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ON INTERMEDIATE LEVELS OF A NESTED OCCUPANCY SCHEME IN RANDOM ENVIRONMENT GENERATED BY STICK-BREAKING: THE CASE OF HEAVY TAILS

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Abstract. We investigate a nested balls-in-boxes scheme in a random environment. The boxes follow a nested hierarchy, with infinitely many boxes in each level, and the hitting probabilities of boxes are random and obtained by iterated fragmentation of a unit mass. The hitting probabilities of the first-level boxes are given by a stick-breaking model $P_k = W_1 W_2 \cdot \ldots \cdot W_{k-1}(1 - W_k)$ for $k \in \mathbb{N}$, where W_1, $W_2, \ldots$ are independent copies of a random variable W taking values in $(0, 1)$. The infinite balls-in-boxes scheme in the first level is known as a Bernoulli sieve. We assume that the mean of $|\log W|$ is infinite and the distribution tail of $|\log W|$ is regularly varying at ∞. Denote by $K_n(j)$ the number of occupied boxes in the jth level provided that there are n balls and call the level j intermediate if $j = j_n \to \infty$ and $j_n = o((\log n)^a)$ as $n \to \infty$ for some $a > 0$. We prove that, for some intermediate levels j, finite-dimensional distributions of the process $(K_n(\lfloor j_n u \rfloor))_{u>0}$, properly normalized, converge weakly as $n \to \infty$ to those of a pathwise Lebesgue–Stieltjes integral, with the integrand being an exponential function and the integrator being an inverse stable subordinator. The present paper continues the line of investigation initiated in the articles of Buraczewski, Dovgay and Iksanov (2020) and Iksanov, Marynych and Samoilenko (2022) in which the random variable $|\log W|$ has a finite second moment, and of Iksanov, Marynych and Rashytov (2022) in which $|\log W|$ has a finite mean and an infinite second moment.

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1. INTRODUCTION

1.1. Definition of the model. Let $P_1, P_2, \ldots$ be nonnegative random variables with an arbitrary joint distribution satisfying $\sum_{k\geq 1} P_k = 1$ a.s. The sequence $(P_k)_{k\geq 1}$ is interpreted as a *random environment*. An *occupancy scheme in random environ-*

ment is defined as follows. Conditionally on $(P_k)_{k\geq 1}$, balls are allocated independently in an infinite array of boxes $1, 2, \ldots$ with probability P_k of hitting box k. The occupancy scheme in random environment is called *infinite* if the number of positive probabilities is infinite a.s. One of the most popular infinite occupancy schemes in random environment is called a *Bernoulli sieve*. It corresponds to $(P_k)_{k\geq 1}$ which follow a stick-breaking model

$$(1.1) \qquad P_k = W_1 W_2 \cdot \ldots \cdot W_{k-1}(1 - W_k), \quad k \in \mathbb{N} := \{1, 2, \ldots\},$$

where $W_1, W_2, \ldots$ are independent copies of a random variable W taking values in $(0, 1)$. The Bernoulli sieve was introduced in [15] and further investigated in many articles. Surveys of earlier works can be found in [14, 18]. An incomplete list of more recent contributions includes [1, 12, 19, 28, 29, 30].

A *nested infinite occupancy scheme in random environment* is a hierarchical generalization of the infinite occupancy scheme in random environment which is defined by producing, in a consistent way, a sequence of occupancy schemes in random environment on the tree of a weighted branching process with positive weights (a multiplicative counterpart of a branching random walk). The scheme was introduced in [3] and further investigated in [8, 9, 13, 20, 23, 25]. Following [3, 13], we now provide the definition of a nested occupancy scheme in random environment. To this end, some preparations are needed. Let $\mathbb{V} := \bigcup_{n\geq 0} \mathbb{N}^n$ be the set of all possible individuals of some population. The initial ancestor is identified with the empty word $\oslash$ and its weight is $P(\oslash) = 1$. On some probability space, let $((P_k(v))_{k\geq 1})_{v\in\mathbb{V}}$ denote a family of independent copies of $(P_k)_{k\geq 1}$. An individual $v = v_1 \ldots v_j$ in the jth level (generation) whose weight is denoted by $P(v)$ produces an infinite number of offspring residing in the $(j+1)$st level. The offspring of the individual v are enumerated as $vk = v_1 \ldots v_j k$, where $k \in \mathbb{N}$, and the weights of the offspring are denoted by $P(vk)$. It is postulated that $P(vk) = P(v)P_k(v)$. Note that, for each $j \in \mathbb{N}$, $\sum_{|v|=j} P(v) = 1$ a.s., where, by convention, the sum $\sum_{|v|=j}$ is taken over all individuals in the jth level.

We are ready to explain the construction of the nested occupancy scheme in random environment. We identify individuals with boxes, so that the random weights (probabilities) of boxes in the subsequent levels are formed by the vectors $(P(v))_{|v|=1} = (P_k)_{k\geq 1}$, $(P(v))_{|v|=2}$ and so on. The collection of balls is the same for all levels. The balls are allocated, conditionally on $(P(v))_{v\in\mathbb{V}}$, according to the following rule. At time 0, infinitely many balls are collected in the box $\oslash$. At time $n \in \mathbb{N}$, one of the balls leaves the root $\oslash$ and falls, independently of the $n - 1$ balls that have left the root earlier, into the box v in the first level with probability $P(v)$. Simultaneously, it falls into the box vi_1 in the second level with probability $P(vi_1)/P(v)$, into the box vi_1i_2 in the third level with probability $P(vi_1i_2)/P(vi_1)$ and so on, indefinitely. At time n, that is, when n balls have already spread over the tree, a box is deemed occupied provided it was hit by some ball (out of n) on its way over the levels.

In what follows, we assume that the probabilities $(P_k)_{k\geq 1}$ are given by (1.1) and that these and the outcome of throwing balls are defined on a common probability space. Let $K_n(j)$ be the number of occupied boxes at time n in the jth level. Assuming that the mean of $|\log W|$ is infinite and the distribution tail of $|\log W|$ is regularly varying at ∞, we investigate the distributional behavior of $K_n(j)$ when $j = j_n \to \infty$ and $j_n = o((\log n)^a)$ as $n \to \infty$ for appropriate $a > 0$. We call the levels j satisfying the last two properties *intermediate*. The analogous problem was studied in [9, 23] when $|\log W|$ has a finite second moment, and in [22] when $|\log W|$ has a finite mean and an infinite second moment.

By [25, Theorem 1], the height of the scheme defined by $\tau_n := \inf\{j \geq 1 : K_n(j) = n\}$ is of order $\log n$, irrespective of the distribution tail of $|\log W|$. The asymptotic behavior of the scheme in the levels close to the height (so called *late* levels) was investigated in [3, 20]. The results obtained in the cited articles apply both in the heavy-tailed scenario considered here and in a light-tailed scenario.

1.2. Assumptions on the distribution of W. Our main result, Theorem 2.1, will be stated under certain assumptions on the distribution of $(|\log W|, |\log(1-W)|)$. We think it is instructive to introduce the assumptions for an arbitrary random vector (ξ, η) and then specialize them for $(\xi, \eta) = (|\log W|, |\log(1-W)|)$.

Let $(\xi_k, \eta_k)_{k\geq 1}$ be independent copies of a random vector (ξ, η) with arbitrary dependent positive components. Let F and G be the distribution functions of ξ and η, respectively. Denote by $(S_k)_{k\geq 0}$ the zero-delayed standard random walk with increments ξ_k, that is, $S_0 := 0$ and $S_k := \xi_1 + \cdots + \xi_k$ for $k \in \mathbb{N}$. Put

$$T_k := S_{k-1} + \eta_k, \qquad k \in \mathbb{N}.$$

The random sequence $T = (T_k)_{k\geq 1}$ is known in the literature as a (globally) perturbed random walk; see [18] for a survey and [2, 6, 7, 8, 24] for more recent contributions. A connection with the occupancy scheme treated in this paper is due to the fact that if $(P_k)_{k\geq 1}$ is given by (1.1), then $(-\log P_k)_{k\geq 1}$ is a perturbed random walk with $(\xi, \eta) = (|\log W|, |\log(1-W)|)$.

Put $N(t) := \sum_{k\geq 1} \mathbb{1}_{\{T_k \leq t\}}$ and $V(t) := \mathbb{E}N(t)$ for $t \geq 0$. Plainly,

$$V(t) = \int_{[0,\,t]} G(t-y)\,\mathrm{d}U(y), \qquad t \geq 0, \tag{1.2}$$

where, for $t \geq 0$, $U(t) = \sum_{i\geq 0} \mathbb{P}\{S_i \leq t\}$ is the renewal function.

Assume that

$$\mathbb{P}\{\xi > t\} \sim ct^{-\alpha}, \qquad t \to \infty, \tag{1.3}$$

for some $c > 0$ and $\alpha \in (0, 1)$. A standard result of renewal theory states that this limit relation is equivalent to

$$U(t) \sim Ct^{\alpha}, \qquad t \to \infty,$$

where $C = C_\alpha := (c\Gamma(1+\alpha)\Gamma(1-\alpha))^{-1}$ and Γ is the Euler gamma-function. To prove this, one uses the equality $\int_{[0,\infty)} \mathrm{e}^{-st}\,\mathrm{d}U(t) = (1-\varphi(s))^{-1}$ for $s>0$, where $\varphi(s) := \mathbb{E}\mathrm{e}^{-s\xi}$ for $s \geq 0$, and Karamata's Tauberian theorem [5, Theorem 1.7.1, p. 37]. Since $\int_{[0,\infty)} \mathrm{e}^{-st}\,\mathrm{d}V(t) = \mathbb{E}\mathrm{e}^{-s\eta}(1-\mathbb{E}\mathrm{e}^{-s\xi})^{-1}$ for $s>0$, the same reasoning enables us to conclude that (1.3) is equivalent to $V(t) \sim Ct^\alpha$ as $t\to\infty$. It will become clear soon that the first-order behavior of V is not sufficient for the purposes of the present work; a two-term expansion of V is needed. It will be shown in Section 3 that the following two assumptions ensure the required expansion of V.

ASSUMPTION A. $U(t) = Ct^\alpha + O(t^\rho)$ for some $\rho \in [0,\alpha)$.

ASSUMPTION B. Either $\limsup_{t\to\infty}(1-G(t))/(1-F(t)) = m_0 \in [0,\infty)$, or $m_0 = \infty$ and $\limsup_{t\to\infty} t^\theta(1-G(t)) = m_1 \in [0,\infty)$ for some $\theta \in (0,\alpha)$.

It seems to be a non-trivial problem to provide necessary and sufficient conditions for Assumption A to hold, formulated in terms of the distribution of ξ. In the case where the distribution of ξ is absolutely continuous (with respect to Lebesgue measure) sufficient conditions in terms of the densities considered as functions of a complex argument can be found in [32]. In Section 3 we give two sets of sufficient conditions, each of a different flavor.

Observe that when applied to the vector $(\xi,\eta) = (|\log W|, |\log(1-W)|)$, Assumptions A and B regulate the behavior of W near 0 and near 1, respectively. If (a) $\mathbb{E}\mathrm{e}^{-s|\log W|} = \exp(-c\Gamma(1-\alpha)s^\alpha)$ for $s\geq 0$, that is, the distribution of $|\log W|$ is α-stable or (b) $\mathbb{E}\mathrm{e}^{-s|\log W|} = (1+(c/\kappa)\Gamma(1-\alpha)s^\alpha)^{-\kappa}$ for $s \geq 0$ and some $\kappa>0$, then Assumption A holds with $\rho = 0$ and Assumption B holds with $m_0 = 0$. This statement will be proved in Section 3.

2. MAIN RESULT

To formulate our main result, we recall the definition of an *inverse stable subordinator*. Let $\mathcal{Z}_\alpha = (\mathcal{Z}_\alpha(u))_{u\geq 0}$ be a drift-free α-stable subordinator with $-\log \mathbb{E}\mathrm{e}^{-s\mathcal{Z}_\alpha(1)} = \Gamma(1-\alpha)s^\alpha$ for $s\geq 0$. The process $\mathcal{Z}_\alpha^{\leftarrow} = (\mathcal{Z}_\alpha^{\leftarrow}(u))_{u\geq 0}$ defined by

$$\mathcal{Z}_\alpha^{\leftarrow}(u) := \inf\{v\geq 0 : \mathcal{Z}_\alpha(v) > u\}, \qquad u \geq 0,$$

is called an *inverse α-stable subordinator.*

We write $\xrightarrow{\text{f.d.d.}}$ to denote weak convergence of finite-dimensional distributions. Also, $\lfloor x \rfloor$ denotes the integer part of $x\in\mathbb{R}$.

THEOREM 2.1. *Under Assumptions A and B imposed on $(\xi,\eta) = (|\log W|, |\log(1-W)|)$, let $(j_n)_{n\geq 1}$ be a sequence of positive numbers satisfying $\lim_{n\to\infty} j_n = \infty$ and*

$$j_n = o\big((\log n)^{\min(1/3,(\alpha-\beta)/(\alpha-\beta+1))}\big), \qquad n\to\infty,$$

where $\beta = \rho$ *if* $m_0 < \infty$ *and* $\beta = \max(\rho, \alpha - \theta)$ *if* $m_0 = \infty$ *and* $m_1 < \infty$. *Then, as* $n \to \infty$,

$$\Big(\frac{c j_n^\alpha}{\rho_{\lfloor j_n u\rfloor -1}(\log n)^{\alpha\lfloor j_n u\rfloor}} K_n(\lfloor j_n u\rfloor)\Big)_{u>0} \xrightarrow{\text{f.d.d.}} \Big(\int_0^\infty \mathrm{e}^{-\alpha u y}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y)\Big)_{u>0},$$

where

$$\rho_i := \frac{(C\Gamma(\alpha+1))^i}{\Gamma(\alpha i+1)}, \qquad i \in \mathbb{N}_0 := \{0, 1, \ldots\}, \tag{2.1}$$

and Γ *is the Euler gamma-function.*

Now we discuss the structure of the limit process in Theorem 2.1. It can be seen from the proof that the integrator $\mathcal{Z}_\alpha^{\leftarrow}$ describes the fluctuations of the number of occupied boxes in the first level, properly normalized. The process $\mathcal{Z}_\alpha^{\leftarrow}$ is a.s. continuous singular with respect to Lebesgue measure. The integrand $y \mapsto \mathrm{e}^{-\alpha u y}$ which is formed by the renewal structure of the underlying tree makes the process $(\int_0^\infty \mathrm{e}^{-\alpha u y}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y))_{u>0}$ a.s. infinitely differentiable. Thus, when passing from the first level to intermediate levels the fluctuations of the number of occupied boxes smooth out.

Since $\mathcal{Z}_\alpha^{\leftarrow}(\mathcal{Z}_\alpha(y)) = y$ a.s., we conclude that

$$\Big(\int_0^\infty \mathrm{e}^{-\alpha u y}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y)\Big)_{u>0} = \Big(\int_0^\infty \mathrm{e}^{-\alpha u \mathcal{Z}_\alpha(y)}\,\mathrm{d}y\Big)_{u>0} \quad \text{a.s.}$$

Further, by the scaling property of $\mathcal{Z}_\alpha$, with $u > 0$ fixed,

$$\int_0^\infty \mathrm{e}^{-\alpha u \mathcal{Z}_\alpha(y)}\,\mathrm{d}y \overset{\mathrm{d}}{=} (\alpha u)^{-\alpha}\int_0^\infty \mathrm{e}^{-\mathcal{Z}_\alpha(y)}\,\mathrm{d}y =: (\alpha u)^{-\alpha} I_\alpha,$$

where $\overset{\mathrm{d}}{=}$ denotes equality in distribution. The latter integral is known in the literature as an exponential functional of the subordinator $\mathcal{Z}_\alpha$. According to [11, Proposition 3.3],

$$\mathbb{E}(I_\alpha)^n = \frac{(n!)^{1-\alpha}}{(\Gamma(1-\alpha))^n}, \qquad n \in \mathbb{N},$$

and furthermore $\mathbb{E}\exp(sI_\alpha) < \infty$ for all $s > 0$. It can be shown (see [11, Proposition 3.4(iv)]) that the moment formula implies that $\log I_\alpha + \log\Gamma(1-\alpha) \overset{\mathrm{d}}{=} X(1-\alpha)$, where $(X(u))_{u\ge 0}$ is a Lévy process with $X(1) \overset{\mathrm{d}}{=} \log\mathcal{E}$, and $\mathcal{E}$ is a random variable with exponential distribution of unit mean. A consequence of this observation is that the distribution of I_α is absolutely continuous with respect to Lebesgue measure.

Next, we intend to set a non-rigorous link between Theorem 2.1 and a limit theorem for $K_n(j)$ with fixed j. Assume that $\mathbb{P}\{|\log W| > x\} \sim x^{-\alpha}\ell(x)$ as

$x \to \infty$ for some $\alpha \in (0,1)$ and some ℓ slowly varying at ∞. A specialization of a functional limit theorem given in [8, Theorem 7] reads

$$\left(\frac{(\ell(\log n))^j K_n(j)}{(\log n)^{\alpha j}}\right)_{j\geq 1} \xrightarrow{\mathrm{d}} \Big(\int_{[0,\,1]} (1-y)^{\alpha(j-1)}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y)\Big)_{j\geq 1}, \quad n\to\infty,$$

where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution. Observe that $(\mathcal{Z}_\alpha^{\leftarrow}(y/j))_{y\geq 0}$ has the same distribution as $(j^{-\alpha}\mathcal{Z}_\alpha^{\leftarrow}(y))_{y\geq 0}$ and, with $y > 0$ fixed, we have $\lim_{j\to\infty}(1-y/j)^{\alpha(j-1)} = \mathrm{e}^{-\alpha y}$. Hence,

$$\begin{aligned} j^\alpha & \int_{[0,\,1]} (1-y)^{\alpha(j-1)}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y) \\ &= j^\alpha \int_{[0,\,j]} (1-y/j)^{\alpha(j-1)}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y/j) \overset{\mathrm{d}}{=} \int_{[0,\,j]} (1-y/j)^{\alpha(j-1)}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y). \end{aligned}$$

It is reasonable to expect that the distributional limit of the last integral as $j\to\infty$ is $\int_{[0,\,\infty)} \mathrm{e}^{-\alpha y}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y)$. This is in line with Theorem 2.1.

The remainder of the article is structured as follows. As already announced, in Section 3 we discuss Assumptions A and B and their consequences. Section 4 contains auxiliary results on a general branching process generated by a perturbed random walk. The proof of Theorem 2.1 is given in Section 5.

3. DISCUSSION OF ASSUMPTIONS A AND B

We start by showing that Assumptions A and B secure a two-term expansion of V.

LEMMA 3.1. *Assume that Assumptions A and B hold. Then there exists $D > 0$ such that*

$$(3.1) \qquad |V(t) - Ct^\alpha| \leq Dt^\beta, \quad t\geq 0,$$

where $\beta = \rho$ if $m_0 < \infty$, $\beta = \max(\rho, \alpha-\theta)$ if $m_0 = \infty$ and $m_1 < \infty$; $C > 0$, $\alpha \in (0,1)$ and $\rho \in [0,\alpha)$ are as defined in Assumption A; m_0, m_1 and $\theta \in (0,\alpha)$ are as defined in Assumption B.

Proof. Assuming that $\limsup_{t\to\infty}(1-G(t))/(1-F(t)) = m_0 \in [0,\infty)$ we shall show that

$$(3.2) \qquad \int_{[0,\,t]} (1-G(t-y))\,\mathrm{d}U(y) = O(1), \quad t\to\infty.$$

Indeed, given $m_2 > m_0$ there exists $t_0 > 0$ such that $1-G(t) \leq m_2(1-F(t))$ whenever $t\geq t_0$. With this at hand,

$$\int_{[0,\,t-t_0]} (1-G(t-y))\,\mathrm{d}U(y) \leq m_2 \int_{[0,\,t]} (1-F(t-y))\,\mathrm{d}U(y) = m_2.$$

The last equality is another form of writing the renewal equation for U. Alternatively, it is equivalent to $\mathbb{P}\{\inf\{k \geq 0 : S_k > t\} < \infty\} = 1$. By subadditivity of U (see, for instance, [26, Theorem 1.7, p. 10]),

$$\int_{(t-t_0,\, t]} (1 - G(t-y))\,\mathrm{d}U(y) \leq U(t) - U(t-t_0) \leq U(t_0),$$

and (3.2) follows. Equality (1.2) is equivalent to

$$V(t) = U(t) - \int_{[0,\, t]} (1 - G(t-y))\,\mathrm{d}U(y), \qquad t \geq 0.$$

Hence, Assumption A entails $V(t) = Ct^\alpha + O(t^\rho)$ as $t \to \infty$. Since $V(0) = 0$, we can find $D > 0$ such that $|V(t) - Ct^\alpha| \leq Dt^\rho$ for all $t \geq 0$.

Assume now that $m_0 = \infty$ and $\limsup_{t\to\infty} t^\theta(1 - G(t)) = m_1 \in [0, \infty)$ for some $\theta \in (0, \alpha)$. We shall prove that

$$\int_{[0,\, t]} (1 - G(t-y))\,\mathrm{d}U(y) = O(t^{\alpha-\theta}), \qquad t \to \infty. \tag{3.3}$$

Given $m_3 > m_1$ and $C^* > C$ there exists t_1 such that $U(t) \leq C^* t^\alpha$ and $1 - G(t) \leq m_3 t^{-\theta}$ whenever $t \geq t_1$. We already know from the previous paragraph that subadditivity of U ensures $\int_{(t-t_1,\, t]} (1 - G(t-y))\,\mathrm{d}U(y) = O(1)$ as $t \to \infty$. Further, by integration by parts we obtain

$$\begin{aligned}
&\int_{[0,\, t-t_1]} (1 - G(t-y))\,\mathrm{d}U(y) \\
&\leq m_3 \int_{[0,\, t-t_1]} (t-y)^{-\theta}\,\mathrm{d}U(y) = m_3 \int_{[t_1,\, t]} y^{-\theta}\,\mathrm{d}_y(U(t) - U(t-y)) \\
&= m_3\Big(t^{-\theta}(U(t) - U(0)) - t_1^{-\theta}(U(t) - U(t-t_1)) \\
&\qquad\qquad + \theta \int_{t_1}^{t} (U(t) - U(t-y))y^{-\theta-1}\,\mathrm{d}y\Big).
\end{aligned}$$

Here and elsewhere, we write d_y rather than d whenever there is ambiguity, that is, a function under the differential depends on y and some other variables. The first term is $O(t^{\alpha-\theta})$ and the second term is $O(1)$. Another appeal to subadditivity of U yields

$$\int_{t_1}^{t} (U(t) - U(t-y))y^{-\theta-1}\,\mathrm{d}y \leq \int_{t_1}^{t} U(y)y^{-\theta-1}\,\mathrm{d}y \leq C^* \int_{t_1}^{t} y^{\alpha-\theta-1}\,\mathrm{d}y = O(t^{\alpha-\theta}),$$

and (3.3) follows. Mimicking the argument from the first part of the proof we arrive at (3.1) with $\beta = \max(\rho, \alpha - \theta)$. ∎

Let $(\mathcal{Z}_\alpha(u))_{u\geq 0}$ be an α-stable subordinator as defined at the beginning of Section 2. Now we discuss two sets of sufficient conditions for Assumption A. The first of these was outlined by one of the referees. Observe that, for $v \in \mathbb{R}$,

$$\Phi_\alpha(v) := \mathbb{E}\mathrm{e}^{\mathrm{i}v\mathcal{Z}_\alpha(1)} = \exp\big(-\Gamma(1-\alpha)|v|^\alpha(\cos(\pi\alpha/2) - \mathrm{i}\sin(\pi\alpha/2)\mathrm{sgn}\, v)\big).$$

LEMMA 3.2. *Assume that there exist constants $a_1, a_2 > 0$ and $r \in (\alpha, 1]$ such that, for all $v \in \mathbb{R}$ satisfying $|v| \leq a_1$,*

$$|\mathbb{E}\mathrm{e}^{\mathrm{i}v\xi} - \Phi_\alpha(v)| \leq a_2|v|^r. \tag{3.4}$$

Then Assumption A holds with $\rho = 0$ if $r > 2\alpha$, $\rho = \delta$ for any $\delta > 0$ if $r = 2\alpha$, and $\rho = \lambda(2\alpha - r)$ for any $\lambda > 1$ such that $\lambda(2\alpha - r) < \alpha$ if $r < 2\alpha$.

Proof. Observe that $\int_0^\infty \mathbb{P}\{\mathcal{Z}_\alpha(cu) \leq t\}\,\mathrm{d}u = Ct^\alpha$ for $t \geq 0$ and

$$0 \leq \sum_{n\geq 0} \mathbb{P}\{\mathcal{Z}_\alpha(cn) \leq t\} - \int_0^\infty \mathbb{P}\{\mathcal{Z}_\alpha(cu) \leq t\}\,\mathrm{d}u \leq 1, \quad t \geq 0.$$

Hence, it is enough to prove that

$$\Big|\sum_{n\geq 0} \mathbb{P}\{S_n \leq t\} - \sum_{n\geq 0} \mathbb{P}\{\mathcal{Z}_\alpha(cn) \leq t\}\Big| = O(t^\rho), \quad t \to \infty. \tag{3.5}$$

By [27, Proposition 1],

$$\begin{aligned} \sup_{x\geq 0} &|\mathbb{P}\{S_n \leq n^{1/\alpha}x\} - \mathbb{P}\{\mathcal{Z}_\alpha(c) \leq x\}| \\ &= \sup_{t\geq 0} |\mathbb{P}\{S_n \leq t\} - \mathbb{P}\{\mathcal{Z}_\alpha(cn) \leq t\}| \leq Bn^{-\alpha^{-1}(r-\alpha)} \end{aligned} \tag{3.6}$$

for each $n \in \mathbb{N}$ and a constant $B > 0$ which does not depend on n.

CASE $r \in (2\alpha, 1)$ (necessarily $\alpha < 1/2$). According to (3.6), relation (3.5) holds with $\rho = 0$.

CASE $r \in (\alpha, 2\alpha)$, in which $0 < \alpha^{-1}(r - \alpha) < 1$. Fix any $\lambda > 1$ such that $\lambda(2\alpha - r) < \alpha$. Using (3.6) we infer that, as $t \to \infty$,

$$\Big|\sum_{n=0}^{\lfloor t^{\lambda\alpha}\rfloor} \mathbb{P}\{S_n \leq t\} - \sum_{n=0}^{\lfloor t^{\lambda\alpha}\rfloor} \mathbb{P}\{\mathcal{Z}_\alpha(cn) \leq t\}\Big| \leq B\sum_{n=1}^{\lfloor t^{\lambda\alpha}\rfloor} n^{-\alpha^{-1}(r-\alpha)} = O(t^{\lambda(2\alpha-r)}).$$

We claim that, as $t \to \infty$,

$$\sum_{n\geq\lfloor t^{\lambda\alpha}\rfloor+1} \mathbb{P}\{S_n \leq t\} = o(1) \quad \text{and} \quad \sum_{n\geq\lfloor t^{\lambda\alpha}\rfloor+1} \mathbb{P}\{\mathcal{Z}_\alpha(cn) \leq t\} = o(1). \tag{3.7}$$

Indeed, by Markov's inequality, for each $u > 0$,

$$\sum_{n\ge \lfloor t^{\lambda\alpha}\rfloor+1} \mathbb{P}\{S_n \le t\} \le \mathrm{e}^{ut} \sum_{n\ge \lfloor t^{\lambda\alpha}\rfloor+1} (\varphi(u))^n \le \frac{\mathrm{e}^{ut+\log\varphi(u)t^{\lambda\alpha}}}{1-\varphi(u)}.$$

Put $u = 1/t$. Then $(1-\varphi(1/t))^{-1} \sim (c\Gamma(1-\alpha))^{-1}t^{\alpha}$ as $t \to \infty$. Since $-t^{\lambda\alpha}\log\varphi(1/t) \sim c\Gamma(1-\alpha)t^{(\lambda-1)\alpha}$ as $t \to \infty$, the numerator decays to 0 super-exponentially fast. This proves the first limit relation in (3.7). The argument for the second relation is analogous.

CASE $r = 2\alpha < 1$. In the proof for the previous case, pick any $\lambda > 1$. Then

$$\Big|\sum_{n=0}^{\lfloor t^{\lambda\alpha}\rfloor} \mathbb{P}\{S_n \le t\} - \sum_{n=0}^{\lfloor t^{\lambda\alpha}\rfloor} \mathbb{P}\{\mathcal{Z}_\alpha(cn) \le t\}\Big| = O(\log t), \quad t \to \infty.$$

Also, relations (3.7) hold true in the present setting. ■

We proceed by discussing a possible application of Lemma 3.2. Assume that $\xi \overset{\mathrm{d}}{=} \mathcal{Z}_\alpha(c) + \theta$ for a nonnegative random variable θ independent of $\mathcal{Z}_\alpha(c)$. If $\mathbb{E}\theta < \infty$, then inequality (3.4) holds with $r = 1$. Further, if $\alpha = 3/4$, say, then Assumption A holds with $\rho = \lambda/2$ for any $\rho \in (1, 3/2)$. If $\mathbb{P}\{\theta > t\} \sim \mathrm{const}\, t^{-\beta}$ as $t \to \infty$ for some $\beta \in (\alpha, 1)$, then inequality (3.4) holds with $r = \beta$. Further, if $\alpha = 1/4$ and $\beta = 3/4$, then Assumption A holds with $\rho = 0$.

Next, we give other sufficient conditions for Assumption A which work for a special class of distributions of ξ. Namely, assume that the distribution of ξ coincides with that of $\mathcal{Z}_\alpha(c)X^{1/\alpha}$, where X is a positive random variable of mean 1 which is independent of $\mathcal{Z}_\alpha(c)$ for some $c > 0$. Equivalently, this assumption means that the standard random walk $(S_n)_{n\ge0}$ has the same distribution as $(\mathcal{Z}_\alpha(c\hat{S}_n))_{n\ge0}$. Here, $\hat{S}_0 := 0$, $\hat{S}_n := X_1 + \cdots + X_n$ for $n \in \mathbb{N}$, $X_1, X_2, \ldots$ are independent copies of X, and $(\hat{S}_n)_{n\ge0}$ is independent of $(\mathcal{Z}_\alpha(u))_{u\ge0}$.

Put $\hat{U}(t) := \sum_{n\ge0}\mathbb{P}\{\hat{S}_n \le t\}$ for $t \ge 0$. Now we obtain a formula which connects U and $\hat{U}$. To this end, we need the following property: for fixed $t > 0$, $\mathcal{Z}_\alpha(ct)$ has the same distribution as $t^{1/\alpha}\mathcal{Z}_\alpha(c)$. As a consequence,

$$\begin{aligned}(3.8)\qquad U(t) &= \sum_{n\ge0}\mathbb{P}\{S_n \le t\} = \sum_{n\ge0}\mathbb{P}\{\mathcal{Z}_\alpha(c\hat{S}_n) \le t\}\\ &= \sum_{n\ge0}\mathbb{P}\{\hat{S}_n^{1/\alpha}\mathcal{Z}_\alpha(c) \le t\} = \sum_{n\ge0}\mathbb{P}\{\hat{S}_n \le (\mathcal{Z}_\alpha(c))^{-\alpha}t^{\alpha}\}\\ &= \mathbb{E}\hat{U}((\mathcal{Z}_\alpha(c))^{-\alpha}t^{\alpha}), \quad t \ge 0.\end{aligned}$$

Let $d > 0$. Recall that the distribution of a positive random variable is called d-*arithmetic* if it is concentrated on the lattice $(nd)_{n\ge1}$, but is not concentrated on $(nd_1)_{n\ge1}$ for any $d_1 > d$. The distribution of Y is called *nonarithmetic* if it is not d-arithmetic for any $d > 0$.

Invoking (3.8) enables us to provide sufficient conditions for Assumption A to hold.

LEMMA 3.3. *Assume that* $\mathbb{E}X^\vartheta < \infty$ *for some* $\vartheta \in (1, 2]$. *Then*

$$U(t) = Ct^\alpha + O(t^{\alpha(2-\theta)}), \quad t \to \infty,$$

that is, Assumption A holds with $\rho = \alpha(2 - \vartheta)$. *Here, as before,* $C = (c\Gamma(1 + \alpha)\Gamma(1 - \alpha))^{-1}$.

Proof. We intend to use Lorden's inequality for $\hat{U}$. Although in many sources it is only stated for nonarithmetic distributions, it holds and takes the same form for both nonarithmetic and arithmetic distributions; see [10] for an elegant proof.

If $\mathbb{E}X^2 < \infty$, then, by Lorden's inequality, $\hat{U}(t) \le t + \mathbb{E}X^2$ for all $t \ge 0$ (recall that $\mathbb{E}X = 1$). Hence,

$$U(t) = \mathbb{E}\hat{U}((\mathcal{Z}_\alpha(c))^{-\alpha}t^\alpha) \le \mathbb{E}(\mathcal{Z}_\alpha(c))^{-\alpha}t^\alpha + \mathbb{E}X^2.$$

It remains to note that, by Lemma 6.1,

$$\mathbb{E}(\mathcal{Z}_\alpha(c))^{-\alpha} = \frac{\alpha}{\Gamma(1+\alpha)} \int_0^\infty s^{\alpha-1} \mathrm{e}^{-c\Gamma(1-\alpha)s^\alpha}\,\mathrm{d}s = C.$$

If $\mathbb{E}X^\vartheta < \infty$ for some $\vartheta \in (1, 2)$, then $\hat{U}(t) = t + O(t^{2-\vartheta})$ as $t \to \infty$; see, for instance, [7, (3.13), p. 433]. In particular, for some $t_0 > 0$ and a constant $C^* > 0$ we have $\hat{U}(t) \le t + C^* t^{2-\vartheta}$ whenever $t \ge t_0$. Using (3.8) yields

$$\begin{aligned} U(t) &\le \mathbb{E}\big((\mathcal{Z}_\alpha(c))^{-\alpha}t^\alpha + C^*(\mathcal{Z}_\alpha(c))^{-\alpha(2-\vartheta)}t^{\alpha(2-\vartheta)}\big)\mathbb{1}_{\{(\mathcal{Z}_\alpha(c))^{-\alpha}t^\alpha \ge t_0\}} \\ &\quad + \mathbb{E}\hat{U}((\mathcal{Z}_\alpha(c))^{-\alpha}t^\alpha)\mathbb{1}_{\{(\mathcal{Z}_\alpha(c))^{-\alpha}t^\alpha < t_0\}} \\ &\le Ct^\alpha + C^*\mathbb{E}(\mathcal{Z}_\alpha(c))^{-\alpha(2-\theta)}t^{\alpha(2-\theta)} + \hat{U}(t_0) \\ &= Ct^\alpha + O(t^{\alpha(2-\vartheta)}), \quad t \to \infty. \end{aligned}$$

Put $\hat{\nu}(t) := \inf\{k \ge 1 : \hat{S}_k > t\}$ for $t \ge 0$. Using Wald's identity we infer that, for all $t \ge 0$, $\hat{U}(t) = \mathbb{E}\hat{S}_{\hat{\nu}(t)} \ge t$, whence $U(t) \ge Ct^\alpha$. This completes the proof. ∎

Next, we show that Assumptions A and B hold true, with $(\xi, \eta) = (|\log W|, |\log(1-W)|)$, if, for instance, (a) $|\log W| = \mathcal{Z}_\alpha(c)$ for some $c > 0$, that is, the distribution of $|\log W|$ is α-stable; or (b) $\varphi(s) := \mathbb{E}\mathrm{e}^{-s\xi} = (1 + (c/\kappa)\Gamma(1-\alpha)s^\alpha)^{-\kappa}$ for $s \ge 0$, with some $\alpha \in (0, 1)$ and some $\kappa > 0$. In both cases, $|\log W|$ has the same distribution as $\mathcal{Z}_\alpha(c)X^{1/\alpha}$, where $\mathbb{P}\{X = 1\} = 1$ in case (a) and X is independent of $\mathcal{Z}_\alpha(c)$ and has a gamma distribution with parameters κ and κ in case (b). To justify the latter claim, observe that $s \mapsto \varphi(((c\Gamma(1-\alpha))^{-1}s)^{1/\alpha}) =$

$(1+s/\kappa)^{-\kappa}$ is the Laplace–Stieltjes transform of the aforementioned gamma distribution. Thus, in both cases $\mathbb{E}X^2<\infty$ and, according to Lemma 3.3, Assumption A holds with $\rho=0$.

Now, temporarily ignoring cases (a) and (b), we show that Assumption B holds true, with $m_0=0$, whenever $1-F(x)=\mathbb{P}\{|\log W|>x\}\sim cx^{-\alpha}$ as $x\to\infty$ and $\mathbb{E}|\log W|^{-\gamma}<\infty$ for some $\gamma>0$. Indeed, by Markov's inequality,

$$\begin{aligned}1-G(x)&=\mathbb{P}\{|\log(1-W)|>x\}=\mathbb{P}\{|\log W|<|\log(1-\mathrm{e}^{-x})|\}\\&=\mathbb{P}\{|\log W|^{-\gamma}>|\log(1-\mathrm{e}^{-x})|^{-\gamma}\}\\&\le\mathbb{E}|\log W|^{-\gamma}(-\log(1-\mathrm{e}^{-x}))^{\gamma}\sim\mathbb{E}|\log W|^{-\gamma}\mathrm{e}^{-\gamma x}\end{aligned}$$

as $x\to\infty$, and the claim follows.

We already know that $\mathbb{E}(\mathcal{Z}_\alpha(c))^{-\alpha}<\infty$. Hence, Assumption B holds, with $m_0=0$, in case (a). Although we do not need such precision, we note in passing that in case (a), with c and α satisfying $c\Gamma(1-\alpha)=1$, by [16, Lemma 1], the following asymptotic relation holds:

$$\mathbb{P}\{|\log(1-W)|>x\}\ \sim\ c_1\mathrm{e}^{-\frac{\alpha x}{2(1-\alpha)}}\exp(-c_2|\log(1-\mathrm{e}^{-x})|^{-\frac{\alpha}{1-\alpha}}),\qquad x\to\infty,$$

for explicit positive constants c_1 and c_2.

In case (b), using Lemma 6.1 we infer $\mathbb{E}X^{-\gamma}<\infty$ for all $\gamma\in(0,\min(\kappa,1))$. Thus, again, Assumption B holds, with $m_0=0$. To make the presentation symmetric, we note, without going into details, that Tauberian and Abelian theorems for Laplace transforms can be used to show that

$$\mathbb{P}\{|\log(1-W)|>x\}\sim\frac{\kappa^\kappa}{(c\Gamma(1-\alpha))^\kappa\Gamma(1+\alpha\kappa)}\mathrm{e}^{-\alpha\kappa x},\qquad x\to\infty.$$

4. AN AUXILIARY GENERAL BRANCHING PROCESS

To prove Theorem 2.1, we need some auxiliary results on a general branching process generated by T (a perturbed random walk). Here is its definition in terms of a population of individuals initiated at time 0 by one individual, the ancestor. An individual born at time $s\ge0$ produces offspring whose birth times have the same distribution as $(s+T_k)_{k\ge1}$. All individuals act independently of each other. An individual resides in the jth generation if it has exactly j ancestors.

For $t\ge0$ and $j\in\mathbb{N}$, denote by $N_j(t)$ the number of the jth generation individuals with birth times smaller than or equal to t and put $V_j(t):=\mathbb{E}N_j(t)$. Then $N_1(t)=N(t)$, and, for $j\ge2$, N_j admits the decomposition

$$N_j(t)=\sum_{r\ge1}N_{j-1}^{(r)}(t-T_r)\mathbb{1}_{\{T_r\le t\}},\quad t\ge0,$$

where, for $r\in\mathbb{N}$, $N_{j-1}^{(r)}(t)$ is the number of the jth generation individuals who are descendants of the first-generation individual with birth time T_r, and whose birth times fall in $[T_r, T_r + t]$. By the branching property, $(N^{(1)}_{j-1}(t))_{t\ge 0}$, $(N^{(2)}_{j-1}(t))_{t\ge 0}, \ldots$ are independent copies of $(N_{j-1}(t))_{t\ge 0}$ which are also independent of T. As a consequence, $V_1(t) = V(t)$, and

$$V_j(t) = (V_{j-1} * V)(t) = \int\limits_{[0,\,t]} V_{j-1}(t-y)\,\mathrm{d}V(y), \qquad j \ge 2,\ t \ge 0, \tag{4.1}$$

that is, $V_j = V^{*(j)}$ is the j-fold Lebesgue–Stieltjes convolution of V with itself. The remainder of this section is concerned with precise and asymptotic estimates for V_j.

According to Lemma 3.1, Assumptions A and B ensure that inequality (3.1) holds true. Now we prove that V_n admits a similar estimate.

LEMMA 4.1. *Suppose* (3.1). *Then, for* $n \in \mathbb{N}$ *and* $t \ge 0$,

$$|V_n(t) - \rho_n t^{\alpha n}| \le \sum_{i=0}^{n-1} \binom{n}{i} \frac{(\Gamma(\alpha+1))^i (\Gamma(\beta+1))^{n-i}}{\Gamma(\alpha i + \beta(n-i)+1)} (Ct^{\alpha})^i (Dt^{\beta})^{n-i}, \tag{4.2}$$

where ρ_n *is as given in* (2.1).

Proof. We use induction. For $n = 1$, (4.2) reduces to (3.1). Assume that (4.2) holds with $n-1$ replacing n. Using (4.1) for the first equality we conclude that

$$\begin{aligned}
|V_n(t) - \rho_n t^{\alpha n}| &= \Big| \int\limits_{[0,\,t]} V_{n-1}(t-y)\,\mathrm{d}V(y) - \rho_n t^{\alpha n} \Big| \\
&\le \int\limits_{[0,\,t]} |V_{n-1}(t-y) - \rho_{n-1}(t-y)^{\alpha(n-1)}|\,\mathrm{d}V(y) \\
&\quad + \Big| \rho_{n-1} \int\limits_{[0,\,t]} (t-y)^{\alpha(n-1)}\,\mathrm{d}V(y) - \rho_n t^{\alpha n} \Big|.
\end{aligned}$$

Invoking the induction assumption and then integration by parts we obtain

$$\begin{aligned}
A_n(t) &:= \int\limits_{[0,\,t]} |V_{n-1}(t-y) - \rho_{n-1}(t-y)^{\alpha(n-1)}|\,\mathrm{d}V(y) \\
&\le \sum_{i=0}^{n-2} \binom{n-1}{i} \frac{(\Gamma(\alpha+1))^i (\Gamma(\beta+1))^{n-1-i}}{\Gamma(\alpha i + \beta(n-1-i)+1)} \\
&\qquad\qquad \times D^{n-1-i} C^i \int\limits_{[0,\,t]} (t-y)^{\alpha i + \beta(n-1-i)}\,\mathrm{d}V(y) \\
&= \sum_{i=0}^{n-2} \binom{n-1}{i} \frac{(\Gamma(\alpha+1))^i (\Gamma(\beta+1))^{n-1-i}}{\Gamma(\alpha i + \beta(n-1-i)+1)} \\
&\qquad\qquad \times D^{n-1-i} C^i \int_0^t V(t-y)\,\mathrm{d}(y^{\alpha i + \beta(n-1-i)}).
\end{aligned}$$

In view of (3.1),

$$\int_0^t V(t-y)\,\mathrm{d}(y^{\alpha i+\beta(n-1-i)}) \le \int_0^t (C(t-y)^\alpha + D(t-y)^\beta)\,\mathrm{d}(y^{\alpha i+\beta(n-1-i)})$$
$$= C\frac{\Gamma(\alpha+1)\Gamma(\alpha i+\beta(n-1-i)+1)}{\Gamma(\alpha(i+1)+\beta(n-1-i)+1)}t^{\alpha(i+1)+\beta(n-1-i)}$$
$$+ D\frac{\Gamma(\beta+1)\Gamma(\alpha i+\beta(n-1-i)+1)}{\Gamma(\alpha i+\beta(n-i)+1)}t^{\alpha i+\beta(n-i)},$$

whence

(4.3) $$A_n(t)$$
$$\le \sum_{i=0}^{n-2}\binom{n-1}{i}\frac{(\Gamma(\alpha+1))^{i+1}(\Gamma(\beta+1))^{n-1-i}}{\Gamma(\alpha(i+1)+\beta(n-1-i)+1)}D^{n-1-i}C^{i+1}t^{\alpha(i+1)+\beta(n-1-i)}$$
$$+\sum_{i=0}^{n-2}\binom{n-1}{i}\frac{(\Gamma(\alpha+1))^{i}(\Gamma(\beta+1))^{n-i}}{\Gamma(\alpha i+\beta(n-i)+1)}D^{n-i}C^{i}t^{\alpha i+\beta(n-i)}$$
$$=(n-1)\frac{(\Gamma(\alpha+1))^{n-1}\Gamma(\beta+1)}{\Gamma(\alpha(n-1)+\beta+1)}DC^{n-1}t^{\alpha(n-1)+\beta}$$
$$+\sum_{i=1}^{n-2}\left(\binom{n-1}{i-1}+\binom{n-1}{i}\right)\frac{(\Gamma(\alpha+1))^{i}(\Gamma(\beta+1))^{n-i}}{\Gamma(\alpha i+\beta(n-i)+1)}D^{n-i}C^{i}t^{\alpha i+\beta(n-i)}$$
$$+\frac{(\Gamma(\beta+1))^n}{\Gamma(\beta n+1)}D^n t^{\beta n}$$
$$=(n-1)\frac{(\Gamma(\alpha+1))^{n-1}\Gamma(\beta+1)}{\Gamma(\alpha(n-1)+\beta+1)}DC^{n-1}t^{\alpha(n-1)+\beta}$$
$$+\sum_{i=0}^{n-2}\binom{n}{i}\frac{(\Gamma(\alpha+1))^{i}(\Gamma(\beta+1))^{n-i}}{\Gamma(\alpha i+\beta(n-i)+1)}D^{n-i}C^{i}t^{\alpha i+\beta(n-i)}.$$

To obtain the first equality we have changed the variable $i\to i-1$ in the first sum, singled out the term which corresponds to $i=n-1$ and also singled out the term of the second sum which corresponds to $i=0$.

Further, we infer that

$$\Big|\rho_{n-1}\int_{[0,\,t]}(t-y)^{\alpha(n-1)}\,\mathrm{d}V(y)-\rho_n t^{\alpha n}\Big|$$
$$=\rho_{n-1}\Big|\int_{[0,\,t]}(t-y)^{\alpha(n-1)}\,\mathrm{d}V(y)-C\int_{[0,\,t]}(t-y)^{\alpha}\,\mathrm{d}(y^{\alpha(n-1)})\Big|$$
$$=\Big|\rho_{n-1}\int_0^t(V(t-y)-C(t-y)^\alpha)\,\mathrm{d}(y^{\alpha(n-1)})\Big|\le D\rho_{n-1}\int_0^t(t-y)^\beta\,\mathrm{d}(y^{\alpha(n-1)})$$
$$=D\frac{(C\Gamma(\alpha+1))^{n-1}}{\Gamma(\alpha(n-1)+1)}\,\frac{\Gamma(\alpha(n-1)+1)\Gamma(\beta+1)}{\Gamma(\alpha(n-1)+\beta+1)}t^{\alpha(n-1)+\beta}$$
$$=\frac{(\Gamma(\alpha+1))^{n-1}\Gamma(\beta+1)}{\Gamma(\alpha(n-1)+\beta+1)}DC^{n-1}t^{\alpha(n-1)+\beta},$$

having utilized integration by parts for the second equality and (3.1) for the first inequality. The sum of the last expression and the first term on the right-hand side of (4.3) is equal to the term of the sum in (4.2) which corresponds to $i = n-1$. The proof of Lemma 4.1 is complete. ■

We proceed with a technical estimate of an analytic flavor.

LEMMA 4.2. *Let α, β, C and D be as in* (3.1). *For $t \geq 0$ and a positive integer j satisfying*

$$2D\Gamma(\beta+1)j(\alpha(j-1)+\beta+1)^{\alpha-\beta} \leq C\Gamma(\alpha+1)t^{\alpha-\beta}, \tag{4.4}$$

we have

$$\sum_{i=0}^{j-1}\binom{j}{i}\frac{(\Gamma(\alpha+1))^i(\Gamma(\beta+1))^{j-i}}{\Gamma(\alpha i+\beta(j-i)+1)}(Ct^\alpha)^i(Dt^\beta)^{j-i} \leq 2DC^{j-1}j\frac{(\Gamma(\alpha+1))^{j-1}\Gamma(\beta+1)}{\Gamma(\alpha(j-1)+\beta+1)}t^{\alpha(j-1)+\beta}.$$

Proof. Write

$$\begin{aligned}\sum_{i=0}^{j-1}\binom{j}{i}&\frac{(\Gamma(\alpha+1))^i(\Gamma(\beta+1))^{j-i}}{\Gamma(\alpha i+\beta(j-i)+1)}(Ct^\alpha)^i(Dt^\beta)^{j-i}\\ &= DC^{j-1}j\frac{(\Gamma(\alpha+1))^{j-1}\Gamma(\beta+1)}{\Gamma(\alpha(j-1)+\beta+1)}t^{\alpha(j-1)+\beta}\\ &\quad+\sum_{i=0}^{j-2}\binom{j}{i}\frac{(\Gamma(\alpha+1))^i(\Gamma(\beta+1))^{j-i}}{\Gamma(\alpha i+\beta(j-i)+1)}(Ct^\alpha)^i(Dt^\beta)^{j-i}.\end{aligned}$$

Assuming that t and j satisfy (4.4) we intend to show that the second term on the right-hand side does not exceed the first term. Indeed, using the inequality

$$\binom{j}{i} \leq j^{j-i} \tag{4.5}$$

and (6.1) we infer that

$$\begin{aligned}&\frac{\Gamma(\alpha(j-1)+\beta+1)}{DC^{j-1}j(\Gamma(\alpha+1))^{j-1}\Gamma(\beta+1)t^{\alpha(j-1)+\beta}}\\ &\qquad\times\sum_{i=0}^{j-2}\binom{j}{i}\frac{(\Gamma(\alpha+1))^i(\Gamma(\beta+1))^{j-i}}{\Gamma(\alpha i+\beta(j-i)+1)}(Ct^\alpha)^i(Dt^\beta)^{j-i}\\ &\leq\frac{C\Gamma(\alpha+1)t^{\alpha-\beta}}{D\Gamma(\beta+1)j}\sum_{i=0}^{j-2}\left(\frac{D\Gamma(\beta+1)j}{C\Gamma(\alpha+1)t^{\alpha-\beta}}\right)^{j-i}\frac{\Gamma(\alpha(j-1)+\beta+1)}{\Gamma(\alpha i+\beta(j-i)+1)}\end{aligned}$$

$$\leq \frac{C\Gamma(\alpha+1)t^{\alpha-\beta}}{D\Gamma(\beta+1)j(\alpha(j-1)+\beta+1)^{\alpha-\beta}}$$
$$\times \sum_{i=0}^{j-2}\Big(\frac{D\Gamma(\beta+1)j(\alpha(j-1)+\beta+1)^{\alpha-\beta}}{C\Gamma(\alpha+1)t^{\alpha-\beta}}\Big)^{j-i}$$
$$\leq \frac{D\Gamma(\beta+1)j(\alpha(j-1)+\beta+1)^{\alpha-\beta}}{C\Gamma(\alpha+1)t^{\alpha-\beta}}$$
$$\times \Big(1-\frac{D\Gamma(\beta+1)j(\alpha(j-1)+\beta+1)^{\alpha-\beta}}{C\Gamma(\alpha+1)t^{\alpha-\beta}}\Big)^{-1} \leq 1.$$

The penultimate inequality is secured by $\sum_{i=0}^{j-2}(\ldots)^{j-i} \leq \sum_{i\geq 2}(\ldots)^{i}$. The function $x \mapsto x(1-x)^{-1}$ is increasing on $[0,1)$ and equal to 1 at $x=1/2$. This fact in combination with (4.4) justifies the last inequality. ■

Now we are ready to provide a simplification of formula (4.2) for convolution powers n and arguments t satisfying (4.4), with $j=n$.

COROLLARY 1. *Suppose* (3.1). *Then, for all* $t\geq 0$ *and every positive integer* j *satisfying* (4.4),

$$|V_j(t)-\rho_j t^{\alpha j}| \leq 2DC^{j-1}j\frac{(\Gamma(\alpha+1))^{j-1}\Gamma(\beta+1)}{\Gamma(\alpha(j-1)+\beta+1)}t^{\alpha(j-1)+\beta} \tag{4.6}$$

and

$$V_j(t) \leq 5\rho_j t^{\alpha j}. \tag{4.7}$$

Proof. Inequality (4.6) is an immediate consequence of Lemmas 4.1 and 4.2. A part of (4.6) reads

$$V_j(t) \leq \rho_j t^{\alpha j} + 2DC^{j-1}j\frac{(\Gamma(\alpha+1))^{j-1}\Gamma(\beta+1)}{\Gamma(\alpha(j-1)+\beta+1)}t^{\alpha(j-1)+\beta}.$$

To prove (4.7) we bound from above the ratio of the terms on the right-hand side of the last inequality:

$$\frac{2DC^{j-1}j(\Gamma(\alpha+1))^{j-1}\Gamma(\beta+1)t^{\alpha(j-1)+\beta}}{\Gamma(\alpha(j-1)+\beta+1)\rho_j t^{\alpha j}}$$
$$= \frac{2Dj\Gamma(\beta+1)\Gamma(\alpha j+1)}{C\Gamma(\alpha+1)t^{\alpha-\beta}\Gamma(\alpha(j-1)+\beta+1)}$$
$$\leq \frac{2Dj\Gamma(\beta+1)}{C\Gamma(\alpha+1)t^{\alpha-\beta}}\cdot(\alpha j+1)^{\alpha-\beta}$$
$$\leq \frac{2(\alpha j+1)^{\alpha-\beta}}{(\alpha(j-1)+\beta+1)^{\alpha-\beta}} \leq 2\Big(1+\frac{\alpha-\beta}{\alpha(j-1)+\beta+1}\Big)^{\alpha} \leq 4.$$

Here, the first inequality follows from Lemma 6.2, the second literally repeats (4.4) and the last is a consequence of $\frac{\alpha-\beta}{\alpha(j-1)+\beta+1} < 1$, which in turn is secured by $\alpha \in (0,1)$ and $\beta \geq 0$. ■

COROLLARY 2. *Suppose* (3.1). *Then, for all* $\gamma > 0$,

$$\lim_{t\to\infty} \sup_{y\geq\gamma t} \left| \frac{V_j(y)}{\rho_j y^{\alpha j}} - 1 \right| = 0 \tag{4.8}$$

whenever $j = j(t)$ *satisfies* $j(t) = o(t^{\frac{\alpha-\beta}{\alpha-\beta+1}})$ *as* $t \to \infty$, *and in particular*

$$V_j(t) \sim \rho_j t^{\alpha j}, \quad t \to \infty. \tag{4.9}$$

Proof. If $j(t) = o(t^{\frac{\alpha-\beta}{\alpha-\beta+1}})$ as $t \to \infty$, then (4.4) holds for large j and t and

$$\sup_{y\geq\gamma t} \frac{DC^{j-1} j \frac{(\Gamma(\alpha+1))^{j-1}\Gamma(\beta+1)}{\Gamma(\alpha(j-1)+\beta+1)} y^{\alpha(j-1)+\beta}}{\rho_j y^{\alpha j}}$$
$$\leq \frac{DC^{j-1} j \frac{(\Gamma(\alpha+1))^{j-1}\Gamma(\beta+1)}{\Gamma(\alpha(j-1)+\beta+1)}}{\rho_j t^{\alpha-\beta}} \sim \frac{D\Gamma(\beta+1)\alpha^{\alpha-\beta} j^{\alpha-\beta+1}}{C\Gamma(\alpha+1) t^{\alpha-\beta}} \to 0$$

as $t \to \infty$. Here, we have used the standard asymptotic relation

$$\Gamma(x+a)/\Gamma(x) \sim x^a, \quad x \to \infty, \tag{4.10}$$

for fixed $a > 0$, with $x = 1 + \alpha j - (\alpha - \beta)$ and $a = \alpha - \beta$. Thus, according to (4.6), the claim follows. ■

5. PROOF OF THEOREM 2.1

We start by collecting several (additional) auxiliary results to be used in the proof of Theorem 2.1.

Recall that $f : [0,\infty) \to [0,\infty)$ is called *directly Riemann integrable* (dRi) if $\overline{\sigma}(h) < \infty$ for each $h > 0$ and $\lim_{h\to 0+}(\overline{\sigma}(h) - \underline{\sigma}(h)) = 0$, where

$$\overline{\sigma}(h) := h \sum_{n\geq 1} \sup_{(n-1)h\leq y<nh} f(y) \quad \text{and} \quad \underline{\sigma}(h) := h \sum_{n\geq 1} \inf_{(n-1)h\leq y<nh} f(y).$$

A function $f : \mathbb{R} \to [0,\infty)$ is called dRi if the same two conditions hold with $n \in \mathbb{Z}$ replacing $n \geq 1$ in the definition of the integral sums.

The next result follows from the proof of Lemma 4.5 in [23].

LEMMA 5.1. *Let* $f : [0,\infty) \to [0,\infty)$ *be dRi and* j *a positive integer, possibly depending on* t *and possibly diverging to* ∞ *together with* t. *Then*

$$\int_{[0,\,t]} f(t-y)\,\mathrm{d}V_j(y) = O(V_{j-1}(t)), \quad t \to \infty.$$

LEMMA 5.2. *Suppose* (3.1) *and let* $j = j(t) = o(t^{\frac{\alpha-\beta}{\alpha-\beta+1}})$ *as* $t \to \infty$. *Then*

$$\int_{(t,\,\infty)} \mathrm{e}^{t-y}\,\mathrm{d}V_j(y) = O(V_{j-1}(t)), \qquad t \to \infty.$$

Proof. Let $h : \mathbb{R} \to [0, \infty)$ be a dRi function satisfying $h(t) = 0$ for $t > 0$. We start as in [23, proof of Lemma 4.6]: for $t \geq 0$,

$$\int_{(t,\,\infty)} h(t-y)\,\mathrm{d}V_j(y) = \int_{[0,\,t]} h_1(t-y)\,\mathrm{d}V_{j-1}(y) + \int_{(t,\,\infty)} h_2(t-y)\,\mathrm{d}V_{j-1}(y),$$

where $h_1(t) = \int_{(t,\,\infty)} h(t-y)\,\mathrm{d}V(y)$ and $h_2(t) = \int_{[0,\,\infty)} h(t-y)\,\mathrm{d}V(y)$ for $t \in \mathbb{R}$. By [23, Lemma A.1], $h_1(t) \leq b$ for some $b > 0$ and all $t \geq 0$, whence

$$\int_{[0,\,t]} h_1(t-y)\,\mathrm{d}V_{j-1}(y) = O(V_{j-1}(t)), \qquad t \to \infty.$$

Now we put $h(t) = \mathrm{e}^t\,\mathbb{1}_{(-\infty,0]}(t)$ and note that all the formulae given in the preceding part of the present proof hold for this h. Plainly, $h_2(t) = \mathrm{e}^t \int_{[0,\,\infty)} \mathrm{e}^{-y}\,\mathrm{d}V(y) = \kappa \mathrm{e}^t$ for $t \leq 0$, where $\kappa := \mathbb{E}\mathrm{e}^{-\eta}(1 - \mathbb{E}\mathrm{e}^{-\xi})^{-1}$. Integrating by parts yields

$$\begin{aligned}\int_{(t,\,\infty)} h_2(t-y)\,\mathrm{d}V_{j-1}(y) &= \kappa \int_{(t,\,\infty)} \mathrm{e}^{t-y}\,\mathrm{d}V_{j-1}(y)\\ &= -\kappa V_{j-1}(t) + \kappa \int_t^\infty \mathrm{e}^{t-y} V_{j-1}(y)\,\mathrm{d}y =: \kappa C_j(t)\end{aligned}$$

for $t \geq 0$. By Corollary 2, (3.1) together with our choice of $j = j(t)$ ensures (4.8). In view of (4.8), given $\varepsilon > 0$ we obtain, for large enough t,

$$\begin{aligned}0 \leq C_j(t) &\leq -V_{j-1}(t) + (1+\varepsilon)\rho_{j-1} \int_t^\infty \mathrm{e}^{t-y} y^{\alpha(j-1)}\,\mathrm{d}y\\ &= -V_{j-1}(t) + (1+\varepsilon)(C\Gamma(\alpha+1))^{j-1}\\ &\times \left(\sum_{k=0}^{\lfloor \alpha(j-1) \rfloor} \frac{t^{\alpha(j-1)-k}}{\Gamma(\alpha(j-1)+1-k)} + \frac{1}{\Gamma(\{\alpha(j-1)\})} \int_t^\infty \mathrm{e}^{t-y} y^{\{\alpha(j-1)\}-1}\,\mathrm{d}y \right),\end{aligned}$$

where $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$. The expression in parentheses is asymptotically equivalent to $t^{\alpha(j-1)}/\Gamma(\alpha(j-1)+1)$ whenever $j(t) = o(t)$ as $t \to \infty$. Hence, recalling that, according to (4.9) (which holds according to Corollary 2), $V_{j-1}(t) \sim \rho_{j-1} t^{\alpha(j-1)}$ as $t \to \infty$, we conclude that $\limsup_{t\to\infty} C_j(t)/V_{j-1}(t) \leq \varepsilon$ and so $C_j(t) = o(V_{j-1}(t))$ as $t \to \infty$ upon letting $\varepsilon \to 0+$. ∎

LEMMA 5.3. *Suppose* (3.1) *and let* $j = j(t) \to \infty$ *and* $j(t) = o(t^{\frac{\alpha-\beta}{\alpha-\beta+1}})$ *as* $t \to \infty$. *Then*

$$\lim_{s\to\infty} \limsup_{t\to\infty} \frac{j^\alpha}{\rho_{j-1} t^{\alpha j}} \int_{(st/j,\, t]} y^\alpha \, \mathrm{d}(-V_{j-1}(t-y)) = 0.$$

Proof. Integrating by parts we conclude that

$$\begin{aligned}\frac{j^\alpha}{\rho_{j-1} t^{\alpha j}} \int_{(st/j,\, t]} y^\alpha \, \mathrm{d}(-V_{j-1}(t-y)) &= \frac{s^\alpha V_{j-1}(t(1-s/j))}{\rho_{j-1} t^{\alpha(j-1)}} \\ &\quad + \frac{\alpha j^\alpha}{\rho_{j-1} t^{\alpha j}} \int_{st/j}^{t} V_{j-1}(t-y) y^{\alpha-1} \, \mathrm{d}y.\end{aligned}$$

In view of (4.8) which holds true according to Corollary 2,

$$\frac{s^\alpha V_{j-1}(t(1-s/j))}{\rho_{j-1} t^{\alpha(j-1)}} \sim s^\alpha (1-s/j)^{\alpha(j-1)} \ \to \ s^\alpha \mathrm{e}^{-\alpha s}, \quad t \to \infty.$$

The right-hand side converges to 0 as $s \to \infty$. Further, with the help of (4.2) we infer that

$$\begin{aligned}&\frac{j^\alpha}{\rho_{j-1} t^{\alpha j}} \int_{st/j}^{t} V_{j-1}(t-y) y^{\alpha-1} \, \mathrm{d}y \\ &\quad \le \frac{j^\alpha}{\rho_{j-1} t^{\alpha j}} \Bigg(\int_{st/j}^{t} \rho_{j-1} (t-y)^{\alpha(j-1)} y^{\alpha-1} \, \mathrm{d}y \\ &\qquad + \int_{st/j}^{t} \sum_{i=0}^{j-2} \binom{j-1}{i} \frac{(\Gamma(\alpha+1))^i (\Gamma(\beta+1))^{j-i-1}}{\Gamma(\alpha i + \beta(j-i-1)+1)} \\ &\qquad\qquad \times (C(t-y)^\alpha)^i (D(t-y)^\beta)^{j-i-1} y^{\alpha-1} \, \mathrm{d}y \Bigg).\end{aligned} \tag{5.1}$$

Changing the variable we obtain for the first term, as $t \to \infty$,

$$\frac{j^\alpha}{t^{\alpha j}} \int_{st/j}^{t} (t-y)^{\alpha(j-1)} y^{\alpha-1} \, \mathrm{d}y = \int_s^j (1-y/j)^{\alpha(j-1)} y^{\alpha-1} \, \mathrm{d}y \ \to \ \int_s^\infty \mathrm{e}^{-\alpha y} y^{\alpha-1} \, \mathrm{d}y,$$

having utilized the Lebesgue dominated convergence theorem for the limit relation. The right-hand side converges to 0 as $s \to \infty$. Invoking $y^{\alpha-1} \le (st/j)^{\alpha-1}$ for $y \in [st/j, t]$ and then evaluating the remaining integral we conclude that the second term in (5.1) does not exceed

$$\frac{j^\alpha}{\rho_{j-1}t^{\alpha j}}\Big(\frac{st}{j}\Big)^{\alpha-1}\sum_{i=0}^{j-2}\binom{j-1}{i}\frac{(C\Gamma(\alpha+1))^i(D\Gamma(\beta+1))^{j-i-1}}{\Gamma(\alpha i+\beta(j-i-1)+1)}$$
$$\times\frac{(t-st/j)^{\alpha i+\beta(j-i-1)+1}}{\alpha i+\beta(j-i-1)+1}$$
$$=s^{\alpha-1}j\sum_{i=0}^{j-2}\binom{j-1}{i}\frac{\Gamma(\alpha(j-1)+1)}{\Gamma(\alpha i+\beta(j-i-1)+1)}\Big(\frac{D\Gamma(\beta+1)t^\beta}{C\Gamma(\alpha+1)t^\alpha}\Big)^{j-i-1}$$
$$\times\frac{(1-s/j)^{\alpha i+\beta(j-i-1)+1}}{\alpha i+\beta(j-i-1)+1}$$
$$\leq\frac{s^{\alpha-1}}{\beta}\sum_{i=0}^{j-2}\binom{j-1}{i}\frac{\Gamma(\alpha(j-1)+1)}{\Gamma(\alpha i+\beta(j-i-1)+1)}\Big(\frac{D\Gamma(\beta+1)t^\beta}{C\Gamma(\alpha+1)t^\alpha}\Big)^{j-i-1}$$
$$\leq\frac{s^{\alpha-1}}{\beta}\sum_{i=0}^{j-2}((j-1)(\alpha(j-1)+1)^{\alpha-\beta}\frac{D\Gamma(\beta+1)}{C\Gamma(\alpha+1)t^{\alpha-\beta}})^{j-i-1}$$
$$\leq\frac{s^{\alpha-1}}{\beta}\,\frac{D\Gamma(\beta+1)}{C\Gamma(\alpha+1)}(j-1)\Big(\frac{\alpha(j-1)+1}{t}\Big)^{\alpha-\beta}$$
$$\times\bigg(1-\frac{D\Gamma(\beta+1)}{C\Gamma(\alpha+1)}(j-1)\Big(\frac{\alpha(j-1)+1}{t}\Big)^{\alpha-\beta}\bigg)^{-1}\ \to\ 0,\quad t\to\infty.$$

We have used $0<(1-s/j)^{\alpha i+\beta(j-i-1)+1}\leq 1$ and $\frac{1}{\alpha i+\beta(j-i-1)+1}\leq\frac{1}{\beta j}$ for the first inequality, and (4.5) and Lemma 6.2 for the second. Our assumption $j(t)=o(t^{\frac{\alpha-\beta}{\alpha-\beta+1}})$ as $t\to\infty$ entails $\lim_{t\to\infty}(j(t)-1)((\alpha(j(t)-1)+1)/t)^{\alpha-\beta}=0$, thereby justifying the limit relation. ∎

A major part of the proof of Theorem 2.1 is covered by Theorems 5.1 and 5.2 given next.

THEOREM 5.1. *Suppose* (3.1) *and let* $j=j(t)\to\infty$ *and* $j(t)=o(t^{\frac{\alpha-\beta}{\alpha-\beta+1}})$ *as* $t\to\infty$. *Then*

$$\Big(\frac{c(j(t))^\alpha}{\rho_{\lfloor j(t)u\rfloor-1}t^{\alpha\lfloor j(t)u\rfloor}}\sum_{r\geq1}V_{\lfloor j(t)u\rfloor-1}(t-T_r)\mathbb{1}_{\{T_r\leq t\}}\Big)_{u>0}$$
$$\xrightarrow{\text{f.d.d.}}\Big(\int_{[0,\infty)}\mathrm{e}^{-\alpha uy}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y)\Big)_{u>0},\quad t\to\infty,$$

where ρ_n *is as given in* (2.1)*, and* $\mathcal{Z}_\alpha^{\leftarrow}$ *is an inverse* α*-stable subordinator.*

Our proof of Theorem 5.1 uses an auxiliary technical result, which is a slight reformulation of [17, Lemma A.5]. Let $\mathcal{D}$ denote the Skorokhod space of càdlàg functions defined on $[0,\infty)$, endowed with the J_1-topology. Comprehensive information about the J_1-topology on $\mathcal{D}$ can be found in [4].

LEMMA 5.4. *For each $k \in \mathbb{N}$, let $y_k : [0,\infty) \to \mathbb{R}$ be a right-continuous bounded and nondecreasing function. Assume that $\lim_{k\to\infty} x_k = x$ on $\mathcal{D}$ and, for each $t \geq 0$, $\lim_{k\to\infty} y_k(t) = y(t)$, where $y : [0,\infty) \to \mathbb{R}$ is a bounded continuous function. Then, for all $a, b \geq 0$ with $a < b$,*

$$\lim_{k\to\infty} \int\limits_{[a,\,b]} x_k(t)\,\mathrm{d}y_k(t) = \int\limits_{[a,\,b]} x(t)\,\mathrm{d}y(t).$$

Proof of Theorem 5.1. For notational simplicity we write j for $j(t)$. In view of the Cramér–Wold device and the equality $\int_{[0,\,\infty)} \mathrm{e}^{-\alpha u y}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y) = \alpha u \times \int_0^\infty \mathcal{Z}_\alpha^{\leftarrow}(y)\mathrm{e}^{-\alpha u y}\,\mathrm{d}y$ for $u > 0$, obtained by integration by parts, it suffices to show that, for any $\ell \in \mathbb{N}$, any positive $\lambda_1, \ldots, \lambda_\ell$ and any $0 < u_1 < \cdots < u_\ell$, as $t \to \infty$,

$$\sum_{i=1}^{\ell} \lambda_i \frac{cj^\alpha \sum_{r\geq 1} V_{\lfloor ju_i\rfloor -1}(t-T_r)\mathbb{1}_{\{T_r\leq t\}}}{\rho_{\lfloor ju_i\rfloor-1} t^{\alpha\lfloor ju_i\rfloor}} \xrightarrow{\mathrm{d}} \sum_{i=1}^{\ell} \lambda_i \alpha u_i \int\limits_0^\infty \mathcal{Z}_\alpha^{\leftarrow}(y)\mathrm{e}^{-\alpha u_i y}\,\mathrm{d}y,$$

where, as before, $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution. Write, for any $s > 0$ and sufficiently large t,

$$\begin{aligned}
\frac{cj^\alpha \sum_{r\geq 1} V_{\lfloor ju_i\rfloor -1}(t-T_r)\mathbb{1}_{\{T_r\leq t\}}}{\rho_{\lfloor ju_i\rfloor-1} t^{\alpha\lfloor ju_i\rfloor}} &= \frac{cj^\alpha \int_{[0,\,t]} V_{\lfloor ju_i\rfloor-1}(t-y)\,\mathrm{d}N(y)}{\rho_{\lfloor ju_i\rfloor-1} t^{\alpha\lfloor ju_i\rfloor}} \\
&= \frac{cj^\alpha \int_{[0,\,t]} N(y)\,\mathrm{d}_y(-V_{\lfloor ju_i\rfloor-1}(t-y))}{\rho_{\lfloor ju_i\rfloor-1} t^{\alpha\lfloor ju_i\rfloor}} \\
= \frac{1}{\rho_{\lfloor ju_i\rfloor-1} t^{\alpha(\lfloor ju_i\rfloor-1)}} &\int\limits_{[0,\,s]} \frac{cN(yt/j)}{(t/j)^\alpha}\,\mathrm{d}_y(-V_{\lfloor ju_i\rfloor-1}(t(1-y/j))) \\
&+ \frac{cj^\alpha}{\rho_{\lfloor ju_i\rfloor-1} t^{\alpha\lfloor ju_i\rfloor}} \int\limits_{(st/j,\,t]} N(y)\,\mathrm{d}_y(-V_{\lfloor ju_i\rfloor-1}(t-y)).
\end{aligned}$$

Using (4.8) we obtain, for each fixed $y > 0$, as $t \to \infty$,

$$\begin{aligned}
V_{\lfloor ju_i\rfloor-1}(t(1-y/j)) &\sim \rho_{\lfloor ju_i\rfloor-1} t^{\alpha(\lfloor ju_i\rfloor-1)}(1-y/j)^{\alpha(\lfloor ju_i\rfloor-1)} \\
&\sim \rho_{\lfloor ju_i\rfloor-1} t^{\alpha(\lfloor ju_i\rfloor-1)}\mathrm{e}^{-\alpha u_i y}.
\end{aligned} \tag{5.2}$$

Under the sole assumption $\mathbb{P}\{\xi > t\} \sim ct^{-\alpha}$ as $t \to \infty$, an application of [1, Theorem 3.2(B.4)] yields

$$(c(t/j)^{-\alpha}N(yt/j))_{y\geq 0} \rightharpoonup (\mathcal{Z}_\alpha^{\leftarrow}(y))_{y\geq 0}, \quad t\to\infty,$$

on $\mathcal{D}$, where $\rightharpoonup$ denotes weak convergence on a function space. Here, we have used the fact that the assumptions imposed on j ensure that $\lim_{t\to\infty} t/j(t) = \infty$.

Let $(t_k)_{k\geq 1}$ be any sequence of positive numbers satisfying $\lim_{k\to\infty} t_k = \infty$. According to the Skorokhod representation theorem, there exist a version $\hat{\mathcal{Z}}_\alpha^{\leftarrow}$ of $\mathcal{Z}_\alpha^{\leftarrow}$ and a version $((\hat{N}_{t_k}(y))_{y\geq 0})_{k\geq 1}$ of $((c(t_k/j(t_k))^{-\alpha}N(yt_k/j(t_k)))_{y\geq 0})_{k\geq 1}$ such that, for all $T>0$,

$$\lim_{k\to\infty}\sup_{y\in[0,\,T]}|\hat{N}_{t_k}(y)-\hat{\mathcal{Z}}_\alpha^{\leftarrow}(y)|=0\quad\text{ a.s.}$$

In view of (5.2),

$$\lim_{k\to\infty}\frac{V_{\lfloor j(t_k)u_i\rfloor-1}(t_k(1-y/j(t_k)))}{\rho_{\lfloor j(t_k)u_i\rfloor-1}t_k^{\alpha(\lfloor j(t_k)u_i\rfloor-1)}}=\mathrm{e}^{-\alpha u_i y}.$$

Using Lemma 5.4 with $x_k=\hat{N}_{t_k}$, $x=\hat{\mathcal{Z}}_\alpha^{\leftarrow}$, and y_k and y given by the functions on the left-hand side and the right-hand side of the last displayed formula, respectively, we obtain

$$\lim_{k\to\infty}\frac{1}{\rho_{\lfloor j(t_k)u_i\rfloor-1}t_k^{\alpha(\lfloor j(t_k)u_i\rfloor-1)}}\int_{[0,\,s]}\hat{N}_{t_k}(y)\,\mathrm{d}_y(-V_{\lfloor j(t_k)u_i\rfloor-1}(t_k(1-y/j(t_k))))$$
$$=\alpha u_i\int_0^s\hat{\mathcal{Z}}_\alpha^{\leftarrow}(y)\mathrm{e}^{-\alpha u_i y}\,\mathrm{d}y\quad\text{ a.s.}$$

Since the divergent sequence $(t_k)_{k\geq 1}$ is arbitrary, this entails

$$\sum_{i=1}^{\ell}\lambda_i\frac{1}{\rho_{\lfloor ju_i\rfloor-1}t^{\alpha(\lfloor ju_i\rfloor-1)}}\int_{[0,\,s]}c(t/j)^{-\alpha}N(yt/j)\,\mathrm{d}_y(-V_{\lfloor ju_i\rfloor-1}(t(1-y/j)))$$
$$\stackrel{\mathrm{d}}{\to}\sum_{i=i}^{\ell}\lambda_i\alpha u_i\int_0^s\mathcal{Z}_\alpha^{\leftarrow}(y)\mathrm{e}^{-\alpha u_i y}\,\mathrm{d}y,\qquad t\to\infty.$$

In view of

$$\lim_{s\to\infty}\sum_{i=1}^{\ell}\lambda_i\alpha u_i\int_0^s\mathcal{Z}_\alpha^{\leftarrow}(y)\mathrm{e}^{-\alpha u_i y}\,\mathrm{d}y=\sum_{i=1}^{\ell}\lambda_i\alpha u_i\int_0^\infty\mathcal{Z}_\alpha^{\leftarrow}(y)\mathrm{e}^{-\alpha u_i y}\,\mathrm{d}y\quad\text{ a.s.,}$$

it remains to show that, for all $\varepsilon>0$,

$$\lim_{s\to\infty}\limsup_{t\to\infty}\mathbb{P}\Big\{\frac{j^\alpha}{\rho_{\lfloor ju_i\rfloor-1}t^{\alpha\lfloor ju_i\rfloor}}\int_{(st/j,\,t]}N(y)\,\mathrm{d}_y(-V_{\lfloor ju_i\rfloor-1}(t-y))>\varepsilon\Big\}=0.$$

Noting that $\mathbb{E}N(y)=V(y)\leq 2Cy^\alpha$ for large y, this limit relation is ensured by Markov's inequality and Lemma 5.3. ∎

THEOREM 5.2. *Suppose* (3.1) *and let* $t \mapsto j(t)$ *be an integer-valued function satisfying* $j(t) = o(t^{1/3})$ *as* $t \to \infty$ *and* $\lim_{t\to\infty} j(t) = \infty$. *Then*

$$\frac{(j(t))^\alpha}{\rho_{j(t)-1} t^{\alpha j(t)}} \Big(N_{j(t)}(t) - \sum_{r\geq 1} V_{j(t)-1}(t - T_r) \mathbb{1}_{\{T_r \leq t\}} \Big) \xrightarrow{\mathbb{P}} 0, \qquad t \to \infty.$$

Proof. Although the scheme of the proof is analogous to that in [23, Theorem 3.1], some technical details are essentially different.

For $j \in \mathbb{N}$ and $t \geq 0$, put $D_j(t) := \operatorname{Var} N_j(t)$ and

$$I_j(t) := \mathbb{E}\Big(\sum_{r\geq 1} V_{j-1}(t - T_r) \mathbb{1}_{\{T_r \leq t\}} - V_j(t) \Big)^2$$

with the convention that $V_0(t) = 1$ for $t \geq 0$, so that $I_1(t) = D_1(t)$. We shall use a formula: for $j \geq 2$ and $t \geq 0$,

$$\begin{aligned} D_j(t) &= \mathbb{E}\Big(\sum_{r\geq 1} \big(N^{(r)}_{j-1}(t - T_r) - V_{j-1}(t - T_r) \big) \mathbb{1}_{\{T_r \leq t\}} \Big)^2 \\ &\quad + \mathbb{E}\Big(\sum_{r\geq 1} V_{j-1}(t - T_r) \mathbb{1}_{\{T_r \leq t\}} - V_j(t) \Big)^2 \\ &= \int_{[0,\,t]} D_{j-1}(t - y)\, \mathrm{d}V(y) + I_j(t). \end{aligned} \tag{5.3}$$

Iterating (5.3) yields

$$\int_{[0,\,t]} D_{j-1}(t - y)\, \mathrm{d}V(y) = \sum_{k=1}^{j-1} \int_{[0,\,t]} I_k(t - y)\, \mathrm{d}V_{j-k}(y), \qquad j \geq 2,\ t \geq 0. \tag{5.4}$$

First we show that I_j is bounded from above by a nonnegative and nondecreasing function. We need an inequality obtained in [23, proof of Theorem 3.1]:

$$\begin{aligned} I_j(t) &\leq V_{j-1}(t)V_j(t) + 2 \int_{[0,\,t]} V_{j-1}(t - y)V_j(t - y)\, \mathrm{d}U(y) - (V_j(t))^2 \\ &\leq V_{j-1}(t)V_j(t) + 2 \int_{[0,\,t]} V_{j-1}(t - y)V_j(t - y)\, \mathrm{d}U(y). \end{aligned} \tag{5.5}$$

Put $\tilde{U}(x) := \sum_{i\geq 1} \mathbb{P}\{S_i \leq x\}$ for $x \in \mathbb{R}$ and note that $U(x) = \tilde{U}(x) + 1$ for $x \geq 0$. Assumption A together with $\tilde{U}(0) = 0$ entails

$$\tilde{U}(x) \leq Cx^\alpha + C_1 x^\rho, \qquad x \geq 0,$$

for some constant $C_1 > 0$. Integrating by parts and then using the last inequality yields

$$\begin{aligned}\int\limits_{[0,\,t]} & V_{j-1}(t-y)V_j(t-y)\,\mathrm{d}U(y) \\ &= V_{j-1}(t)V_j(t) + \int\limits_{[0,\,t]} V_{j-1}(t-y)V_j(t-y)\,\mathrm{d}\tilde{U}(y) \\ &= V_{j-1}(t)V_j(t) + \int\limits_{[0,\,t]} \tilde{U}(t-y)\,\mathrm{d}(V_{j-1}(y)V_j(y)) \\ &\le V_{j-1}(t)V_j(t) + C\int\limits_{[0,\,t]} (t-y)^{\alpha}\,\mathrm{d}(V_{j-1}(y)V_j(y)) \\ &\quad + C_1\int\limits_{[0,\,t]} (t-y)^{\rho}\,\mathrm{d}(V_{j-1}(y)V_j(y)).\end{aligned}$$

With this at hand, integrating by parts once again we arrive at

$$\begin{aligned}I_j(t) &\le V_{j-1}(t)V_j(t) + 2\int\limits_{[0,\,t]} V_{j-1}(t-y)V_j(t-y)\,\mathrm{d}U(y) \\ &\le 3V_{j-1}(t)V_j(t) + 2C\alpha\int_0^t V_{j-1}(y)V_j(y)(t-y)^{\alpha-1}\,\mathrm{d}y \\ &\quad + 2C_1\rho\int_0^t V_{j-1}(y)V_j(y)(t-y)^{\rho-1}\,\mathrm{d}y =: h_j(t).\end{aligned} \tag{5.6}$$

Here h_j is a nonnegative and nondecreasing function that we were looking for. Combining (5.6) with (5.3) and (5.4) we infer that

$$\begin{aligned}D_{j-1}(t) &= \sum_{k=1}^{j-2}\int_0^t I_k(t-y)\,\mathrm{d}V_{j-k-1}(y) + I_{j-1}(t) \\ &\le \sum_{k=1}^{j-2} h_k(t)V_{j-k-1}(t) + h_{j-1}(t), \quad j\ge 2,\ t\ge 0.\end{aligned} \tag{5.7}$$

Next, we obtain an upper bound for h_j (and for D_j) which is valid for large arguments. Fix $j\in\mathbb{N}$ and $s\ge 0$ satisfying

$$2D\Gamma(\beta+1)j(\alpha(j-1)+\beta+1)^{\alpha-\beta} \le C\Gamma(\alpha+1)s^{\alpha-\beta}, \tag{5.8}$$

$$2(\alpha(j-2)+1)^{2\alpha} \le C\Gamma(\alpha+1)s^{\alpha}, \tag{5.9}$$

$$3(2\alpha j+1)^{\alpha} \le 2C\Gamma(\alpha+1)s^{\alpha}, \tag{5.10}$$

$$C_1\Gamma(\rho)(2\alpha j+1)^{\alpha-\rho} \le \Gamma(\alpha+1)s^{\alpha-\rho}, \tag{5.11}$$

$$j\ge 1/\alpha+2. \tag{5.12}$$

Plainly, these inequalities trivially hold for large enough s whenever $j=j(s)=o(s^{1/2})$ as $s\to\infty$.

Put $r:=5^2$. In view of (5.8), we conclude using (4.7) that, for $1\le k\le j$,

$$V_{k-1}(s)V_k(s) \le r\rho_{k-1}\rho_k s^{\alpha(2k-1)}.$$

Substituting this into (5.6) we obtain

$$h_k(s) \le 3r\rho_{k-1}\rho_k s^{\alpha(2k-1)} + 2C\alpha r\rho_{k-1}\rho_k \int_0^s s^{\alpha(2k-1)}(s-y)^{\alpha-1}\,\mathrm{d}y + 2C_1\rho r\rho_{k-1}\rho_k \int_0^s s^{\alpha(2k-1)}(s-y)^{\rho-1}\,\mathrm{d}y$$
$$= 3r\rho_{k-1}\rho_k s^{\alpha(2k-1)} + 2rC\alpha\rho_{k-1}\rho_k \mathrm{B}(\alpha(2k-1)+1,\alpha)s^{2\alpha k} + 2rC_1\rho\rho_{k-1}\rho_k\mathrm{B}(\alpha(2k-1)+1,\rho)\mathrm{B}(\alpha(2k-1)+1,\rho)s^{2\alpha k-\alpha+\rho}, \tag{5.13}$$

where B is the Euler beta-function. The second term on the right-hand side is of the highest order. Consider the ratio of the first and the second terms in (5.13):

$$\frac{3r\rho_{k-1}\rho_k s^{\alpha(2k-1)}}{2rC\alpha\rho_{k-1}\rho_k\mathrm{B}(\alpha(2k-1)+1,\alpha)s^{2\alpha k}} = \frac{3\Gamma(2\alpha k+1)}{2C\Gamma(\alpha+1)\Gamma(\alpha(2k-1)+1)s^{\alpha}} \le \frac{3(2\alpha k+1)^{\alpha}}{2C\Gamma(\alpha+1)s^{\alpha}} \le \frac{3(2\alpha j+1)^{\alpha}}{2C\Gamma(\alpha+1)s^{\alpha}} \le 1,$$

where the first inequality is justified by Lemma 6.2 and the second one by (5.10). The ratio of the third and second terms in (5.13) can be estimated as follows:

$$\frac{2rC_1\rho\rho_{k-1}\rho_k\mathrm{B}(\alpha(2k-1)+1,\rho)s^{2\alpha k-\alpha+\rho}}{2rC\alpha\rho_{k-1}\rho_k\mathrm{B}(\alpha(2k-1)+1,\alpha)s^{2\alpha k}} = \frac{C_1\Gamma(\rho)\Gamma(2\alpha k+1)}{\Gamma(\alpha+1)s^{\alpha-\rho}\Gamma(2\alpha k-\alpha+\rho+1)} \le \frac{C_1\Gamma(\rho)(2\alpha k+1)^{\alpha-\rho}}{\Gamma(\alpha+1)s^{\alpha-\rho}} \le \frac{C_1\Gamma(\rho)(2\alpha j+1)^{\alpha-\rho}}{\Gamma(\alpha+1)s^{\alpha-\rho}} \le 1,$$

where the first inequality follows from Lemma 6.2 and the last one from (5.11). Inequality (5.13) in combination with the two estimates enables us to conclude that, for $1 \le k \le j$,

$$h_k(s) \le 6rC\alpha\rho_{k-1}\rho_k\mathrm{B}(\alpha(2k-1)+1,\alpha)s^{2\alpha k} =: C_2\rho_{k-1}\rho_k\mathrm{B}(\alpha(2k-1)+1,\alpha)s^{2\alpha k}.$$

Invoking (5.7) we further obtain

$$D_{j-1}(s) \le C_2\rho_{j-2}\rho_{j-1}\mathrm{B}(\alpha(2j-3)+1,\alpha)s^{2\alpha(j-1)} + \sum_{k=1}^{j-2} C_2\rho_{k-1}\rho_k\mathrm{B}(\alpha(2k-1)+1,\alpha)s^{2\alpha k}V_{j-k-1}(s). \tag{5.14}$$

Write, with the help of (2.1) and (4.7),

$$
\begin{aligned}
(5.15)\quad & \sum_{k=1}^{j-2} \rho_{k-1}\rho_k \mathrm{B}(\alpha(2k-1)+1,\alpha) s^{2\alpha k} V_{j-k-1}(s) \\
&\leq 5\sum_{k=1}^{j-2} \rho_{k-1}\rho_k \mathrm{B}(\alpha(2k-1)+1,\alpha) s^{2\alpha k}\rho_{j-k-1} s^{\alpha(j-k-1)} \\
&= 5\Gamma(\alpha)\sum_{k=1}^{j-2} \frac{(C\Gamma(\alpha+1))^{j+k-2}\Gamma(\alpha(2k-1)+1)}{\Gamma(\alpha(k-1)+1)\Gamma(\alpha k+1)\Gamma(\alpha(j-k-1)+1)\Gamma(2\alpha k+1)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times s^{\alpha(j+k-1)} \\
&\leq 5(2\alpha+1)\Gamma(\alpha)\sum_{k=1}^{j-2} \frac{(C\Gamma(\alpha+1))^{j+k-2}}{(\Gamma(\alpha(k-1)+1))^2\Gamma(\alpha(j-k-1)+1)} s^{\alpha(j+k-1)}.
\end{aligned}
$$

To justify the last inequality, we use the fact that Γ is increasing on $[2,\infty)$, whence, for $k\geq 1$,

$$
\begin{aligned}
\frac{1}{\Gamma(\alpha k+1)} = \frac{\alpha k+1}{\Gamma(\alpha k+2)} &\leq \frac{\alpha k+1}{\Gamma(\alpha(k-1)+2)} \\
&= \frac{\alpha k+1}{(\alpha(k-1)+1)\Gamma(\alpha(k-1)+1)} \leq \frac{\alpha+1}{\Gamma(\alpha(k-1)+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\Gamma(\alpha(2k-1)+1)}{\Gamma(2\alpha k+1)} &= \frac{\Gamma(\alpha(2k-1)+2)}{\Gamma(2\alpha k+2)}\frac{2\alpha k+1}{\alpha(2k-1)+1} \\
&\leq \frac{2\alpha k+1}{\alpha(2k-1)+1} \leq \frac{2\alpha+1}{\alpha+1}.
\end{aligned}
$$

The right-hand side of (5.15), with the multiplicative constant omitted, can be bounded from above as follows:

$$
\begin{aligned}
(5.16)\quad & \\
&\frac{1}{(\Gamma(\alpha(j-2)+1))^2}\sum_{k=1}^{j-2}\frac{(C\Gamma(\alpha+1))^{j+k-2}(\Gamma(\alpha(j-2)+1))^2}{(\Gamma(\alpha(k-1)+1))^2\Gamma(\alpha(j-k-1)+1)} s^{\alpha(j+k-1)} \\
&\leq \frac{\mathrm{const}}{(\Gamma(\alpha(j-2)+1))^2}\sum_{k=1}^{j-2}(C\Gamma(\alpha+1))^{j+k-2}(\alpha(j-2)+1)^{2\alpha(j-k-1)} s^{\alpha(j+k-1)}.
\end{aligned}
$$

Hereinafter, const denotes a constant whose value is of no importance and may change from one appearance to another. The last inequality is a consequence of

$$
\frac{\Gamma(\alpha(j-2)+1)}{\Gamma(\alpha(k-1)+1)} \leq (\alpha(j-2)+1)^{\alpha(j-k-1)},
$$

which follows from Lemma 6.2, and $\frac{1}{\Gamma(\alpha(j-k-1)+1)} \leq \frac{1}{\min_{z\in[1,2]}\Gamma(z)} < \infty$ for $1\leq k\leq j-2$ (the minimum cannot be 0 because $\Gamma(z)$ is the moment of order

$z-1$ of an exponentially distributed random variable with mean 1). We proceed by bounding from above the right-hand side of (5.16), up to a multiplicative constant:

$$\begin{aligned}
&\frac{(C\Gamma(\alpha+1))^{2j-3}s^{2\alpha(j-1)}}{(\Gamma(\alpha(j-2)+1))^2}\sum_{k=1}^{j-2}\Big(\frac{(\alpha(j-2)+1)^{2\alpha}}{C\Gamma(\alpha+1)s^\alpha}\Big)^{j-k-1}\\
&\qquad=\frac{(C\Gamma(\alpha+1))^{2j-3}s^{2\alpha(j-1)}}{(\Gamma(\alpha(j-2)+1))^2}\sum_{k=1}^{j-2}\Big(\frac{(\alpha(j-2)+1)^{2\alpha}}{C\Gamma(\alpha+1)s^\alpha}\Big)^{k}\\
&\qquad\leq\frac{(C\Gamma(\alpha+1))^{2j-3}s^{2\alpha(j-1)}}{(\Gamma(\alpha(j-2)+1))^2}\sum_{k\geq 1}\Big(\frac{(\alpha(j-2)+1)^{2\alpha}}{C\Gamma(\alpha+1)s^\alpha}\Big)^{k}\\
&\qquad\leq\frac{(C\Gamma(\alpha+1))^{2j-3}s^{2\alpha(j-1)}}{(\Gamma(\alpha(j-2)+1))^2}.
\end{aligned}$$

Here, the last inequality is secured by (5.9). Combining this with (5.14) we conclude that

$$\begin{aligned}
D_{j-1}(s)\leq C_2\rho_{j-2}\rho_{j-1}\mathrm{B}(\alpha(2j-3)+1,\alpha)s^{2\alpha(j-1)}&\\
+\frac{\mathrm{const}(C\Gamma(\alpha+1))^{2j-3}s^{2\alpha(j-1)}}{(\Gamma(\alpha(j-2)+1))^2}&.
\end{aligned}$$

Condition (5.12) entails $2j\geq 1/\alpha+3$. This together with monotonicity of the gamma-function on $[2,\infty)$ proves

$$\begin{aligned}
&\frac{\Gamma(\alpha)(C\Gamma(\alpha+1))^{2j-3}}{\Gamma^2(\alpha(j-2)+1)\rho_{j-2}\rho_{j-1}\mathrm{B}(\alpha(2j-3)+1,\alpha)}\\
&\qquad=\frac{\Gamma(\alpha(j-1)+1)\Gamma(\alpha(2j-2)+1)}{\Gamma(\alpha(j-2)+1)\Gamma(\alpha(2j-3)+1)}\geq 1
\end{aligned}$$

and so

$$D_{j-1}(s)\leq\mathrm{const}\,\frac{(C\Gamma(\alpha+1))^{2j-3}s^{2\alpha(j-1)}}{(\Gamma(\alpha(j-2)+1))^2}. \tag{5.17}$$

Using (5.17) we infer that

$$\begin{aligned}
&\int_{[0,\,t]}D_{j-1}(t-y)\,\mathrm{d}V(y)\\
&\qquad=\int_{[0,\,t-j]}D_{j-1}(t-y)\,\mathrm{d}V(y)+\int_{(t-j,\,t]}D_{j-1}(t-y)\,\mathrm{d}V(y)\\
&\qquad\leq\mathrm{const}\,\frac{(C\Gamma(\alpha+1))^{2j-3}}{(\Gamma(\alpha(j-2)+1))^2}\int_{[0,\,t]}(t-y)^{2\alpha(j-1)}\,\mathrm{d}V(y)\\
&\qquad\quad+\max_{s\in[0,\,j]}D_{j-1}(s)\,U(j)
\end{aligned} \tag{5.18}$$

having utilized $V(t)-V(t-j)\le U(j)$; see [7, (40) and its proof]. Integrating by parts and then invoking (3.1) we estimate the last integral as follows:

$$\begin{aligned}\int\limits_{[0,\,t]}(t-y)^{2\alpha(j-1)}\,\mathrm{d}V(y)&\le 2\alpha(j-1)\int\limits_0^t(Cy^\alpha+Dy^\beta)(t-y)^{2\alpha(j-1)-1}\,\mathrm{d}y\\&=2\alpha(j-1)(C\mathrm{B}(2\alpha(j-1),\alpha+1)t^{\alpha(2j-1)}\\&\quad+D\mathrm{B}(2\alpha(j-1),\beta+1)t^{2\alpha(j-1)+\beta})\\&=O(j\mathrm{B}(2\alpha(j-1),\alpha+1)t^{\alpha(2j-1)}),\qquad t\to\infty.\end{aligned}$$

To justify the last equality, note that, as $j,t\to\infty$,

$$\begin{aligned}\mathrm{B}(2\alpha(j-1),\alpha+1)t^{\alpha(2j-1)}&\sim\mathrm{const}\,\frac{t^{\alpha(j-1)}}{j^{\alpha+1}},\\ \mathrm{B}(2\alpha(j-1),\beta+1)t^{2\alpha(j-1)+\beta}&\sim\mathrm{const}\,\frac{t^{2\alpha(j-1)+\beta}}{j^{\beta+1}}.\end{aligned}$$

Hence, $\mathrm{B}(2\alpha(j-1),\beta+1)t^{2\alpha(j-1)+\beta}=o(\mathrm{B}(2\alpha(j-1),\alpha+1)t^{\alpha(2j-1)})$ as a consequence of $j=j(t)=o(t^{1/3})$. Thus, the first term on the right-hand side of (5.18) is of the order

$$\frac{(C\Gamma(\alpha+1))^{2j-3}}{(\Gamma(\alpha(j-2)+1))^2}j\mathrm{B}(2\alpha(j-1),\alpha+1)t^{\alpha(2j-1)}.$$

The product of this and the squared normalization (appearing in the theorem) vanishes under the assumption $j=j(t)=o(t^{1/3})$ as $t\to\infty$:

$$\begin{aligned}&\frac{(C\Gamma(\alpha+1))^{2j-3}j\mathrm{B}(2\alpha(j-1),\alpha+1)t^{\alpha(2j-1)}}{(\Gamma(\alpha(j-2)+1))^2}\,\frac{j^{2\alpha}}{\rho_{j-1}^2t^{2\alpha j}}\\&=\frac{\Gamma(2\alpha(j-1))}{\Gamma(\alpha(2j-1)+1)}\,\frac{(\Gamma(\alpha(j-1)+1))^2}{(\Gamma(\alpha(j-2)+1))^2}\frac{j^{2\alpha+1}}{Ct^\alpha}\sim\frac{\alpha^{2\alpha}}{C2^{\alpha+1}}\frac{j^{3\alpha}}{t^\alpha}\ \to\ 0,\qquad t\to\infty.\end{aligned}$$

Further, using (5.17) and (4.7),

$$\begin{aligned}\max_{s\in[0,\,j]}D_{j-1}(s)U(j)&\le(D_{j-1}(j)+(V_{j-1}(j))^2)U(j)\\&\le\left(\mathrm{const}\,\frac{(C\Gamma(\alpha+1))^{2j-3}j^{2\alpha(j-1)}}{(\Gamma(\alpha(j-2)+1))^2}+r\rho_{j-1}^2j^{2\alpha(j-1)}\right)U(j).\end{aligned}\tag{5.19}$$

Assumption A entails $U(j)\sim Cj^\alpha$ as $j\to\infty$. Now we show that the right-hand side of (5.19) times the squared normalization vanishes whenever $j=j(t)=o(t)$

and $j(t) \to \infty$ as $t \to \infty$. We start with the first term:

$$\frac{j^{2\alpha j}(C\Gamma(\alpha+1))^{2j-3}}{\rho_{j-1}^2 t^{2\alpha j}(\Gamma(\alpha(j-2)+1))^2}U(j) = \frac{1}{C\Gamma(\alpha+1)}\left(\frac{j}{t}\right)^{2\alpha j}\frac{(\Gamma(\alpha(j-1)+1))^2}{(\Gamma(\alpha(j-2)+1))^2}U(j)$$
$$\sim \frac{\alpha^{2\alpha}}{C\Gamma(\alpha+1)}\left(\frac{j}{t}\right)^{2\alpha j} j^{3\alpha} \to 0, \quad t \to \infty,$$

because $(j/t)^{2\alpha j}$ converges to zero faster than any negative power of j. As for the second term in (5.19), arguing similarly we infer that

$$\frac{j^{2\alpha j}}{t^{2\alpha j}}U(a_j) \sim \left(\frac{j}{t}\right)^{2\alpha j} Cj^\alpha \to 0, \quad t \to \infty.$$

An application of Markov's inequality completes the proof of Theorem 5.2. ■

Proof of Theorem 2.1. Equality (1.1) is equivalent to

$$|\log P_k| = |\log W_1| + \cdots + |\log W_{k-1}| + |\log(1 - W_k)|, \quad k \in \mathbb{N}.$$

Thus, $(|\log P_k|)_{k\geq 1}$ is a globally perturbed random walk generated by $(|\log W|,\ |\log(1-W)|)$, that is, in the notation of Section 1.2, $(\xi, \eta) = (|\log W|, |\log(1-W)|)$.

For $j \in \mathbb{N}$ and $t > 0$, put $\psi_j(t) := \sum_{|v|=j} \mathbb{1}_{\{P(v)\geq 1/t\}}$ and observe that $\psi_j(n) = N_j(\log n)$ for $j, n \in \mathbb{N}$. We shall use the decomposition

$$\begin{aligned} K_n(\lfloor j_n u\rfloor) &= (K_n(\lfloor j_n u\rfloor) - \psi_{\lfloor j_n u\rfloor}(n)) \\ &\quad + \Big(\psi_{\lfloor j_n u\rfloor}(n) - \sum_{r\geq 1} V_{\lfloor j_n u\rfloor-1}(\log n - T_r)\mathbb{1}_{\{T_r\leq \log n\}}\Big) \\ &\quad + \sum_{r\geq 1} V_{\lfloor j_n u\rfloor-1}(\log n - T_r)\mathbb{1}_{\{T_r\leq \log n\}} \\ &=: Y_1(n,u) + Y_2(n,u) + Y_3(n,u). \end{aligned}$$

It suffices to show that, as $n \to \infty$,

$$\text{(5.20)} \qquad \frac{j_n^\alpha Y_i(n,u)}{\rho_{\lfloor j_n u\rfloor-1}(\log n)^{\alpha\lfloor j_n u\rfloor}} \xrightarrow{\mathbb{P}} 0, \quad i = 1, 2,$$

where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability, and

$$\left(\frac{cj_n^\alpha Y_3(n,u)}{\rho_{\lfloor j_n u\rfloor-1}(\log n)^{\alpha\lfloor j_n u\rfloor}}\right)_{u>0} \xrightarrow{\text{f.d.d.}} \Big(\int_0^\infty \mathrm{e}^{-\alpha u y}\,\mathrm{d}\mathcal{Z}_\alpha^{\leftarrow}(y)\Big)_{u>0}.$$

We are going to apply Theorems 5.1 and 5.2 and Lemmas 5.1 and 5.2. While doing so, we replace t with $\log n$ and choose any diverging positive function $t \mapsto j(t)$

satisfying $j(\log n) = j_n$ and $j(t) = o(t^{\min(\frac{1}{3}, \frac{\alpha-\beta}{\alpha-\beta+1})})$ as $t \to \infty$. By Lemma 3.1, Assumptions A and B entail (3.1). With this at hand, relation (5.20) with $i = 2$ follows from Theorem 5.2. The limit relation involving Y_3 is ensured by Theorem 5.1. In view of Markov's inequality, relation (5.20) with $i = 1$ follows if we can prove that, with $u > 0$ fixed,

$$\lim_{n\to\infty} \frac{j_n^\alpha \mathbb{E}|K_n(\lfloor j_n u\rfloor) - \psi_{\lfloor j_n u\rfloor}(n)|}{\rho_{\lfloor j_n u\rfloor -1}(\log n)^{\alpha\lfloor j_n u\rfloor}} = 0.$$

In [9, Section 6, top of p. 21], it was shown that

$$\begin{aligned}\mathbb{E}|K_n(\lfloor j_n u\rfloor) - \psi_{\lfloor j_n u\rfloor}(n)| \\ \le n \int_{(n,\infty)} x^{-1}\,\mathrm{d}(\mathbb{E}\psi_{\lfloor j_n u\rfloor}(x)) + \int_{[1,n]} \mathrm{e}^{-n/x}\,\mathrm{d}(\mathbb{E}\psi_{\lfloor j_n u\rfloor}(x)).\end{aligned}$$

By Lemma 5.2 applied in the particular setting $(\xi, \eta) = (|\log W|, |\log(1-W)|$, so that $V_{\lfloor j_n u\rfloor}(\log x) = \mathbb{E}\psi_{\lfloor j_n u\rfloor}(x)$ for $x \ge 1$, we obtain, as $n \to \infty$,

$$n \int_{(n,\infty)} x^{-1}\,\mathrm{d}(\mathbb{E}\psi_{\lfloor j_n u\rfloor}(x)) = \int_{(\log n,\infty)} \mathrm{e}^{\log n - x}\,\mathrm{d}V_{\lfloor j_n u\rfloor}(x) = O(V_{\lfloor j_n u\rfloor -1}(\log n))$$

for each fixed $u > 0$. The function f defined on $[0,\infty)$ by $f(x) = \exp(-\mathrm{e}^x)$ is decreasing and Lebesgue integrable. Hence, it is dRi; see, for instance, [18, Lemma 6.2.1(a)]. By Lemma 5.1, with f so defined, as $n \to \infty$,

$$\begin{aligned}&\int_{[1,n]} \mathrm{e}^{-n/x}\,\mathrm{d}(\mathbb{E}\psi_{\lfloor j_n u\rfloor}(x)) \\ &= \int_{[0,\log n]} \exp(-\mathrm{e}^{\log n - x})\,\mathrm{d}V_{\lfloor j_n u\rfloor}(x) = O(V_{\lfloor j_n u\rfloor -1}(\log n))\end{aligned}$$

for each fixed $u > 0$. Invoking (4.9) and the assumption on the growth rate of j_n, which in particular entails $j_n = o(\log n)$ as $n \to \infty$, we infer that

$$\frac{j_n^\alpha V_{\lfloor j_n u\rfloor -1}(\log n)}{\rho_{\lfloor j_n u\rfloor -1}(\log n)^{\alpha\lfloor j_n u\rfloor}} \sim \frac{j_n^\alpha}{(\log n)^\alpha} \to 0, \quad n \to \infty,$$

thereby arriving at (5.20) with $i = 1$. The proof of Theorem 2.1 is complete. ■

6. APPENDIX

The following formula is rather standard; see, for instance, [21, (A7)].

LEMMA 6.1. *Let $\gamma > 0$ and η be a positive random variable with Laplace transform ℓ. Then*

$$\mathbb{E}\eta^{-\gamma} = \frac{\gamma}{\Gamma(1+\gamma)} \int_0^\infty s^{\gamma-1}\ell(s)\,\mathrm{d}s,$$

where Γ *is the Euler gamma-function. Here, both sides of the equality may be infinite.*

Finally, we give an estimate for the gamma-function.

LEMMA 6.2. *For* $x, y \geq 0$*,*

$$\frac{\Gamma(x+1+y)}{\Gamma(x+1)} \leq (x+1+y)^y. \tag{6.1}$$

Proof. If $y = 0$, the inequality holds trivially. Assume that $y \in \mathbb{N}$. Using the equality

$$\Gamma(z+1) = z\Gamma(z), \quad z > 0, \tag{6.2}$$

we obtain

$$\frac{\Gamma(x+1+y)}{\Gamma(x+1)} = (x+y) \cdot \ldots \cdot (x+1) \leq (x+y)^y \leq (x+1+y)^y.$$

Assume now that $y \notin \mathbb{N}$. Then, by Wendel's inequality [31],

$$\frac{\Gamma(x+1+\{y\})}{\Gamma(x+1)} \leq (x+1)^{\{y\}},$$

where $\{y\}$ is the fractional part of y. This in combination with (6.2) entails[1]

$$\begin{aligned}\frac{\Gamma(x+1+y)}{\Gamma(x+1)} &= (x+y) \cdot \ldots \cdot (x+1+\{y\}) \frac{\Gamma(x+1+\{y\})}{\Gamma(x+1)} \\ &\leq (x+y)^{\lfloor y \rfloor}(x+1)^{\{y\}} \leq (x+1+y)^y. \quad \blacksquare\end{aligned}$$

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