(1), MA(∞) and $t(4)$). For the two cases of AR(1), MA(∞), the estimators perform similarly, while for the $t(4)$ error structure, the estimators exhibit

larger errors, indicating that the heavy-tailed nature of the $t(4)$ distribution significantly affects the precision of the estimators. Under AR(1) and MA(∞) error structures, the estimators show relatively stable performance, and the errors decrease as the sample size increases.

(ii) The choice of bandwidth significantly affects the precision of the estimators. Smaller bandwidths (e.g., $h_{1n} = n^{-1/3}$ and $h_{2n} = n^{-5/12}$) perform better with larger sample sizes, indicating that bandwidth selection plays a crucial role in the convergence rate of the estimators.

Simulation results show that the G-M estimators can quantify the model parameters well when the model errors are weakly dependent. In general, these simulations show a good fit of our theoretical results established in Section 2.

3.2. Real data analysis. To illustrate the methodology described in this paper, we consider the Boston house price data set relating to 14 explanatory variables, which has been analyzed by various authors. The purpose of this research is to evaluate the relationship between house prices and air quality in neighboring areas by using regression models. There are a total of 506 observations. Here we consider the variable RM (average number of rooms per dwelling) due to its nonlinear relationship with LSTAT ($\%$ lower status of the population); see Figure 2.

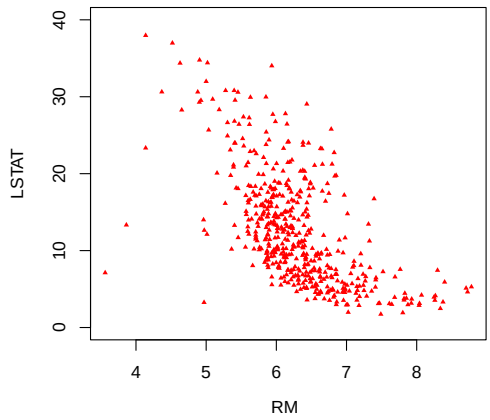

FIGURE 2. The scatter plot of LSTAT versus RM

We employ the Gaussian kernel function $K(u) = \frac{1}{\sqrt{2\pi}} \exp(-u^2/2)$ and the bandwidth $h_{1n} = n^{-1/3}$, and obtain $\widehat{y}_i = \widehat{g}(x_i),\ i = 1, \ldots, 506$, where x_i denotes the average number of rooms per dwelling. At this point, we draw the scatter plot of the fitted values and the absolute values of residuals, as shown in Figure 3. Given $h_{2n} = n^{-5/12}$, it follows that $\widehat{g}(\cdot)$ and $\widehat{f}(\cdot)$ are given in Figures 4 and 5.

In particular, we draw the auto-correlation function plot and partial auto-correlation function plot about $\widehat{\varepsilon}_i$ in Figure 6. After performing a goodness-of-fit test on the residuals of the real data, they fitted an ARIMA(1,0,1) model, which includes

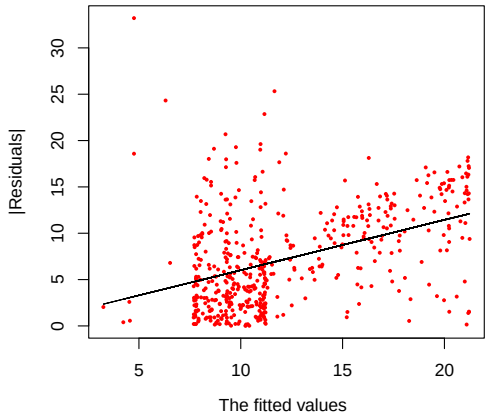

FIGURE 3. The scatter plot of the absolute values of residuals versus the fitted values

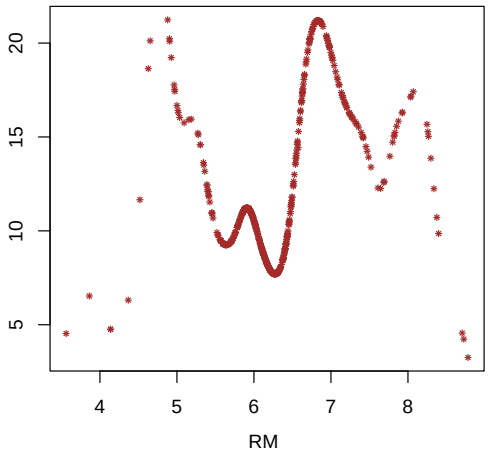

FIGURE 4. Estimates of g, where the symbols $*$ are the estimated values

both AR(1) and MA(1) components. This indicates that the residuals contain both autoregressive (AR) and moving average (MA) components. The p-value of the Ljung-Box test is extremely small (p-value $< 2.2e{-}16$), strongly rejecting the null hypothesis that the residuals are white noise. This indicates significant autocorrelation in the residuals. And the ARMA(1,1) process is stationary since the AR(1) coefficient $\phi_1 = 0.8928$ satisfies $|\phi_1| < 1$, i.e., the residuals form an α-mixing sequence (because a stationary ARMA$(\cdot,\cdot)$ process is α-mixing according to Fan and Yao [7]). This is consistent with our theoretical conditions on error sequences.

Concluding remarks. For nonparametric regression models with heteroscedastic errors, we have studied the G-M estimators for the nonparametric regression function and volatility function. G-M estimation is based on a kernel function. Compared with alternative estimators such as wavelet, it is simpler to calculate and

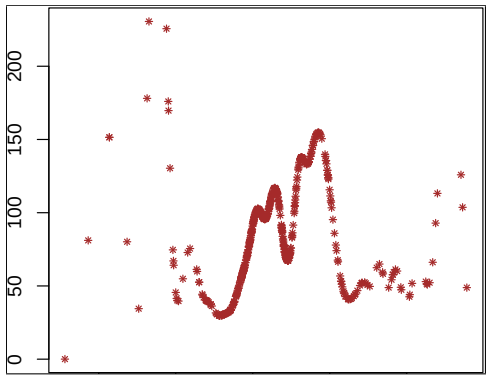

FIGURE 5. Estimates of f, where the symbols $*$ are the estimated values

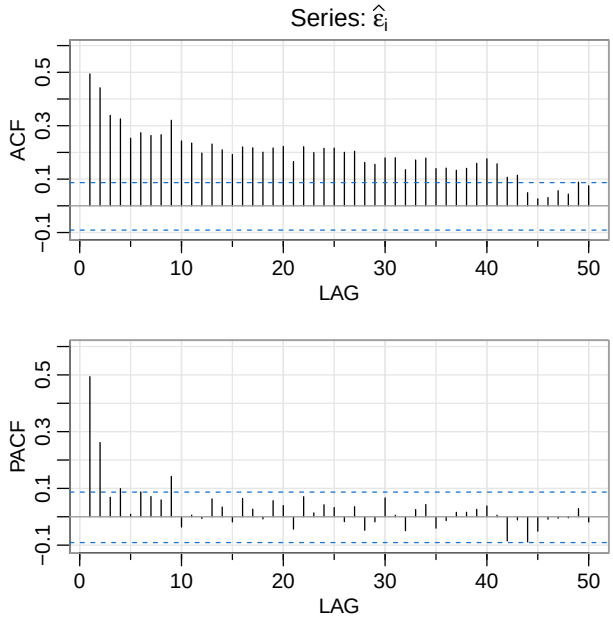

FIGURE 6. ACF plot and PACF plot of $\widehat{\varepsilon}_i$

easier to realize. For example, one does not need to make assumptions about the spectral function and carry out the wavelet transform of the data.

4. PRELIMINARY LEMMAS

In this section, we provide some lemmas that will be used to prove our main results. The first lemma is the moment inequality for α-mixing random variables with $r > 2$.

LEMMA 4.1 (cf. Yang [25, Theorem 2.2]). *Let $r > 2$, $\delta > 0$ and $\{X_i,\ i \geqslant 1\}$ be a sequence of α-mixing random variables with $EX_i = 0$ and $E|X_i|^{r+\delta} < \infty$ for each $i \geqslant 1$. Suppose that $\alpha(n) \leqslant Cn^{-\beta_1}$ for some $\beta_1 > r(r+\delta)/(2\delta)$, where C is a positive constant. Then, for any $\epsilon > 0$, there exists a positive constant $K = K(\epsilon, r, \delta, \beta_1, C) < \infty$ such that*

$$E\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} X_i\Big|^r\Big) \leqslant K\Big[n^{\epsilon}\sum_{i=1}^{n} E|X_i|^r + \Big(\sum_{i=1}^{n}\|X_i\|_{r+\delta}^2\Big)^{r/2}\Big].$$

The following lemma is the moment inequality for α-mixing random variables with $r = 2$.

LEMMA 4.2 (cf. Zhang and Liang [27, Lemma 5.2(b)]). *Let $\{X_i,\ i \geqslant 1\}$ be a sequence of α-mixing random variables with $EX_i = 0$ and $E|X_i|^{2+\delta} < \infty$ for some $\delta > 0$ and each $i \geqslant 1$. Then*

$$E\Big(\sum_{i=1}^{n} X_i\Big)^2 \leqslant \Big[1 + 16\sum_{l=1}^{n}\alpha^{\delta/(2+\delta)}(l)\Big]\sum_{i=1}^{n}\|X_i\|_{2+\delta}^2.$$

The next lemma is an important property of the kernel function, which is useful in proving the main results.

LEMMA 4.3. *Let l_{ni} be fixed design points from $[0,1]$ satisfying $0 = l_{n0} < l_{n1} \leqslant \cdots \leqslant l_{n,n-1} < l_{nn} = 1$. Suppose that $\int_{-\infty}^{\infty}|K(u)|\,du < \infty$ and $0 < b_n \to 0$ as $n \to \infty$. Then for any $l \in (0,1)$,*

$$\lim_{n\to\infty} b_n^{-1}\sum_{i=1}^{n}\int_{l_{n,i-1}}^{l_{ni}}\left|K\left(\frac{l-s}{b_n}\right)\right| ds = \int_{-\infty}^{\infty}|K(u)|\,du. \tag{4.1}$$

Proof. Inspired by the proof of Lemma 4 in Yang and Yang [26], we argue as follows. For any $l \in (0,1)$, we obtain

$$\begin{aligned}
0 &\leqslant \left|b_n^{-1}\sum_{i=1}^{n}\int_{l_{n,i-1}}^{l_{ni}}\left|K\left(\frac{l-s}{b_n}\right)\right| ds - \int_{-\infty}^{\infty}|K(u)|\,du\right| \\
&= \left|b_n^{-1}\int_{0}^{1}\left|K\left(\frac{l-s}{b_n}\right)\right| ds - b_n^{-1}\int_{-\infty}^{\infty}\left|K\left(\frac{l-s}{b_n}\right)\right| ds\right| \\
&= \left|-b_n^{-1}\int_{-\infty}^{0}\left|K\left(\frac{l-s}{b_n}\right)\right| ds - b_n^{-1}\int_{1}^{\infty}\left|K\left(\frac{l-s}{b_n}\right)\right| ds\right| \\
&\leqslant \int_{l/b_n}^{\infty}|K(u)|\,du + \int_{-\infty}^{(l-1)/b_n}|K(u)|\,du.
\end{aligned}$$

Note that $l/b_n \to \infty$ and $(l-1)/b_n \to -\infty$ as $n \to \infty$, which combined with $\int_{-\infty}^{\infty} |K(u)|\,du < \infty$ implies that as $n \to \infty$,

$$\int_{l/b_n}^{\infty} |K(u)|\,du \to 0 \quad \text{and} \quad \int_{-\infty}^{(l-1)/b_n} |K(u)|\,du \to 0.$$

This yields (4.1). ∎

REMARK 4.1. From the above proof, we can note that (4.1) still holds if $l_{ni} = x_{ni}$, $b_n = h_{1n}$ and $l = x$, or if $l_{ni} = u_{ni}$, $b_n = h_{2n}$ and $l = u$.

The following Bernstein-type inequality is useful in proving the subsequent lemmas.

LEMMA 4.4 (cf. Liebscher [14, Proposition 5.1]). *Let $\{X_i,\ i \geqslant 1\}$ be a sequence of α-mixing random variables with $EX_i = 0$ and $|X_i| \leqslant S < \infty$ a.s., $i = 1, \ldots, n$, where S is a positive constant. Denote*

$$D_m = \max_{1 \leqslant j \leqslant 2m} \operatorname{Var}\Big(\sum_{i=1}^{j} X_i\Big).$$

Then, for $n, m \in \mathbb{N}$, $0 < m \leqslant n/2$, and any $\epsilon > 0$,

$$P\Big(\Big|\sum_{i=1}^{n} X_i\Big| > \epsilon\Big) \leqslant 4\exp\Big\{-\frac{\epsilon^2}{16}\Big(nm^{-1}D_m + \frac{1}{3}\epsilon S m\Big)^{-1}\Big\} + 32\frac{S}{\epsilon}n\alpha(m).$$

LEMMA 4.5. *In the model (1.1), assume that $\{\varepsilon_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ have the same distribution as $\{\vartheta_i,\ i \geqslant 1\}$, which are identically distributed α-mixing random variables with $E\vartheta_1 = 0$. Suppose that (A1)–(A3) and (A4)′ are satisfied and $\alpha(n) = O(n^{-\lambda})$ for some $\lambda > \max\big\{\frac{2+\delta}{\delta}, \frac{(3-\theta_1)(2+\delta)+1}{\theta_1(2+\delta)-1}, \frac{(3-\theta_2)(2+\delta)+1}{\theta_2(2+\delta)-1}\big\}$ with $\delta > 0$. If $E|\vartheta_1|^{2+\delta} < \infty$, then*

$$\max_{1 \leqslant k \leqslant n} \Big| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\Big(\frac{x_{nk} - s}{h_{1n}}\Big)\,ds\, f^{1/2}(u_{ni})\vartheta_i \Big| \xrightarrow[n\to\infty]{a.s.} 0.$$

Proof. Set $\vartheta_i^{(1)} = \vartheta_i I(|\vartheta_i| \leqslant n^{\frac{1}{2+\delta}})$ and $\vartheta_i^{(2)} = \vartheta_i I(|\vartheta_i| > n^{\frac{1}{2+\delta}})$. It is easy to check that

$$\begin{aligned}(4.2)\qquad &\max_{1 \leqslant k \leqslant n} \Big| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\Big(\frac{x_{nk} - s}{h_{1n}}\Big)\,ds\, f^{1/2}(u_{ni})\vartheta_i \Big| \\ &\leqslant \max_{1 \leqslant k \leqslant n} \Big| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\Big(\frac{x_{nk} - s}{h_{1n}}\Big)\,ds\, f^{1/2}(u_{ni})\vartheta_i^{(1)} \Big| \\ &\quad + \max_{1 \leqslant k \leqslant n} \Big| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\Big(\frac{x_{nk} - s}{h_{1n}}\Big)\,ds\, f^{1/2}(u_{ni})\vartheta_i^{(2)} \Big| =: I_1 + I_2.\end{aligned}$$

For I_2, by (A1)(ii) and (A2)(i) we have

$$\max_{1\leqslant k\leqslant n}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}K\left(\frac{x_{nk}-s}{h_{1n}}\right)ds\,f^{1/2}(u_{ni})\vartheta_i^{(2)}\right| \leqslant C\frac{\delta_{1n}}{h_{1n}}\sum_{i=1}^{n}|\vartheta_i|I(|\vartheta_i|>n^{\frac{1}{2+\delta}}). \tag{4.3}$$

Noting that $E|\vartheta_1|^{2+\delta}<\infty$, we have

$$\sum_{i=1}^{\infty}P(|\vartheta_i|>i^{\frac{1}{2+\delta}})<\infty, \tag{4.4}$$

and thus

$$\sum_{i=1}^{n}|\vartheta_i|I(|\vartheta_i|>n^{\frac{1}{2+\delta}})\leqslant\sum_{i=1}^{n}|\vartheta_i|I(|\vartheta_i|>i^{\frac{1}{2+\delta}})<\infty \quad \text{a.s.} \tag{4.5}$$

Hence, by (A3) we get

$$\max_{1\leqslant k\leqslant n}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}K\left(\frac{x_{nk}-s}{h_{1n}}\right)ds\,f^{1/2}(u_{ni})\vartheta_i^{(2)}\right|\xrightarrow{\text{a.s.}}0. \tag{4.6}$$

Due to $E\vartheta_i=0$, $E|\vartheta_1|^{2+\delta}<\infty$ and Lemma 4.3, we have

$$\begin{aligned}&\max_{1\leqslant k\leqslant n}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}K\left(\frac{x_{nk}-s}{h_{1n}}\right)ds\,f^{1/2}(u_{ni})E\vartheta_i^{(1)}\right|\\ &\quad=\max_{1\leqslant k\leqslant n}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}K\left(\frac{x_{nk}-s}{h_{1n}}\right)ds\,f^{1/2}(u_{ni})E\vartheta_iI(|\vartheta_i|>n^{\frac{1}{2+\delta}})\right|\\ &\quad\leqslant CK_1\sup_i[E|\vartheta_i|I(|\vartheta_i|>n^{\frac{1}{2+\delta}})]\\ &\quad\leqslant Cn^{-\frac{1+\delta}{2+\delta}}E|\vartheta_1|^{2+\delta}\xrightarrow[n\to\infty]{}0.\end{aligned} \tag{4.7}$$

In order to complete the proof, we only need to deduce from (4.2)–(4.7) that

$$\max_{1\leqslant k\leqslant n}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}K\left(\frac{x_{nk}-s}{h_{1n}}\right)ds\,f^{1/2}(u_{ni})(\vartheta_i^{(1)}-E\vartheta_i^{(1)})\right|\xrightarrow[n\to\infty]{\text{a.s.}}0. \tag{4.8}$$

According to the Borel–Cantelli lemma, it suffices to prove that

$$\sum_{n=1}^{\infty}P\left(\max_{1\leqslant k\leqslant n}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}K\left(\frac{x_{nk}-s}{h_{1n}}\right)ds\,f^{1/2}(u_{ni})(\vartheta_i^{(1)}-E\vartheta_i^{(1)})\right|>\epsilon\right)<\infty$$

for any $\epsilon>0$.

In fact, we denote, for $i = 1, \dots, n$,

$$\zeta_{nik} = n^{\theta_1 - 1/(2+\delta)} h_{1n}^{-1} \int_{x_{n,i-1}}^{x_{ni}} K\Big(\frac{x_{nk} - s}{h_{1n}}\Big)\, ds\, f^{1/2}(u_{ni})(\vartheta_i^{(1)} - E\vartheta_i^{(1)}).$$

We can easily see that

$$\max_{1 \leqslant i,k \leqslant n} |\zeta_{nik}| \leqslant C n^{\theta_1 - 1/(2+\delta)} \cdot n^{-\theta_1} \cdot n^{1/(2+\delta)} = C < \infty.$$

Furthermore, for some $\lambda > \frac{2+\delta}{\delta}$, by Lemma 4.2 and Jensen's inequality we get

$$D_m := \max_{1 \leqslant k \leqslant n} \max_{1 \leqslant j \leqslant 2m} \mathrm{Var}\Big(\sum_{i=1}^{j} \zeta_{nik}\Big) \leqslant C \sum_{i=1}^{2m} \|\zeta_{nik}\|_{2+\delta}^2 \leqslant C m n^{-2/(2+\delta)}.$$

Applying Lemma 4.4 with $m = \lfloor n^{\theta_1 - 1/(2+\delta)} \log^{-2} n \rfloor$, for some

$$\lambda > \frac{(3-\theta_1)(2+\delta)+1}{\theta_1(2+\delta)-1} \quad \text{with } \frac{1}{2} < \theta_1 < 1$$

we get, for any $\epsilon > 0$ and some sufficiently large N,

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\Big(\max_{1 \leqslant k \leqslant n} \Big| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\Big(\frac{x_{nk} - s}{h_{1n}}\Big)\, ds\, f^{1/2}(u_{ni})(\vartheta_i^{(1)} - E\vartheta_i^{(1)}) \Big| > \epsilon \Big) \\
&\quad \leqslant \sum_{n=1}^{\infty} \sum_{k=1}^{n} P\Big(\Big| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\Big(\frac{x_{nk} - s}{h_{1n}}\Big)\, ds\, f^{1/2}(u_{ni})(\vartheta_i^{(1)} - E\vartheta_i^{(1)}) \Big| > \epsilon \Big) \\
&\quad = \sum_{n=1}^{\infty} \sum_{k=1}^{n} P\Big(\Big| \sum_{i=1}^{n} \zeta_{nik} \Big| > \epsilon n^{\theta_1 - \frac{1}{2+\delta}} \Big) \\
&\quad \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \Big[\exp\Big(\frac{-C\epsilon^2 n^{2\theta_1 - \frac{2}{2+\delta}}}{n^{1 - \frac{2}{2+\delta}} + \epsilon n^{2\theta_1 - \frac{2}{2+\delta}} \log^{-2} n} \Big) + \frac{Cn \cdot \alpha(n^{\theta_1 - \frac{1}{2+\delta}} \log^{-2} n)}{\epsilon n^{\theta_1 - \frac{1}{2+\delta}}} \Big] \\
&\quad \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} \Big[\exp\Big(\frac{-C\epsilon^2 n^{2\theta_1}}{n + \epsilon n^{2\theta_1} \log^{-2} n} \Big) + \frac{C n^{1 - \theta_1 + \frac{1}{2+\delta}}}{(n^{\theta_1 - \frac{1}{2+\delta}} \log^{-2} n)^{\lambda}} \Big] \\
&\quad \leqslant C \sum_{n=1}^{\infty} n \exp(-C \min\{\epsilon^2 n^{2\theta_1 - 1}, \epsilon \log^2 n\}) + \sum_{n=1}^{\infty} \frac{C n^{2 - \theta_1 + \frac{1}{2+\delta}}}{(n^{\theta_1 - \frac{1}{2+\delta}} \log^{-2} n)^{\lambda}} \\
&\quad \leqslant C + C \sum_{n=N}^{\infty} n \exp(-C\epsilon \log^2 n) + \sum_{n=N}^{\infty} \frac{C n^{2 - \theta_1 + \frac{1}{2+\delta}}}{(n^{\theta_1 - \frac{1}{2+\delta}} \log^{-2} n)^{\lambda}} < \infty,
\end{aligned}$$

which yields (4.8). This completes the proof of the lemma. ∎

In order to simplify the proof of the main results, we give the following two lemmas, which play important roles in proving Theorems 2.3 and 2.4.

LEMMA 4.6. *Suppose that* (A1)–(A3) *and* (A4)′ *are satisfied and* $\alpha(n) = O(n^{-\lambda})$ *for some* $\lambda > \max\left\{\frac{2+\delta}{\delta}, \frac{(2-\theta_1)(2+\delta)+1}{\theta_1(2+\delta)-1}, \frac{(2-\theta_2)(2+\delta)+1}{\theta_2(2+\delta)-1}\right\}$ *with* $\delta > 0$. *If* $E|\vartheta_1|^{2+\delta} < \infty$, *then for any* $x \in (0,1)$,

$$\left| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\, f^{1/2}(u_{ni})\vartheta_i \right| \xrightarrow[n\to\infty]{a.s.} 0. \tag{4.9}$$

Proof. The proof is similar to that of Lemma 4.5. ∎

LEMMA 4.7. *Suppose that the assumptions of Lemma* 4.6 *hold. If* $E|\vartheta_1|^{4+2\delta} < \infty$, *then for any* $x \in (0,1)$,

$$\left| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\, f(u_{ni})(\vartheta_i^2 - E\vartheta_i^2) \right| \xrightarrow[n\to\infty]{a.s.} 0. \tag{4.10}$$

Proof. Taking $\vartheta_i := \vartheta_i^2 - E\vartheta_i^2$ in the proof of Lemma 4.6, we can derive the result analogously to that proof. ∎

LEMMA 4.8. *Suppose that the assumptions of Lemma* 4.5 *hold. Then for any* $x \in (0,1)$,

$$\sum_{i=1}^{n} \left| h_{2n}^{-1} \int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f^{1/2}(u_{ni}) \right| (|\vartheta_i| - E|\vartheta_i|) \xrightarrow[n\to\infty]{a.s.} 0. \tag{4.11}$$

Proof. Taking $\kappa_i = |\vartheta_i| - E|\vartheta_i|$, we note that $E\kappa_i = 0$. The proof is analogous to the proof of Lemma 4.5. ∎

5. THE PROOFS OF THE MAIN RESULTS

5.1. The proof of Theorem 2.1. For any $x \in (0,1)$, we have

$$\begin{aligned} E|\widehat{g}_n(x) - g(x)|^r &= E|\widehat{g}_n(x) - E\widehat{g}_n(x) + E\widehat{g}_n(x) - g(x)|^r \\ &\leqslant 2^{r-1}E|\widehat{g}_n(x) - E\widehat{g}_n(x)|^r + 2^{r-1}E|E\widehat{g}_n(x) - g(x)|^r \\ &= 2^{r-1}E\left| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\, f^{1/2}(u_{ni})\varepsilon_{ni} \right|^r \\ &\quad + 2^{r-1}\left| h_{1n}^{-1} \sum_{i=1}^{n} \int_{x_{n,i-1}}^{x_{ni}} g(x_{ni}) K\left(\frac{x-s}{h_{1n}}\right) ds - g(x) \right|^r \\ &=: 2^{r-1}G_{1n}(x) + 2^{r-1}G_{2n}(x). \end{aligned} \tag{5.1}$$

Firstly, we deal with $G_{1n}(x)$. Let $0 < \epsilon < \theta_1(r-1)$. By (A1)(ii), (A4), and Lemmas 4.1 and 4.3 we obtain

$$
\begin{aligned}
(5.2)\quad & E\left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\, f^{1/2}(u_{ni})\varepsilon_{ni}\right|^r \\
&\leqslant K\left[n^{\epsilon}\sum_{i=1}^{n} E\left|h_{1n}^{-1}\int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\, f^{1/2}(u_{ni})\varepsilon_{ni}\right|^r\right] \\
&\quad + K\left[\sum_{i=1}^{n}\left(E\left|h_{1n}^{-1}\int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\, f^{1/2}(u_{ni})\varepsilon_{ni}\right|^{r+\delta}\right)^{\frac{2}{r+\delta}}\right]^{r/2} \\
&\leqslant Cn^{\epsilon}\sum_{i=1}^{n}\left|h_{1n}^{-1}\int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\right|^r E|\varepsilon_{ni}|^r \\
&\quad + C\left[\sum_{i=1}^{n}\left|h_{1n}^{-1}\int_{x_{n,i-1}}^{x_{ni}} K\left(\frac{x-s}{h_{1n}}\right) ds\, f^{1/2}(u_{ni})\right|^2 (E|\varepsilon_{ni}|^{r+\delta})^{\frac{2}{r+\delta}}\right]^{r/2} \\
&\leqslant CK_1 n^{\epsilon} E|\vartheta_1|^r n^{-\theta_1(r-1)} \\
&\quad + CK_1^{r/2}(E|\vartheta_1|^{r+\delta})^{\frac{r}{r+\delta}} n^{-\theta_1 r/2} \xrightarrow[n\to\infty]{} 0,
\end{aligned}
$$

which yields

$$
(5.3)\qquad G_{1n}(x) \xrightarrow[n\to\infty]{} 0.
$$

Next, we consider $G_{2n}(x)$. By the C_r-inequality we have

$$
\begin{aligned}
(5.4)\quad & \left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} g(x_{ni})K\left(\frac{x-s}{h_{1n}}\right) ds - g(x)\right|^r \\
&\leqslant 2^{r-1}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} g(x_{ni})K\left(\frac{x-s}{h_{1n}}\right) ds - h_n^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} g(s)K\left(\frac{x-s}{h_{1n}}\right) ds\right|^r \\
&\quad + 2^{r-1}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} g(s)K\left(\frac{x-s}{h_{1n}}\right) ds - g(x)\right|^r \\
&=: 2^{r-1}G_{2n}^{(1)}(x) + 2^{r-1}G_{2n}^{(2)}(x).
\end{aligned}
$$

For $G_{2n}^{(1)}(x)$, by (A1)(i), (A2), (A3) and Lemma 4.3 we get

$$
\begin{aligned}
(5.5)\quad & \left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} g(x_{ni})K\left(\frac{x-s}{h_{1n}}\right) ds - h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} g(s)K\left(\frac{x-s}{h_{1n}}\right) ds\right|^r \\
&= \left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} [g(x_{ni}) - g(s)]K\left(\frac{x-s}{h_{1n}}\right) ds\right|^r \leqslant CK_1^r\delta_{1n}^r \xrightarrow[n\to\infty]{} 0.
\end{aligned}
$$

For $G_{2n}^{(2)}(x)$, by the C_r-inequality again we obtain

$$\begin{aligned}
&\left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}} g(s)K\left(\frac{x-s}{h_{1n}}\right)ds - g(x)\right|^r \\
&\quad= \left|h_{1n}^{-1}\int_{0}^{1} g(s)K\left(\frac{x-s}{h_{1n}}\right)ds - g(x)\right|^r \\
&\quad= \Big|\int_{(x-1)/h_{1n}}^{x/h_{1n}} g(x-\nu h_{1n})K(\nu)\,d\nu - g(x)\int_{-\infty}^{\infty} K(\nu)\,d\nu\Big|^r \\
&\quad= \Big|\int_{(x-1)/h_{1n}}^{x/h_{1n}} [g(x-\nu h_{1n})-g(x)]K(\nu)\,d\nu - g(x)\int_{-\infty}^{(x-1)/h_{1n}} K(\nu)\,d\nu \\
&\qquad\qquad - g(x)\int_{x/h_{1n}}^{\infty} K(\nu)\,d\nu\Big|^r,
\end{aligned}$$

which is less than or equal to

$$\begin{aligned}
&3^{r-1}\Big|\int_{(x-1)/h_{1n}}^{x/h_{1n}} [g(x-\nu h_{1n})-g(x)]K(\nu)\,d\nu\Big|^r + 3^{r-1}\Big|g(x)\int_{-\infty}^{(x-1)/h_{1n}} K(\nu)\,d\nu\Big|^r \\
&\qquad + 3^{r-1}\Big|g(x)\int_{x/h_{1n}}^{\infty} K(\nu)\,d\nu\Big|^r \\
&\quad =: 3^{r-1}G_{2n}^{(21)}(x) + 3^{r-1}G_{2n}^{(22)}(x) + 3^{r-1}G_{2n}^{(23)}(x).
\end{aligned}$$

For $G_{2n}^{(21)}(x)$, by (A2)(ii) we find that for any $\zeta > 0$,

$$\begin{aligned}
\int_{(x-1)/h_{1n}}^{x/h_{1n}} |g(x-\nu h_{1n})-g(x)|\,|K(\nu)|\,d\nu &\leqslant \int_{-\infty}^{\infty} |g(x-\nu h_{1n})-g(x)|\,|K(\nu)|\,d\nu \\
&= \int_{|\nu|<\zeta/h_{1n}} |g(x-\nu h_{1n})-g(x)|\,|K(\nu)|\,d\nu \\
&\quad + \int_{|\nu|\geqslant\zeta/h_{1n}} |g(x-\nu h_{1n})-g(x)|\,|K(\nu)|\,d\nu \\
&\leqslant CK_1 \sup_{|\nu|<\zeta/h_{1n}} |g(x-\nu h_{1n})-g(x)| + C\int_{|\nu|\geqslant\zeta/h_{1n}} |K(\nu)|\,d\nu \to 0.
\end{aligned}$$

The limit is zero because of (A1)(i), (A2)(ii) and letting $\zeta = \zeta_n \downarrow 0$ such that $\zeta_n/h_{1n} \to \infty$ as $n \to \infty$. Thus

$$G_{2n}^{(21)}(x) \xrightarrow[n\to\infty]{} 0. \tag{5.6}$$

Note that $\int_{-\infty}^{\infty}|K(\nu)|\,d\nu<\infty$ from (A2)(ii). For $G_{2n}^{(22)}(x)$ and $G_{2n}^{(23)}(x)$, it follows from (A2)(ii) and (A3) that for any $x\in(0,1)$,

$$\int_{-\infty}^{(x-1)/h_{1n}}|K(\nu)|\,d\nu\xrightarrow[n\to\infty]{}0\quad\text{and}\quad\int_{x/h_{1n}}^{\infty}|K(\nu)|\,d\nu\xrightarrow[n\to\infty]{}0,$$

which implies that

$$G_{2n}^{(2j)}(x)\xrightarrow[n\to\infty]{}0,\quad j=2,3. \tag{5.7}$$

From (5.4)–(5.7), we derive

$$G_{2n}(x)\xrightarrow[n\to\infty]{}0. \tag{5.8}$$

Hence, we attain (2.1) according to (5.1), (5.3) and (5.8). ∎

5.2. The proof of Theorem 2.2. For any $u\in(0,1)$, we have

$$\begin{aligned}
&E|\widehat{f}_n(u)-f(u)|^r \qquad (5.9)\\
&=E\Big|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\Big(\frac{u-s}{h_{2n}}\Big)\,ds\,\big(f^{1/2}(u_{ni})\varepsilon_{ni}+g(x_{ni})-\widehat{g}_n(x_{ni})\big)^2-f(u)\Big|^r\\
&\leqslant 3^{r-1}E\Big|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\Big(\frac{u-s}{h_{2n}}\Big)\,ds\,f(u_{ni})\varepsilon_{ni}^2-f(u)\Big|^r\\
&\quad+3^{r-1}E\Big|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\Big(\frac{u-s}{h_{2n}}\Big)\,ds\,(g(x_{ni})-\widehat{g}_n(x_{ni}))^2\Big|^r\\
&\quad+3^{r-1}E\Big|2h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\Big(\frac{u-s}{h_{2n}}\Big)\,ds\,f^{1/2}(u_{ni})\varepsilon_{ni}(g(x_{ni})-\widehat{g}_n(x_{ni}))\Big|^r\\
&=:3^{r-1}F_{1n}(u)+3^{r-1}F_{2n}(u)+3^{r-1}F_{3n}(u).
\end{aligned}$$

Firstly, we deal with $F_{1n}(u)$. Referring to the proof of Theorem 2.1, we make the following transformations: ε_{ni}, x_{ni}, h_{1n} and $f^{1/2}(u_{ni})$ are replaced by $(\varepsilon_{ni}^2-E\varepsilon_{ni}^2)$, u_{ni}, h_{2n} and $f(u_{ni})$ in (5.2), respectively, and at the same time, x_{ni}, h_{1n}, $g(x_{ni})$ and $g(x)$ are replaced by u_{ni}, h_{2n}, $f(u_{ni})$ and $f(u)$ in (5.4), respectively. Repeating the proofs of (5.2) and (5.4)–(5.8) in Section 5.1, we deduce that for $r>2$,

$$\begin{aligned}
&E\Big|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\Big(\frac{u-s}{h_{2n}}\Big)\,ds\,f(u_{ni})\varepsilon_{ni}^2-f(u)\Big|^r \qquad (5.10)\\
&\qquad\leqslant 2^{r-1}E\Big|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\Big(\frac{u-s}{h_{2n}}\Big)\,ds\,f(u_{ni})(\varepsilon_{ni}^2-E\varepsilon_{ni}^2)\Big|^r\\
&\qquad\quad+2^{r-1}\Big|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\Big(\frac{u-s}{h_{2n}}\Big)\,ds\,f(u_{ni})-f(u)\Big|^r\xrightarrow[n\to\infty]{}0,
\end{aligned}$$

which yields

$$F_{1n}(u) \xrightarrow[n\to\infty]{} 0. \tag{5.11}$$

Next, we consider $F_{2n}(u)$. We get

$$\begin{aligned}
&E\left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\,(g(x_{ni})-\widehat{g}_n(x_{ni}))^2\right|^r \qquad (5.12)\\
&= E\left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\,\big(g(x_{ni})-E\widehat{g}_n(x_{ni})+E\widehat{g}_n(x_{ni})-\widehat{g}_n(x_{ni})\big)^2\right|^r\\
&\leqslant 2^{r-1}\left(h_{2n}^{-1}\sum_{i=1}^{n}\left|\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\right|\right.\\
&\qquad\qquad \left.\times\left(g(x_{ni})-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}} g(x_{nk})K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds\right)^2\right)^r\\
&\quad + 2^{r-1}E\left(h_{2n}^{-1}\sum_{i=1}^{n}\left|\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\right|\right.\\
&\qquad\qquad \left.\times\left(h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds\, f^{1/2}(u_{nk})\varepsilon_{nk}\right)^2\right)^r\\
&=: 2^{r-1}T_1(u)+2^{r-1}T_2(u) =: T_1+T_2.
\end{aligned}$$

Note that

$$\begin{aligned}
T_1 &\leqslant C\sup_{x\in(0,1)}\left|g(x)-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}} g(x_{nk})K\left(\frac{x-s}{h_{1n}}\right) ds\right|^{2r} \qquad (5.13)\\
&\qquad\times\left(h_{2n}^{-1}\sum_{i=1}^{n}\left|\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\right|\right)^r\\
&\leqslant CK_1^r\sup_{x\in(0,1)}\left|g(x)-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}} g(x_{nk})K\left(\frac{x-s}{h_{1n}}\right) ds\right|^{2r}
\end{aligned}$$

and

$$\begin{aligned}
T_2 &\leqslant CE\max_{1\leqslant i\leqslant n}\left|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds\, f^{1/2}(u_{nk})\varepsilon_{nk}\right|^{2r} \qquad (5.14)\\
&\qquad\times\left(h_{2n}^{-1}\sum_{i=1}^{n}\left|\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\right|\right)^r\\
&\leqslant CK_1^rE\max_{1\leqslant i\leqslant n}\left|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds\, f^{1/2}(u_{nk})\varepsilon_{nk}\right|^{2r}.
\end{aligned}$$

The decomposition is as follows:

$$
\begin{aligned}
(5.15)\qquad &\sup_{x\in(0,1)}\left|g(x)-h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}}g(x_{ni})K\left(\frac{x-s}{h_{1n}}\right)ds\right| \\
&=\sup_{x\in(0,1)}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}}g(x_{ni})K\left(\frac{x-s}{h_{1n}}\right)ds-h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}}g(s)K\left(\frac{x-s}{h_{1n}}\right)ds\right. \\
&\qquad\left.+h_{1n}^{-1}\int_{-\infty}^{\infty}g(s)K\left(\frac{x-s}{h_{1n}}\right)ds-g(x)\right| \\
&=\sup_{x\in(0,1)}\left|h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}}(g(x_{ni})-g(s))K\left(\frac{x-s}{h_{1n}}\right)ds\right. \\
&\qquad\left.+\int_{-\infty}^{\infty}g(x-uh_{1n})K(u)\,du-g(x)\int_{-\infty}^{\infty}K(u)\,du\right|,
\end{aligned}
$$

which is less than or equal to

$$
\begin{aligned}
(5.16)\qquad &\sup_{x\in(0,1)}\left(h_{1n}^{-1}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}}|g(x_{ni})-g(s)|\left|K\left(\frac{x-s}{h_{1n}}\right)\right|ds\right. \\
&\qquad\left.+\int_{-\infty}^{\infty}|g(x-uh_{1n})-g(x)|\,|K(u)|\,du\right) \\
&\leqslant\sup_{x\in(0,1)}\left(\frac{C\delta_{1n}}{h_{1n}}\sum_{i=1}^{n}\int_{x_{n,i-1}}^{x_{ni}}\left|K\left(\frac{x-s}{h_{1n}}\right)\right|ds\right. \\
&\qquad+\int_{|u|<\zeta/h_{1n}}|g(x-uh_{1n})-g(x)|\,|K(u)|\,du \\
&\qquad\left.+C\int_{|u|\geqslant\zeta/h_{1n}}|g(x-uh_{1n})-g(x)|\,|K(u)|\,du\right) \\
&\leqslant\sup_{x\in(0,1)}\left(\frac{C\delta_{1n}}{h_{1n}}\sum_{i=1}^{n}(x_{ni}-x_{n,i-1})\right. \\
&\qquad\left.+K_1\sup_{|u|<\zeta/h_{1n}}|g(x-uh_{1n})-g(x)|+C\int_{|u|\geqslant\zeta/h_{1n}}|K(u)|\,du\right) \\
&\leqslant\frac{C\delta_{1n}}{h_{1n}}+C\sup_{x\in(0,1)}\Big(\sup_{|u|<\zeta/h_{1n}}|g(x-uh_{1n})-g(x)|\Big)+C\int_{|u|\geqslant\zeta/h_{1n}}|K(u)|\,du\to0,
\end{aligned}
$$

by letting $\zeta=\zeta_n\to0$ such that $\zeta_n/h_{1n}\to\infty$ as $n\to\infty$.

Taking $0<\epsilon<\theta_1(2r-1)-1$, by Lemma 4.1 and $\theta_1>\frac{1}{2}$ $(>\frac{1}{r}>\frac{1}{2r-1})$ we get

$$
\begin{aligned}
(5.17)\qquad &E\max_{1\leqslant i\leqslant n}\left|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}K\left(\frac{x_{ni}-s}{h_{1n}}\right)ds\,f^{1/2}(u_{nk})\varepsilon_{nk}\right|^{2r} \\
&\qquad\leqslant\sum_{i=1}^{n}E\left|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}K\left(\frac{x_{ni}-s}{h_{1n}}\right)ds\,f^{1/2}(u_{nk})\varepsilon_{nk}\right|^{2r}
\end{aligned}
$$

$$\leqslant \sum_{i=1}^{n} Kn^{\epsilon} \sum_{k=1}^{n} \left| h_{1n}^{-1} \sum_{k=1}^{n} \int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds\, f^{1/2}(u_{nk}) \right|^{2r} E|\varepsilon_{nk}|^{2r}$$
$$+ \sum_{i=1}^{n} K\left(\sum_{k=1}^{n} \left| h_{1n}^{-1} \int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds\, f^{1/2}(u_{nk}) \right|^{2} (E|\varepsilon_{nk}|^{2r+\delta})^{\frac{2}{2r+\delta}} \right)^{r},$$

which is less than or equal to

$$\sum_{i=1}^{n} CKn^{\epsilon} \sum_{k=1}^{n} \left| h_{1n}^{-1} \sum_{k=1}^{n} \int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds \right| \times \left| \int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds \right|^{2r-1} E|\varepsilon_{nk}|^{2r}$$
$$+ \sum_{i=1}^{n} CK\left(\sum_{k=1}^{n} \left| h_{1n}^{-1} \int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds \right| \times \left| h_{1n}^{-1} \int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds \right| (E|\varepsilon_{nk}|^{2r+\delta})^{\frac{2}{2r+\delta}} \right)^{r}$$
$$\leqslant CKK_1 n^{\epsilon} n^{1-\theta_1(2r-1)} E|\vartheta_1|^{2r} + CKK_1^{r} n^{1-\theta_1 r} (E|\vartheta_1|^{2r+\delta})^{\frac{2r}{2r+\delta}} \xrightarrow[n\to\infty]{} 0. \tag{5.18}$$

Therefore, we conclude from (5.12)–(5.18) that

$$F_{2n}(u) \xrightarrow[n\to\infty]{} 0. \tag{5.19}$$

For $F_{3n}(u)$, we have

$$F_{3n}(u) \tag{5.20}$$
$$= 2E \left| h_{2n}^{-1} \sum_{i=1}^{n} \int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f^{1/2}(u_{ni}) \varepsilon_{ni} (g(x_{ni}) - \widehat{g}_n(x_{ni})) \right|^{r}$$
$$\leqslant 2^r E \left| h_{2n}^{-1} \sum_{i=1}^{n} \int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f^{1/2}(u_{ni}) \varepsilon_{ni} \times \left(g(x_{ni}) - h_{1n}^{-1} \sum_{k=1}^{n} \int_{x_{n,k-1}}^{x_{nk}} g(x_{nk}) K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds \right) \right|^{r}$$
$$+ 2^r E \left| h_{2n}^{-1} \sum_{i=1}^{n} \int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f^{1/2}(u_{ni}) \varepsilon_{ni} \times \left(h_{1n}^{-1} \sum_{k=1}^{n} \int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds\, f^{1/2}(u_{nk}) \varepsilon_{nk} \right) \right|^{r}$$
$$=: 2^r I_1(u) + 2^r I_2(u) =: I_1 + I_2.$$

Now we deal with I_1 and I_2. By Lemmas 4.1 and 4.3, we obtain

$$
\begin{aligned}
&E\bigg|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\,f^{1/2}(u_{ni})\varepsilon_{ni}\\
&\qquad\qquad\times\bigg(g(x_{ni})-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}g(x_{nk})K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\bigg)\bigg|^r\\
&=E\bigg|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\,f^{1/2}(u_{ni})\\
&\qquad\qquad\times\bigg(g(x_{ni})-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}g(x_{nk})K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\bigg)\varepsilon_{ni}\bigg|^r\\
&\leqslant Kn^{\epsilon}\sum_{i=1}^{n}\bigg|Ch_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\,f^{1/2}(u_{ni})\\
&\qquad\qquad\times\bigg(g(x_{ni})-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}g(x_{nk})K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\bigg)\bigg|^r E|\varepsilon_{ni}|^r\\
&\quad+K\bigg(\sum_{i=1}^{n}\bigg|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\\
&\qquad\times\bigg(g(x_{ni})-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}g(x_{nk})K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\bigg)\bigg|^2(E|\varepsilon_{ni}|^{r+\delta})^{\frac{2}{r+\delta}}\bigg)^{r/2}\\
&=:I_{11}+I_{12}.
\end{aligned}
$$

On the one hand, taking $0<\epsilon\leqslant\theta_2(r-1)$ it follows by (5.16) that

$$
\begin{aligned}
I_{11}&\leqslant CKn^{\epsilon}\sum_{i=1}^{n}h_{2n}^{-1}\bigg|\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\bigg|\max_{1\leqslant i\leqslant n}\bigg|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\bigg|^{r-1}\\
&\qquad\times\bigg|g(x_{ni})-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}g(x_{nk})K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\bigg|^r E|\varepsilon_{ni}|^r\\
&\leqslant CKn^{\epsilon}\max_{1\leqslant i\leqslant n}\bigg|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\bigg|^{r-1}\sum_{i=1}^{n}h_{2n}^{-1}\bigg|\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\bigg|\\
&\qquad\times\sup_{x\in(0,1)}\bigg|g(x)-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}g(x_{nk})K\bigg(\frac{x-s}{h_{1n}}\bigg)\,ds\bigg|^r E|\vartheta_1|^r\\
&\leqslant CKK_1n^{\epsilon}n^{-\theta_2(r-1)}\\
&\qquad\times\sup_{x\in(0,1)}\bigg|g(x)-h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}g(x_{nk})K\bigg(\frac{x-s}{h_{1n}}\bigg)\,ds\bigg|^r E|\vartheta_1|^r\\
&\xrightarrow[n\to\infty]{}0.
\end{aligned}
$$

On the other hand, it follows by (5.16) that

$$\begin{aligned}
I_{12} &\leqslant K\biggl(h_{2n}^{-1}\sum_{i=1}^{n}\biggl|\int\limits_{u_{n,i-1}}^{u_{ni}} K\biggl(\frac{u-s}{h_{2n}}\biggr)\,ds\biggr| \max_{1\leqslant i\leqslant n}\biggl|h_{2n}^{-1}\int\limits_{u_{n,i-1}}^{u_{ni}} K\biggl(\frac{u-s}{h_{2n}}\biggr)\,ds\biggr| \\
&\quad\times\biggl(g(x_{ni})-h_{1n}^{-1}\sum_{k=1}^{n}\int\limits_{x_{n,k-1}}^{x_{nk}} g(x_{nk})K\biggl(\frac{x_{ni}-s}{h_{1n}}\biggr)\,ds\biggr)^2 (E|\varepsilon_{ni}|^{r+\delta})^{\frac{2}{r+\delta}}\biggr)^{r/2} \\
&\leqslant KK_1^{r/2}n^{-\theta_2 r/2} \\
&\quad\times\sup_{x\in(0,1)}\biggl|g(x)-h_{1n}^{-1}\sum_{k=1}^{n}\int\limits_{x_{n,k-1}}^{x_{nk}} g(x_{nk})K\biggl(\frac{x-s}{h_{1n}}\biggr)\,ds\biggr|^r (E|\vartheta_1|^{r+\delta})^{\frac{r}{r+\delta}} \\
&\xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Thus we have

$$I_1 \xrightarrow[n\to\infty]{} 0. \tag{5.21}$$

As for I_2, it follows by the Cauchy inequality that

$$\begin{aligned}
&E\biggl|h_{2n}^{-1}\sum_{i=1}^{n}\int\limits_{u_{n,i-1}}^{u_{ni}} K\biggl(\frac{u-s}{h_{2n}}\biggr)\,ds\, f^{1/2}(u_{ni})\varepsilon_{ni} \\
&\qquad\qquad\times\biggl(h_{1n}^{-1}\sum_{k=1}^{n}\int\limits_{x_{n,k-1}}^{x_{nk}} K\biggl(\frac{x_{ni}-s}{h_{1n}}\biggr)\,ds\, f^{1/2}(u_{nk})\varepsilon_{nk}\biggr)\biggr|^r \\
&\leqslant E\biggl(\sum_{i=1}^{n}|h_{2n}^{-1}\int\limits_{u_{n,i-1}}^{u_{ni}} K\biggl(\frac{u-s}{h_{2n}}\biggr)\,ds\, f^{1/2}(u_{ni})\varepsilon_{ni}|^2 \\
&\qquad\qquad\times\sum_{i=1}^{n}\biggl|h_{1n}^{-1}\sum_{k=1}^{n}\int\limits_{x_{n,k-1}}^{x_{nk}} K\biggl(\frac{x_{ni}-s}{h_{1n}}\biggr)\,ds\, f^{1/2}(u_{nk})\varepsilon_{nk}\biggr|^2\biggr)^{r/2} \\
&\leqslant\biggl\{E\biggl(\sum_{i=1}^{n}\biggl|h_{2n}^{-1}\int\limits_{u_{n,i-1}}^{u_{ni}} K\biggl(\frac{u-s}{h_{2n}}\biggr)\,ds\, f^{1/2}(u_{ni})\varepsilon_{ni}\biggr|^2\biggr)^r \\
&\qquad\qquad\times E\biggl(\sum_{i=1}^{n}\biggl|h_{1n}^{-1}\sum_{k=1}^{n}\int\limits_{x_{n,k-1}}^{x_{nk}} K\biggl(\frac{x_{ni}-s}{h_{1n}}\biggr)\,ds\, f^{1/2}(u_{nk})\varepsilon_{nk}\biggr|^2\biggr)^r\biggr\}^{1/2}.
\end{aligned} \tag{5.22}$$

Note that $1/2<\theta_1,\theta_2<1$. For a sufficiently small positive number

$$0<\epsilon_1<\theta_2(2r-1)+\theta_1 r-r \quad (<2\theta_1(2r-1)-r),$$

from (5.18) and $I_{11}\to 0$ we get

$$
\begin{aligned}
(5.23)\quad & E\Bigg(\sum_{i=1}^{n}\bigg|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\,f^{1/2}(u_{ni})\varepsilon_{ni}\bigg|^2\Bigg)^r\\
&\leqslant 2^{r-1}E\Bigg\{\sum_{i=1}^{n}\Bigg(\bigg|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\,f^{1/2}(u_{ni})\varepsilon_{ni}\bigg|^2\\
&\qquad\qquad - E\bigg|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\,f^{1/2}(u_{ni})\varepsilon_{ni}\bigg|^2\Bigg)\Bigg\}^r\\
&\quad + 2^{r-1}\Bigg(\sum_{i=1}^{n}\bigg|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}}K\bigg(\frac{u-s}{h_{2n}}\bigg)\,ds\,f^{1/2}(u_{ni})\bigg|^2E|\varepsilon_{ni}|^2\Bigg)^r\\
&\leqslant (CKK_1n^{\epsilon_1}n^{-\theta_2(2r-1)}+CK_1^rn^{-\theta_2r})E|\vartheta_1|^{2r}\\
&\quad + CKK_1^rn^{-3\theta_2r/2}(E|\vartheta_1|^{2(r+\delta)})^{\frac{r}{r+\delta}}\\
&= O(n^{\epsilon_1-\theta_2(2r-1)}n^{-\theta_2r}+n^{-3\theta_2r/2}).
\end{aligned}
$$

In order to prove the desired result, we argue as follows. Firstly, taking $0<\epsilon<\min\{(\theta_1+\theta_2)(2r-1)-r-\epsilon_1,\theta_1(2r-1)+\theta_2r-r\}\leqslant(\theta_1+\theta_2)(2r-1)-r-\epsilon_1$, by (5.18) we get

$$
\begin{aligned}
(5.24)\quad & Cn^{\epsilon_1-\theta_2(2r-1)}\\
&\quad\times E\Bigg(\sum_{i=1}^{n}\bigg|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\,f^{1/2}(u_{nk})\varepsilon_{nk}\bigg|^2\Bigg)^r\\
&\leqslant Cn^{\epsilon_1-\theta_2(2r-1)}n^{r-1}\cdot\sum_{i=1}^{n}E\bigg|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\,f^{1/2}(u_{nk})\varepsilon_{nk}\bigg|^{2r}\\
&\leqslant CKK_1n^{\epsilon_1-\theta_2(2r-1)}n^{r-1+\epsilon+1-\theta_1(2r-1)}E|\vartheta_1|^{2r}\\
&\quad + CKK_1^rn^{r-\theta_1r+\epsilon_1-\theta_2(2r-1)}(E|\vartheta_1|^{2r+\delta})^{\frac{2r}{2r+\delta}}=o(1).
\end{aligned}
$$

Secondly, taking $0<\epsilon<\min\{(\theta_1+\theta_2)(2r-1)-r-\epsilon_1,\theta_1(2r-1)+\theta_2r-r\}\leqslant\theta_1(2r-1)+\theta_2r-r$, we find analogously that

$$
\begin{aligned}
(5.25)\quad & Cn^{-\theta_2r}E\Bigg(\sum_{i=1}^{n}\bigg|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\,f^{1/2}(u_{nk})\varepsilon_{nk}\bigg|^2\Bigg)^r\\
&\leqslant Cn^{-\theta_2r}n^{r-1}\sum_{i=1}^{n}E\bigg|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}}K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\,f^{1/2}(u_{nk})\varepsilon_{nk}\bigg|^{2r}\\
&\leqslant CKK_1n^{\epsilon}n^{r-\theta_1(2r-1)-\theta_2r}E|\vartheta_1|^{2r}\\
&\quad + CKK_1^rn^{r-\theta_1r-\theta_2r}(E|\vartheta_1|^{2r+\delta})^{\frac{2r}{2r+\delta}}=o(1).
\end{aligned}
$$

Lastly, taking $0<\epsilon<\min\{(\theta_1+\theta_2)(2r-1)-r-\epsilon_1,\theta_1(2r-1)+\theta_2r-r\}<$

$\theta_1(2r-1)+3\theta_2 r/2-r$, we conclude analogously that

$$
\begin{aligned}
(5.26)\quad & Cn^{-3\theta_2 r/2}E\Bigg(\sum_{i=1}^{n}\bigg|h_{1n}^{-1}\sum_{k=1}^{n}\int\limits_{x_{n,k-1}}^{x_{nk}}K\bigg(\frac{x_{ni}-s}{h_{1n}}\bigg)\,ds\,f^{1/2}(u_{nk})\varepsilon_{nk}\bigg|^2\Bigg)^r \\
&\leqslant CKK_1 n^{\epsilon}n^{r-\theta_1(2r-1)-3\theta_2 r/2}E|\vartheta_1|^{2r} \\
&\quad + CKK_1^r n^{r-\theta_1 r-3\theta_2 r/2}(E|\vartheta_1|^{2r+\delta})^{\frac{2r}{2r+\delta}} = o(1).
\end{aligned}
$$

Then (5.22)–(5.26) yield

$$
(5.27)\qquad I_2 \xrightarrow[n\to\infty]{} 0.
$$

Therefore, we deduce from (5.20), (5.21) and (5.27) that

$$
(5.28)\qquad F_{3n}(u) \xrightarrow[n\to\infty]{} 0.
$$

By (5.9), (5.11), (5.19) and (5.28), the proof is complete. ∎

5.3. The proof of Theorem 2.3. For any $x\in(0,1)$, we have

$$
\begin{aligned}
(5.29)\quad |\widehat{g}_n(x)-g(x)| &\leqslant |\widehat{g}_n(x)-E\widehat{g}_n(x)|+|E\widehat{g}_n(x)-g(x)| \\
&= \bigg|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}K\bigg(\frac{x-s}{h_{1n}}\bigg)\,ds\,f^{1/2}(u_{ni})\varepsilon_{ni}\bigg| \\
&\quad + \bigg|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}g(x_{ni})K\bigg(\frac{x-s}{h_{1n}}\bigg)\,ds-g(x)\bigg|.
\end{aligned}
$$

Thus we only need to prove that

$$
(5.30)\qquad \bigg|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}K\bigg(\frac{x-s}{h_{1n}}\bigg)\,ds\,f^{1/2}(u_{ni})\varepsilon_{ni}\bigg| \xrightarrow[n\to\infty]{\text{a.s.}} 0,
$$

and

$$
(5.31)\qquad \bigg|h_{1n}^{-1}\sum_{i=1}^{n}\int\limits_{x_{n,i-1}}^{x_{ni}}g(x_{ni})K\bigg(\frac{x-s}{h_{1n}}\bigg)\,ds-g(x)\bigg| \xrightarrow[n\to\infty]{} 0.
$$

These follow by Lemma 4.6 and (5.8). ∎

5.4. The proof of Theorem 2.4. For any $u \in (0,1)$, from (5.9) and (5.10) we have

$$
\begin{aligned}
(5.32)\quad & |\widehat{f}_n(u) - f(u)| \\
&\leqslant 3^{r-1}\cdot 2^{r-1}\left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f(u_{ni})(\varepsilon_{ni}^2 - E\varepsilon_{ni}^2)\right| \\
&\quad + 3^{r-1}\cdot 2^{r-1}\left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f(u_{ni}) - f(u)\right| \\
&\quad + 3^{r-1}\left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds (g(x_{ni}) - \widehat{g}_n(x_{ni}))^2\right| \\
&\quad + 3^{r-1}\left|2h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f^{1/2}(u_{ni})\varepsilon_{ni}(g(x_{ni}) - \widehat{g}_n(x_{ni}))\right|.
\end{aligned}
$$

Thus we only need to prove that

$$
(5.33)\qquad \left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f(u_{ni})(\varepsilon_{ni}^2 - E\varepsilon_{ni}^2)\right| \xrightarrow[n\to\infty]{\text{a.s.}} 0,
$$

$$
(5.34)\qquad \left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f(u_{ni}) - f(u)\right| \xrightarrow[n\to\infty]{} 0,
$$

$$
(5.35)\qquad \left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, (g(x_{ni}) - \widehat{g}_n(x_{ni}))^2\right| \xrightarrow[n\to\infty]{\text{a.s.}} 0,
$$

and

$$
(5.36)\qquad \left|h_{2n}^{-1}\sum_{i=1}^{n}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right) ds\, f^{1/2}(u_{ni})\varepsilon_{ni}(g(x_{ni}) - \widehat{g}_n(x_{ni}))\right| \xrightarrow[n\to\infty]{\text{a.s.}} 0.
$$

Firstly, we get (5.33) from Lemma 4.7. Secondly, (5.34) follows from (5.10). Thirdly, by (5.16) and Lemma 4.5,

$$
\begin{aligned}
(5.37)\quad & \max_{1\leqslant i\leqslant n} |g(x_{ni}) - \widehat{g}_n(x_{ni})| \\
&\leqslant \sup_{x\in(0,1)}\left|g(x) - h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}} g(x_{nk})K\left(\frac{x-s}{h_{1n}}\right) ds\right| \\
&\quad + \max_{1\leqslant i\leqslant n}\left|h_{1n}^{-1}\sum_{k=1}^{n}\int_{x_{n,k-1}}^{x_{nk}} K\left(\frac{x_{ni}-s}{h_{1n}}\right) ds\, f^{1/2}(u_{nk})\varepsilon_{nk}\right| \\
&\xrightarrow[n\to\infty]{\text{a.s.}} 0,
\end{aligned}
$$

which combined with Lemma 4.3 shows that the left side of (5.35) is less than or equal to

$$\max_{1\leqslant i\leqslant n}(g(x_{ni})-\widehat{g}_n(x_{ni}))^2 h_{2n}^{-1}\sum_{i=1}^{n}\left|\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right)ds\right| \xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{5.38}$$

Lastly, note that the left side of (5.36) is less than

$$\max_{1\leqslant i\leqslant n}|g(x_{ni})-\widehat{g}_n(x_{ni})|h_{2n}^{-1}\sum_{i=1}^{n}\left|\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right)ds\, f^{1/2}(u_{ni})\varepsilon_{ni}\right| \tag{5.39}$$

and

$$\begin{aligned}\sum_{i=1}^{n}&\left|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right)ds\, f^{1/2}(u_{ni})\right||\varepsilon_{ni}| \\ &=\sum_{i=1}^{n}\left|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right)ds\, f^{1/2}(u_{ni})\right|(|\varepsilon_{ni}|-E|\varepsilon_{ni}|) \\ &\quad+\sum_{i=1}^{n}\left|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right)ds\, f^{1/2}(u_{ni})\right|E|\varepsilon_{ni}|.\end{aligned} \tag{5.40}$$

Clearly, by Lemmas 4.3 and 4.8 respectively it follows that

$$\begin{aligned}\sum_{i=1}^{n}&\left|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right)ds\, f^{1/2}(u_{ni})\right|E|\varepsilon_{ni}| \\ &=\sum_{i=1}^{n}\left|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right)ds\, f^{1/2}(u_{ni})\right|E|\vartheta_i| \\ &\leqslant CK_1E|\vartheta_1|<\infty\end{aligned} \tag{5.41}$$

and

$$\sum_{i=1}^{n}\left|h_{2n}^{-1}\int_{u_{n,i-1}}^{u_{ni}} K\left(\frac{u-s}{h_{2n}}\right)ds\, f^{1/2}(u_{ni})\right|(|\varepsilon_{ni}|-E|\varepsilon_{ni}|)\xrightarrow[n\to\infty]{\text{a.s.}} 0. \tag{5.42}$$

Combining these with (5.39), (5.40) and (5.37) yields (5.36). The proof is complete. ∎

REFERENCES

[1] P. Asghari and V. Fakoor, *A Berry–Esseen type bound for the kernel density estimator based on a weakly dependent and randomly left truncated data*, J. Inequal. Appl. 2017, art. 1, 19 pp.

[2] K. Benhenni, M. Rachdi and Y. C. Su, *The effect of the regularity of the error process on the performance of kernel regression estimators*, Metrika 76 (2013), 765–781.
[3] X. Chen, *Asymptotic properties for estimates of nonparametric regression model with martingale difference errors*, Statistics 46 (2012), 687–696.
[4] J. Dedecker and F. Merlevède, *Almost sure invariance principle for the Kantorovich distance between the empirical and the marginal distributions of strong mixing sequences*, Statist. Probab. Lett. 171 (2021), art. 108991, 9 pp.
[5] L. W. Ding and Y. M. Li, *Berry–Esseen bounds of weighted kernel estimator for a nonparametric regression model based on linear process errors under a LNQD sequence*, J. Inequal. Appl. 2018, art. 10, 12 pp.
[6] M. Ekström, *A general central limit theorem for strong mixing sequences*, Statist. Probab. Lett. 94 (2014), 236–238.
[7] J. Q. Fan and Q. W. Yao, *Nonlinear Time Series: Nonparametric and Parametric Methods*, Springer, New York, 2003.
[8] T. Gasser and H. G. Müller, *Kernel estimation of regression functions*, in: Smoothing Techniques for Curve Estimation, Springer, Heidelberg, 1979, 23–68.
[9] A. A. Georgiev and W. Greblicki. *Nonparametric function recovering from noisy observations*, J. Statist. Planning Inference 13 (1986), 1–14.
[10] H. Y. Liang, *Asymptotic normality of wavelet estimator in heteroscedastic model with α-mixing errors*, J. Systems Sci. Complexity 24 (2011), 725–737.
[11] H. Y. Liang and B. Y. Jing, *Asymptotic properties for estimates of nonparametric regression models based on negatively associated sequences*, J. Multivariate Anal. 95 (2005), 227–245.
[12] H. Y. Liang and Y. M. Liu, *Asymptotic normality of variance estimator in a heteroscedastic model with dependent errors*, J. Nonparametric Statist. 23 (2011), 351–365.
[13] H. Y. Liang and Y. Lu, *Asymptotic normality of wavelet estimator in heteroscedastic regression model*, Appl. Math. J. Chinese Univ. 22 (2007), 453–457.
[14] E. Liebscher, *Estimation of the density and the regression function under mixing conditions*, Statistics Decisions 19 (2001), 9–26.
[15] H. G. Müller, *Weak and universal consistency of moving weighted averages*, Period. Math. Hungar. 18 (1987), 241–250.
[16] M. B. Priestley and M. T. Chao, *Non-parametric function fitting*, J. Roy. Statist. Soc. Ser. B 34 (1972), 385–392.
[17] M. Rosenblatt, *A central limit theorem and a strong mixing condition*, Proc. Nat. Acad. Sci. USA 42 (1956), 43–47.
[18] L. Tabacu, *Weak convergence of the linear rank statistics under strong mixing conditions*, Statist. Probab. Lett. 132 (2018), 28–34.
[19] L. V. Thanh and G. Yin, *Weighted sums of strongly mixing random variables with an application to nonparametric regression*, Statist. Probab. Lett. 105 (2015), 195–202.
[20] R. Wang and A. T. Shen, *Consistency properties for the nearest neighbour estimator of the density function and applications based on α-mixing samples*, Statistics 56 (2022), 91–1011.
[21] X. J. Wang, Y. Wu and W. Wang, *Nonparametric estimation of expected shortfall for α-mixing financial losses*, Comput. Statist. 39 (2024), 3157–3179.
[22] Y. Wang, X. J. Wang and A. T. Shen, *Asymptotic properties for generalized edge frequency polygon of nonparametric density estimation*, J. Statist. Comput. Simulation 94 (2024), 4210–4233.
[23] Y. Wu, T. C. Hu, A. Volodin and X. J. Wang, *Some improved results on Berry–Esseen bounds for strong mixing random variables and applications*, Statistics 57 (2023), 740–760.
[24] Q. Yang, Y. N. Li and Y. Zhang, *Change point detection for nonparametric regression under strongly mixing process*, Statist. Papers 61 (2020), 1465–1506.
[25] S. C. Yang, *Maximal moment inequality for partial sums of strong mixing sequences and application*, Acta Math. Sinica (English Ser.) 23 (2007), 1013–1024.

[26] X. T. Yang and S. C. Yang, *Strong consistency of integral weight regression estimator for α-mixing samples*, J. Math. Wuhan Univ. 39 (2019), 878–888.

[27] J. J. Zhang and H. Y. Liang, *Asymptotic normality of estimators in heteroscedastic semi-parametric model with strong mixing errors*, Comm. Statist. Theory Methods 41 (2012), 2172–2201.

Yan Wang
School of Big Data and Artificial Intelligence
Chizhou University
Chizhou, 247000, P. R. China
and
School of Big Data and Statistics
Anhui University
Hefei, 230601, P. R. China
E-mail: wyahdx@126.com

Xuejun Wang
School of Big Data and Statistics
Anhui University
Hefei, 230601, P. R. China
E-mail: wxjahdx@126.com

Aiting Shen (corresponding author)
School of Big Data and Statistics
Anhui University
Hefei, 230601, P. R. China
E-mail: empress201010@126.com

Received 13.10.2024;
accepted 6.6.2025