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BOOLEAN AND FERMI CENTRAL LIMIT THEOREMS

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Abstract. We provide a necessary and sufficient condition on a sequence of random variables for the Boolean and Fermi central limit theorems to hold true. The result allows us to connect the classical and the non-commutative analogues of the central limit theorems. We show that the Boolean domain of attraction of the symmetric Bernoulli distribution, the free domain of attraction of the semicircle law and the classical domain of attraction of the Gaussian distribution coincide with each other.

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1. INTRODUCTION

In non-commutative probability there are several notions of independence; tensor, Boolean, free, monotone, anti-monotone are some of them (see [9]). It was shown in [14] that the only three universal notions of independence are tensor, Boolean and free. Tensor independence and classical independence give rise to the same convolutions of probability measures, while Boolean and free independence provide different notions of convolutions.

Central limit theorems essentially study the sum of many independent identically distributed random variables where each summand has high probability of being small. When the random variables are chosen to be classically independent, the sum (with a proper scaling and centring) will approximately have a Gaussian distribution (also called the normal distribution). In non-commutative probability, if we start with freely independent random variables then the sum distribution of properly tuned terms tends to the semicircle law (see [8, 11, 18, 20]), and in the case of a Boolean independent sequence of random variables the limit law for the sum of adjusted terms turns out to be the centered Bernoulli distribution (see [15, 17]). The speed of convergence in the above mentioned setup has been extensively studied in the literature. Andrew Campbell Berry (in 1941) and Carl-Gustav Esseen (in 1942) independently reported that given a sequence of classically independent and identically distributed random variables, (under some moment assumptions) the cumulative distribution functions of the standardized sample mean and the standard normal distribution differ by no more than a specified amount (in particular, the difference is $\mathbf{O}(n^{-1/2})$). A similar bound in the free case was obtained in [5, Theorems 2.4, 2.5] under the finite fourth moment assumption. It was proved in [1] that the speed of convergence in the Boolean central limit theorem is of order $\mathbf{O}(n^{-1/2})$ for measures with finite sixth moment, and the result was improved in [13] by assuming finite fourth moment.

The characterization of the laws satisfying the central limit theorem is also an important object of study in both classical and non-commutative probability theory. Classically it is known that a law satisfies the classical central limit theorem if and only if it has almost finite second moment. An analogous result in the free case was detailed in [12]. We have proved parallel results in this direction for Boolean and Fermi convolutions. One of the most important upshots of comparing the results in classical, free and Boolean cases is that a law satisfies the classical central limit theorem if and only if it satisfies the free central limit theorem if and only if it satisfies the Boolean central limit theorem. In particular, the classical domain of attraction of the normal distribution, the free domain of attraction of the semicircle law and the Boolean domain of attraction of the centered Bernoulli distribution are identical.

To state the main results of this article, we use the notations from [12]. Let μ be a probability measure on the real line and $\alpha \geqslant 0$, and define

$$\mu^{(\alpha)} = \int_{-\infty}^{\infty} t^{\alpha}\, d\mu(t)$$

whenever the integral exists. The notation $\mathrm{Var}[\mu]$ will be used for the variance of μ. When μ is a finite positive measure, the function H_μ is defined on the non-negative part of the real line in the following way:

$$H_\mu(y) = \int_{-y}^{y} t^2\, d\mu(t). \tag{1.1}$$

A measure μ is *non-degenerate* if it is not a Dirac measure at a real number. Let D_λ be the dilation of probability measures by a factor $\lambda \geqslant 0$, defined as

$$(D_\lambda \mu)(A) = \mu(\lambda^{-1} A) \tag{1.2}$$

for any Borel measurable subset A of the real line.

We are now ready to state the main theorems of the article.

THEOREM 1.1 (General Boolean CLT). *Let μ be a non-degenerate probability measure on the real line and let*

$$\nu_n := \underbrace{D_{\frac{1}{B_n}}\mu \uplus \cdots \uplus D_{\frac{1}{B_n}}\mu}_{n \text{ times}} \uplus \delta_{-A_n}$$

for any $n > 0$ ($\uplus$ denotes the additive Boolean convolution). Then the sequence $\{\nu_n\}_{n=1}^{\infty}$ of measures converges weakly as $n \to \infty$ to $\boldsymbol{b}_1 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ for some choices of positive constants $\{B_n\}_{n=1}^{\infty}$ and real numbers $\{A_n\}_{n=1}^{\infty}$ if and only if the function H_μ is slowly varying.

We now note down the obvious central limit theorem for the Fermi convolution which has not been mentioned before in the literature. Observing that μ has finite variance implies $H_\mu(y)$ tends to a constant when $y \to \infty$, we have the following central limit theorem for Fermi convolutions:

THEOREM 1.2 (General Fermi CLT). *Let μ be a non-degenerate probability measure on the real line with finite second moment and let*

$$\nu_n := \underbrace{D_{\frac{1}{B_n}}\mu \bullet \cdots \bullet D_{\frac{1}{B_n}}\mu}_{n \text{ times}} \bullet \delta_{-A_n}$$

for any $n > 0$ ($\bullet$ denotes the Fermi convolution). Then the sequence $\{\nu_n\}_{n=1}^{\infty}$ of measures converges weakly as $n \to \infty$ to $\boldsymbol{b}_1 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ for some choices of positive constants $\{B_n\}_{n=1}^{\infty}$ and real numbers $\{A_n\}_{n=1}^{\infty}$.

In the above situation, B_n may be chosen to be $c\sqrt{n}$ and $A_n = c^{-1}\sqrt{n}\mu^{(1)}$, where $\mu^{(1)}$ is the mean of μ and the constant c is $(\mathrm{Var}[\mu])^{-1/2}$.

The next theorem is a particular case of the Boolean central limit theorem.

THEOREM 1.3 (Boolean CLT). *Let μ be a non-degenerate probability measure on the real line and let*

$$\nu_n := \underbrace{D_{\frac{1}{\sqrt{n}c}}\mu \uplus \cdots \uplus D_{\frac{1}{\sqrt{n}c}}\mu}_{n \text{ times}} \uplus \delta_{-A_n}$$

for any $n > 0$. Then the sequence $\{\nu_n\}_{n=1}^{\infty}$ of measures converges weakly as $n \to \infty$ to $\boldsymbol{b}_1$ for some positive constant c and real numbers $\{A_n\}_{n=1}^{\infty}$ if and only if the measure μ has finite variance.

In the above situation, the constants c and A_n may be chosen to be $c = (\mathrm{Var}[\mu])^{-1/2}$ and $A_n = c^{-1}\sqrt{n}\mu^{(1)}$.

The proof of Theorem 1.2 is similar to those of Theorems 1.1 and 1.3 and therefore will be skipped. In the next section we introduce the notions that are used throughout this article. Section 3 describes the notion of domain of attraction and the final section consists of the proofs of the main theorems.

2. PRELIMINARIES AND SOME KNOWN RESULTS

For a complex number $z = x + iy$, the imaginary part y of z will be denoted by $\Im z$. We denote the upper complex half plane by $\mathbb{C}^+ := \{z \in \mathbb{C} : \Im z > 0\}$. The set of all probability measures on $\mathbb{R}$ is represented by $\mathcal{M}$. For two sequences of numbers $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ we write $a_n = \mathrm{o}(b_n)$ if $|a_n/b_n| \to 0$ as $n \to \infty$, and $a_n \sim b_n$ if $|a_n/b_n| \to 1$ as $n \to \infty$.

A complex normed algebra $(\mathcal{A}, \|\cdot\|)$ is a *Banach algebra* if the norm $\|\cdot\|$ induces a complete metric. A Banach algebra is a C^**-algebra* if it has an involution $a \mapsto a^*$ that satisfies $\|aa^*\| = \|a\|^2$. A quadruple $(\mathcal{A}, \|\cdot\|, *, \phi)$ is called a C^**-probability space* if $(\mathcal{A}, \|\cdot\|, *)$ is a C^*-algebra and ϕ is a linear functional such that $\phi(\mathbf{1}) = 1$. Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H. A C^*-algebra $\mathcal{A} \subseteq B(H)$ is said to be a *von Neumann algebra* (or a W^**-algebra*) if it is closed with respect to the weak operator topology. The pair $(\mathcal{A}, \phi)$ is called a W^**-probability space* if $\mathcal{A}$ is a W^*-algebra and ϕ is a linear functional which can be written as $\phi(a) = \langle a\zeta, \zeta\rangle$ for some unit vector $\zeta \in H$. Let $(\mathcal{A}, \phi)$ be a W^*-probability space and let T be a self-adjoint operator affiliated with $\mathcal{A}$, i.e. there exists a complex number z in the spectrum of T such that $(T - z)^{-1} \in \mathcal{A}$. Then the *law of* T is the unique probability measure μ_T on $\mathbb{R}$ such that

$$\phi(u(T)) = \int u(\lambda)\, d\mu_T(\lambda)$$

for any bounded measurable function u on the spectrum of T.

2.1. Boolean convolution and Fermi convolution. The notion of interval partition was introduced in [19, 4]. Later that notion was extended and Boolean convolution was first defined in [15]. We are interested in the Boolean case. The following is Definition 4.1 of [6].

DEFINITION 2.1 (Boolean independence). Subalgebras $A_1, \ldots, A_L$ of $(\mathcal{A}, \phi)$ (non-unital in general) are said to be *Boolean independent* (with respect to a unit vector ζ where $\phi(a) = \langle \zeta, a\zeta\rangle$) whenever for any $n \geqslant 1$,

$$\phi(a_1 \cdots a_n) = \phi(a_1) \cdots \phi(a_n),$$

where $a_i \in A_{k(i)}$ and $k(i) \neq k(i+1)$ for all $i \in \{1, \ldots, n-1\}$.

Let X and Y be two self-adjoint, Boolean independent elements in a non-commutative probability space $(\mathcal{A}, \phi)$ with laws μ_X and μ_Y respectively. Then the law of $X + Y$ is denoted by $\mu_X \uplus \mu_Y$. Also for any two probability measures, there exists a pair of Boolean independent random variables in some W^*-probability space such that the convolutions are well defined. For a measure space $(\Omega, \mathcal{F}, P)$, denote by $L^2(\Omega, P)_0$ the orthogonal complement of the constant functions, i.e.

$$L^2(\Omega, P)_0 = \Big\{g \in L^2(\Omega, P) : \int_\Omega g\, dP = 0\Big\}.$$

Let μ_1, $\mu_2 \in \mathcal{M}$. It was shown in [6] that there exist two self-adjoint operators X and Y, Boolean independent (with respect to the column vector $\zeta = (1, 0, 0)^t$) on the space $\mathbb{C} \oplus L^2(\Omega, \mu_1)_0 \oplus L^2(\Omega, \mu_2)_0$ having laws μ_1 and μ_2 respectively.

Now, we describe the additive Boolean convolution analytically. Set

$$\Gamma_\alpha := \{z = x + iy \in \mathbb{C}^+ : x < \alpha y\} \quad \text{and} \quad \Gamma_{\alpha,\beta} := \{z \in \Gamma_\alpha : |z| > \beta\}$$

for positive real numbers α and β. The region $\Gamma_{\alpha,\beta}$ is known as a *Stolz angle* at ∞. We say that $f(z) \to l$ as $z \to \infty$ non-tangentially if for any $\epsilon > 0$ and $\alpha > 0$, there exists $\beta \equiv \beta(\alpha, \epsilon) > 0$ such that $|f(z) - l| < \epsilon$ whenever $z \in \Gamma_{\alpha,\beta}$. We shall write $f(z) \sim g(z)$ as $z \to \infty$ to mean that $f(z)/g(z)$ converges to 1 as $z \to \infty$ non-tangentially.

Classically moment generating functions (whenever they exist) and characteristic functions determine measures uniquely. In non-commutative probability, there are several transforms which do the job of describing a probability measure. Suppose $\mu \in \mathcal{M}$. Then its *Cauchy transform* $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z - t}\, d\mu(t), \qquad z \in \mathbb{C}^+.$$

The Cauchy transform characterizes a probability measure uniquely. The reciprocal of the Cauchy transform is defined as

$$F_\mu(z) = \frac{1}{G_\mu(z)},$$

which maps $\mathbb{C}^+$ to $\mathbb{C}^+$ analytically. The Nevanlinna representation of F_μ is given by

$$F_\mu(z) = a + z + \int_{-\infty}^{\infty} \frac{1 + tz}{t - z}\, d\sigma(t), \qquad z \in \mathbb{C}^+, \tag{2.1}$$

for some positive finite measure σ and some $a \in \mathbb{R}$. The Boolean additive convolution is determined by the transform K_μ (also called the *energy transform*) which is defined for any $z \in \mathbb{C}^+$ by the formula

$$K_\mu(z) = z - F_\mu(z). \tag{2.2}$$

For two probability measures $\mu, \nu \in \mathcal{M}$, the *additive Boolean convolution* $\mu \uplus \nu$ is determined by

$$K_{\mu \uplus \nu}(z) = K_\mu(z) + K_\nu(z), \qquad z \in \mathbb{C}^+,$$

and $\mu \uplus \nu$ is again a probability measure.

Following [10], the $\widetilde{B}$*-transform* of a probability measure μ having finite second moment is defined in the following way:

$$\widetilde{B}_\mu(z) = \lambda z + z K_{\mu^0}\left(\frac{1}{z}\right), \qquad z \in \mathbb{C}^-, \tag{2.3}$$

where λ is the mean of μ and μ^0 is the zero mean shift of μ.

Let μ and ν be two probability measures on the real line both having finite variances. The *Fermi convolution* $\mu \bullet \nu$ is determined by

$$\widetilde{B}_{\mu\bullet\nu}(z) = \widetilde{B}_\mu(z) + \widetilde{B}_\nu(z), \quad z \in \mathbb{C}^-, \tag{2.4}$$

and $\mu \bullet \nu$ is again a probability measure.

The following proposition describing that $\mu \mapsto K_\mu$ behaves well relative to weak convergence will be used in this article.

PROPOSITION 2.1 ([2, Proposition 6.2]). *Let $\mu_n \in \mathcal{M}$ be a sequence of probability measures. The following are equivalent:*

(1) *μ_n converges weakly to a probability measure μ.*

(2) *There exist $\alpha, \beta > 0$ such that the sequence K_{μ_n} converges uniformly on $\Gamma_{\alpha,\beta}$ to a function K, and $K_{\mu_n}(z) = o(|z|)$ uniformly in n as $z \to \infty$ on $\Gamma_{\alpha,\beta}$.*

(3) *There exists a $\beta_0 > 0$ such that the limit of $K_{\mu_n}(iy)$ exists as $n \to \infty$ for every $y > \beta_0$ and $K_{\mu_n}(iy) = o(iy)$ uniformly in n as $y \to \infty$.*

In particular, in the above situation, we have $K = K_\mu$.

The following proposition is an immediate consequence of the above proposition.

PROPOSITION 2.2. *Let $\mu_n \in \mathcal{M}$ be a sequence of probability measures having finite second moments. The following are equivalent:*

(1) *μ_n converges weakly to a probability measure μ.*

(2) *There exist $\alpha, \beta > 0$ such that the sequence $\widetilde{B}_{\mu_n}$ converges uniformly on $\Gamma_{\alpha,\beta}$ to a function $\widetilde{B}$, and the limit of $\widetilde{B}_{\mu_n}(z^{-1})$ exists non-tangentially uniformly in n as $z \to \infty$ on $\Gamma_{\alpha,\beta}$.*

(3) *There exists a $\beta_0 > 0$ such that the limit of $\widetilde{B}_{\mu_n}(iy)$ exists as $n \to \infty$ for every $y > \beta_0$ and the limit of $\widetilde{B}_{\mu_n}((iy)^{-1})$ exists non-tangentially uniformly in n as $y \to \infty$.*

In particular, in the above situation, we have $\widetilde{B} = \widetilde{B}_\mu$.

It has already been mentioned that Theorem 1.2 may be proved using similar arguments to those for Theorems 1.1 and 1.3 with the help of Proposition 2.2.

2.2. Slowly varying functions and some known results. A measurable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be *regularly varying of index* α for some $\alpha > 0$ if there exists an $M > 0$ such that $r((M, \infty)) \subseteq (0, \infty)$ and for every $t > 0$,

$$r(ty) \sim t^{-\alpha} r(y) \quad \text{as } y \to \infty.$$

A regularly varying function of index 0 is called a *slowly varying function*. A regularly varying function r of index α can be written as $r(y) = y^\alpha l(y)$, where l is a slowly varying function. Let us denote by $\mathcal{RV}_\alpha$ the set of all regularly varying functions of index α ($\alpha \geqslant 0$). A positive sequence $\{h_n\}_{n=1}^{\infty}$ is said to be *slowly varying* if for every $k \in \mathbb{N}$,

$$h_{kn} \sim h_n \quad \text{as } n \to \infty.$$

In general, a slowly varying sequence cannot be extended to a slowly varying function. But, if h_n is a slowly varying sequence such that $h_n \sim h_{n+1}$ as $n \to \infty$, then there exists a continuous function $l \in \mathcal{RV}_0$ such that $l(n) = h_n$ for all $n \in \mathbb{N}$.

To keep the paper self-contained we record some known results. The following proposition gives the tail asymptotics of the functions H_μ and another function.

PROPOSITION 2.3 ([12, Proposition 3.3]). *Let μ be a finite positive measure supported on the real line. Then the following are equivalent:*

(1) $H_\mu \in \mathcal{RV}_0$.

(2) $L_\mu \in \mathcal{RV}_0$.

Here H_μ is defined in (1.1) *and L_μ is defined by*

$$L_\mu(y) = \int_{-\infty}^{\infty} y^2 t^2 (y^2 + t^2)^{-1} \, d\mu(t).$$

In the above situation, $H_\mu(y) \sim L_\mu(y)$ as $y \to \infty$.

The following result is [12, Corollary 3.7].

PROPOSITION 2.4. *Let $\mu \in \mathcal{M}$ with the Nevanlinna representation of F_μ given by* (2.1). *Then $\mu^{(2)} < \infty$ if and only if $\sigma^{(2)} < \infty$. In this case $\mu^{(1)} = \sigma^{(1)} - a$ and* $\mathrm{Var}[\mu] = \sigma^{(0)} + \sigma^{(2)}$.

We record a result characterizing the behaviour of the normalizing constants that appear in the standardized sum of Boolean independent random variables.

PROPOSITION 2.5. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Boolean independent identically distributed random variables. Also, let $\{B_n\}_{n=1}^{\infty}$ and $\{A_n\}_{n=1}^{\infty}$ be sequences of positive and real numbers respectively. If the distribution of*

$$\frac{X_1 + \cdots + X_n}{B_n} - A_n$$

converges in distribution to the symmetric Bernoulli distribution $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, then the constants B_n are necessarily of the form $B_n = \sqrt{n} h_n$, where $\{h_n\}_{n=1}^{\infty}$ is a slowly varying sequence such that $h_{n+1} \sim h_n$ as $n \to \infty$

This is proved by a computation similar to that for [11, Theorem 5.5].

3. DOMAIN OF ATTRACTION

We recall the dilation of a probability measure defined in (1.2). It follows from the definition of the energy transform that

$$K_{D_\lambda \mu}(z) = \lambda K_\mu(z/\lambda) \tag{3.1}$$

for any $z \in \mathbb{C}^+$.

The central limit theorem for Boolean additive convolution was first proved by Roland Speicher and Reza Woroudi for measures with mean zero and finite second moment. Theorem 3.4 in [15] states the following: Given a probability measure μ with mean zero and finite variance σ^2, the measure

$$\underbrace{D_{\frac{1}{\sqrt{n}}}\mu \uplus \cdots \uplus D_{\frac{1}{\sqrt{n}}}\mu}_{n \text{ times}}$$

converges weakly to the centered Bernoulli distribution $\mathbf{b}_\sigma = \frac{1}{2}(\delta_{-\sigma} + \delta_\sigma)$ as $n \to \infty$. The result was further extended to a more general case in [17]. A simple calculation shows that

$$K_{\mathbf{b}_\sigma}(z) = \sigma^2/z \tag{3.2}$$

and

$$K_{\mathbf{b}_{\sigma_1}}(z) + K_{\mathbf{b}_{\sigma_2}}(z) = K_{\mathbf{b}_{\sqrt{\sigma_1+\sigma_2}}}(z),$$

which implies that $\mathbf{b}_{\sigma_1} \uplus \mathbf{b}_{\sigma_2} = \mathbf{b}_{\sqrt{\sigma_1+\sigma_2}}$.

Now we are going to define the Boolean domain of attraction of the centered Bernoulli distribution in an analogous way to the classical and free domains of attraction. To that end, let $\{X_n\}_{n=1}^\infty$ be Boolean independent (with respect to some unit vector ζ) sequence of random variables having common distribution μ in a W^*-probability space. Also let $\{B_n\}_{n=1}^\infty$ be a sequence of positive real numbers and $\{A_n\}_{n=1}^\infty$ be a sequence of real numbers. Define the random variables Z_n as

$$Z_n = Z_n(\mu, B_n, A_n) = \frac{X_1 + \cdots + X_n}{B_n} - A_n.$$

We are interested in the asymptotic distribution of Z_n under minimal assumptions on the unknown measure μ.

DEFINITION 3.1. If for some choices of the constants A_n and B_n, the random variable $Z_n = Z_n(\mu, B_n, A_n)$ converges in distribution to $\mathbf{b}_1$, then μ is said to be *in the Boolean domain of attraction* of the centered Bernoulli distribution and we write $\mu \in \mathcal{D}[\mathbf{b}_1]$.

It follows that $\mu \in \mathcal{D}[\mathbf{b}_1]$ if and only if $\mu \in \mathcal{D}[\mathbf{b}_\sigma]$ for any $\sigma \in \mathbb{R}$, by appropriate choices of B_n and A_n. Proposition 2.5 allows us to conclude that the constant B_n is of the form $B_n = \sqrt{n}h_n$ for all $n \in \mathbb{N}$, where $\{h_n\}_{n=1}^{\infty}$ is a slowly varying sequence such that $h_{n+1} \sim h_n$ as $n \to \infty$. An interesting case is when h_n is a constant function. In that case, we have the following definition.

DEFINITION 3.2. If for some choices of the constants A_n and some positive constant c, the random variable $Z_n = Z_n(\mu, \sqrt{n}c, A_n)$ converges in distribution to $\mathbf{b}_1$, then μ is said to be *in the normal Boolean domain of attraction* of the centered Bernoulli distribution and we denote as $\mu \in \mathcal{ND}[\mathbf{b}_1]$.

Again, $\mu \in \mathcal{ND}[\mathbf{b}_1]$ if and only if $\mu \in \mathcal{ND}[\mathbf{b}_\sigma]$ for any $\sigma \in \mathbb{R}$.

REMARK 3.1. We characterize the Boolean domain of attraction of the centered Bernoulli distribution. This provides a general form of the central limit theorem for Boolean additive convolution. Analogously to the results in the classical case [7, Theorem 2.6.2] and in the free case [12, Theorems 4.3, 4.4], the result here (Theorem 1.1) shows that $\mu \in \mathcal{D}[\mathbf{b}_1]$ if and only if μ has almost finite variance, and Theorem 1.3 shows that $\mu \in \mathcal{ND}[\mathbf{b}_1]$ if and only if μ has finite variance. In combination we may conclude that the classical (normal) domain of attraction of the Gaussian distribution, the free (normal) domain of attraction of the semicircle distribution and the Boolean (normal) domain of attraction of the centered Bernoulli distribution are the same.

4. PROOF OF THEOREMS 1.1 AND 1.3

The proofs of Theorems 1.1 and 1.3 may be carried out using [2, Theorem 6.3] but to prove Theorem 1.2 one needs to follow the following procedure and therefore the following way of proof is described. The following lemma is going to be useful in the proofs of the main results of this article.

LEMMA 4.1. *Let $\mu \in \mathcal{M}$ be non-degenerate and suppose μ is in the Boolean domain of attraction of $\boldsymbol{b}_1$, the centered Bernoulli distribution. Then there exists a function $l \in \mathcal{RV}_0$ such that*

$$\Im F_\mu(iy) = y + \frac{1}{y}l(y) - \frac{1}{y}l(y)o(1)$$

as $y \to \infty$.

Proof. Let ν_n be the distribution of $Z_n = Z_n(\mu, \sqrt{n}h_n, A_n)$ where $\{h_n\}_{n=1}^{\infty}$ is a sequence of slowly varying functions and $\{A_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Assume that ν_n converges weakly to $\mathbf{b}_1$. Then by Proposition 2.1 it follows that

$$\lim_{n\to\infty} K_{\nu_n}(z) = K_{\mathbf{b}_1}(z) = \frac{1}{z}$$

uniformly on compact subsets of some truncated cone $\Gamma_{\alpha,\beta}$ for some $\alpha, \beta > 0$. Also using (3.1), we get

$$K_{\nu_n}(z) = \frac{\sqrt{n}}{h_n} K_\mu(\sqrt{n} h_n z) - A_n$$

for $z \in \Gamma_{\alpha,\beta}$. Now, taking the imaginary parts on both sides of the above equation, we get

$$\Im K_{\nu_n}(\sqrt{n} h_n z) = \frac{h_n}{\sqrt{n}} \Im\left(\frac{1}{z}\right) + \frac{h_n}{\sqrt{n}} \mathrm{o}(1) \tag{4.1}$$

uniformly on compact subsets of $\Gamma_{\alpha,\beta}$.

We have assumed that $h_{n+1} \sim h_n$ as $n \to \infty$. Therefore, there exists an $h \in \mathcal{RV}_0$ such that $h(n) = h_n$ for all $n \in \mathbb{N}$. Consequently, the function $f(y) = \sqrt{y} h(y)$ is in $\mathcal{RV}_{1/2}$. Now using [3, Theorem 1.5.12], there exists a function $f(y) = y^2 k(y)$ where $k \in \mathcal{RV}_0$ such that $g(f(y)) \sim y$ as $y \to \infty$. Hence,

$$(h(y))^2 k(\sqrt{y} h(y)) \sim 1 \tag{4.2}$$

as $y \to \infty$. Let us define a function $l(y) = (k(y))^{-1} \in \mathcal{RV}_0$. Applying [3, Theorem 1.2.1], we get

$$l(\sqrt{n} h_n |z|) \sim (h_n)^2 \tag{4.3}$$

as $n \to \infty$ uniformly on compact subsets of $\mathbb{C}^+$. Observing that $\sqrt{n} h_n \to \infty$ as $n \to \infty$ and replacing $\sqrt{n} h_n z$ by z in (4.1), we get

$$\begin{aligned}\Im K_\mu(z) &= \frac{h_n}{\sqrt{n}} \Im\left(\frac{\sqrt{n} h_n}{z}\right) + \frac{h_n}{\sqrt{n}} \mathrm{o}(1) \\ &\overset{(4.3)}{=} (h_n)^2 \Im\left(\frac{1}{z}\right) + \frac{h_n}{\sqrt{n}} \mathrm{o}(1)\end{aligned}$$

as $n \to \infty$ uniformly on compact subsets of $\Gamma_{\alpha,\beta}$. Now the above with the asymptotic (4.2) and observing that $z \to \infty$ (when $\sqrt{n} h_n z$ is replaced by z) if $n \to \infty$, we may write

$$\Im K_\mu(z) = l(|z|) \Im\left(\frac{1}{z}\right) + \frac{1}{|z|} l(|z|) \mathrm{o}(1) \tag{4.4}$$

as $|z| \to \infty$, $z \in \Gamma_{\alpha,\beta}$. Now from the definition (2.2) of the energy transform it is immediate that

$$\Im K_\mu(iy) = y - \Im(F_\mu(iy)) \tag{4.5}$$

for any $y > 0$. Putting $z = iy$ in (4.4) and using (4.5), we get

$$\begin{aligned}\Im(F_\mu(iy)) &= y - l(y)\Im\left(\frac{1}{iy}\right) - \frac{1}{y}l(y)\mathrm{o}(1)\\ &= y + \frac{1}{y}l(y) - \frac{1}{y}l(y)\mathrm{o}(1)\end{aligned}$$

as $y \to \infty$. ■

We are now ready to prove the main results of this article.

Proof of Theorem 1.1. Assume that $\mu \in \mathcal{D}[\mathbf{b}_1]$. The Nevanlinna representation of F_μ (given in (2.1)) yields

$$\begin{aligned}\Im F_\mu(iy) &= y + \frac{1}{y}L_\sigma(y) + \int_{\infty}^{\infty} \frac{y}{y^2+t^2}\,d\sigma(t)\\ &= y + \frac{1}{y}L_\sigma(y) + \frac{1}{y}\sigma^{(0)} + \frac{1}{y}\mathrm{o}(1)\end{aligned} \tag{4.6}$$

as $y \to \infty$. Also Lemma 4.1 shows that there exists an $l \in \mathcal{RV}_0$ such that

$$\Im F_\mu(iy) = y + \frac{1}{y}l(y) - \frac{1}{y}l(y)\mathrm{o}(1) \tag{4.7}$$

as $y \to \infty$. Comparing (4.6) and (4.7), we get

$$L_\sigma(y) + \sigma^{(0)} = l(y)(1 + \mathrm{o}(1)) \tag{4.8}$$

as $y \to \infty$. This shows that $L_\sigma \in \mathcal{RV}_0$. An appeal to Proposition 2.3 gives us that $H_\sigma \in \mathcal{RV}_0$.

Conversely, suppose that $H_\mu \in \mathcal{RV}_0$. It follows from Proposition 2.3 that $L_\mu \in \mathcal{RV}_0$. Without loss of generality we may assume that μ has mean zero (since the term δ_{A_n} is there just to adjust the mean of μ). The following expression is obtained from [12, Proposition 3.6, (3.7) and (3.8)]:

$$F_\mu(iy) = iy - \frac{1}{iy}H_\mu(y) + \frac{1}{y}H_\mu(y)\mathrm{o}(1)$$

as $y \to \infty$. Therefore,

$$K_\mu(iy) = \frac{1}{iy}H_\mu(iy) - \frac{1}{y}H_\mu(y) \tag{4.9}$$

as $y \to \infty$. We define u_n as the asymptotic inverse of the function $R(u) = u^2/H_\mu(u)$, which is regularly varying with index 2 since $H_\mu \in \mathcal{RV}_0$. As a result, $R(u) \to \infty$ and an asymptotic inverse exists. Setting $u_n = R^{\leftarrow}(ny^2)$, it follows that $u_n \to \infty$ and

$$\frac{ny^2}{u_n^2}H_\mu(u_n) \to 1 \quad \text{as } n \to \infty,$$

due to the property $R(R^{\leftarrow}(u)) \sim u$ as $u \to \infty$. Defining the normalizing sequence $B_n := u_n/y$, we get $B_n \to \infty$ and

$$\frac{n}{B_n^2} H_\mu(B_n y) \sim 1 \quad \text{as } n \to \infty. \tag{4.10}$$

Given that the measure μ is centered, we consider

$$\nu_n := \underbrace{D_{\frac{1}{B_n}}\mu \uplus \cdots \uplus D_{\frac{1}{B_n}}\mu}_{n \text{ times}}.$$

The functions K_{ν_n} are well defined in $\Gamma_{\alpha,\beta}$ for some α and β and for all $y > \beta$, we have

$$K_{\nu_n}(iy) = \frac{n}{B_n} K_\mu(iB_n y).$$

Using (4.9) and the fact that $B_n \to \infty$ as $n \to \infty$, the above equation simplifies to

$$K_{\nu_n}(iy) = \frac{1}{iy}\,\frac{n}{B_n^2} H_\mu(B_n y)(1 + \mathrm{o}(1))$$

as $y \to \infty$ (or $n \to \infty$). Using (4.10), we see that

$$\lim_{n\to\infty} K_{\nu_n}(iy) = \frac{1}{iy}. \tag{4.11}$$

Applying [3, Theorem 1.3.1], for large values of y and n we get

$$\frac{H_\mu(B_n y)}{H_\mu(B_n)} = \frac{c(B_n y)}{c(B_n)} \exp\left(\int_{B_n}^{B_n y} \frac{\epsilon(t)}{t}\, dt\right),$$

where $|\epsilon(t)| < 1$ for $t \geqslant B_n$ and $c(\cdot)$ is a measurable function such that $c(x) \to c \in (0, \infty)$ as $x \to \infty$. For those large values of n, we conclude that

$$\frac{H_\mu(B_n y)}{H_\mu(B_n)} \leqslant y(1 + \mathrm{o}(1))$$

as $y \to \infty$ because of the bound

$$\exp\left(\int_{B_n}^{B_n y} \frac{\epsilon(t)}{t}\, dt\right) \leqslant \exp\left|\int_{B_n}^{B_n y} \frac{\epsilon(t)}{t}\, dt\right| \leqslant \exp\left|\int_{B_n}^{B_n y} \frac{1}{t}\, dt\right| = y.$$

It is clear from (4.10) that $nH_n^{-2}(B_n)$ is bounded for large values of n. Hence,

$$\left|\frac{K_{\nu_n}(iy)}{y}\right| = \frac{nH_\mu(B_n y)}{y^2 B_n^2}(1 + \mathrm{o}(1)) \to 0 \tag{4.12}$$

uniformly as $y \to \infty$. Applying part (iii) of Proposition 2.1 together with (4.11) and (4.12), we conclude that ν_n converges weakly to a measure ν and $K_{\nu_n}(iy) = (iy)^{-1}$ for every $y > \beta$. Corollary 4.9 of [16] and the above observation allow us to deduce that $K_\nu(z) = z^{-1}$ for every $z \in \Gamma_{\alpha,\beta}$ and now it is clear from (3.2) that $\nu = \mathbf{b}_1$. ∎

Theorem 1.3 follows directly from [17, Theorem 1] (taking $g(n) = \sqrt{n}$ and h as the constant function taking value c only). Another proof of Theorem 1.3 can be given by imitating the proof of Theorem 1.1; we give an outline below.

Proof of Theorem 1.3. Suppose $\mu \in \mathcal{ND}[\mathbf{b}_1]$. The slowly varying function l in this case can be chosen as $l(y) = c^2$ and we shall be able to conclude that the measure σ (appearing in the Nevanlinna representation of F_μ) has finite variance. In place of (4.8) we shall have $\sigma^{(2)} + \sigma^{(0)} = c^2$. Finally, Proposition 2.4 will imply that the variance of μ is c^2.

Conversely, if we assume that μ has mean zero and finite variance c^2, then after proceeding as in the previous case, (4.9) simplifies to

$$K_\mu(iy) = \frac{c^2}{iy} + \frac{1}{y}\mathrm{o}(1)$$

as $y \to \infty$. The rest of the proof is same as in the proof of Theorem 1.1 with $B_n = \sqrt{n}c$. ∎

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REFERENCES

[1] O. Arizmendi and M. Salazar, *A Berry–Esseen type limit theorem for Boolean convolution*, Arch. Math. (Basel) 111 (2018), 101–111.
[2] H. Bercovici, V. Pata, and P. Biane, *Stable laws and domains of attraction in free probability theory*, Ann. of Math. 149 (1999), 1023–1060.
[3] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation*, Encyclopedia Math Appl. 27, Cambridge Univ. Press, Cambridge, 1987.
[4] M. Bożejko, *Uniformly bounded representations of free groups*, J. Reine Angew. Math. 377 (1987), 170–186.
[5] G. P. Chistyakov and F. Götze, *Limit theorems in free probability theory. I*, Ann. Probab. 36 (2008), 54–90.
[6] U. Franz, *Monotone and Boolean convolutions for non-compactly supported probability measures*, Indiana Univ. Math. J. 58 (2009), 1151–1185.
[7] I. A. Ibragimov and Y. V. Linnik, *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff, Groningen, 1971.
[8] H. Maassen, *Addition of freely independent random variables*, J. Funct. Anal. 106 (1992), 409–438.

[9] N. Muraki, *The five independences as natural products*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6 (2003), 337–371.
[10] F. Oravecz, *Fermi convolution*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5 (2002), 235–242.
[11] V. Pata, *Lévy type characterization of stable laws for free random variables*, Trans. Amer. Math. Soc. 347 (1995), 2457–2472.
[12] V. Pata, *The central limit theorem for free additive convolution*, J. Funct. Anal. 140 (1996), 359–380.
[13] M. Salazar, *On a Berry–Esseen type limit theorem for boolean convolution*, Electron. Comm. Probab. 27 (2022), art. 11, 10 pp.
[14] R. Speicher, *On universal products*, in: Free Probability Theory (Waterloo, ON, 1995), Fields Inst. Comm. 12, Amer. Math. Soc., Providence, RI, 1997, 257–266.
[15] R. Speicher and R. Woroudi, *Boolean convolution*, in: Free Probability Theory (Waterloo, ON, 1995), Fields Inst. Comm. 12, Amer. Math. Soc., Providence, RI, 1997, 267–279.
[16] E. M. Stein and R. Shakarchi, *Complex Analysis*, Princeton Lectures in Anal. 2, Princeton Univ. Press, Princeton, NJ, 2003.
[17] G. Stoica, *Limit laws for normed and weighted Boolean convolutions*, J. Math. Anal. Appl. 309 (2005), 369–374.
[18] D. Voiculescu, *Symmetries of some reduced free product C^*-algebras*, in: Operator Algebras and Their Connections with Topology and Ergodic Theory (Buşteni, 1983), Lecture Notes in Math. 1132, Springer, Berlin, 1985, 556–588.
[19] W. von Waldenfels, *An approach to the theory of pressure broadening of spectral lines*, in: Probability and Information Theory, II, Lecture Notes in Math. 296, Springer, 1973, 19–69.
[20] J. C. Wang, *Local limit theorems in free probability theory*, Ann. Probab. 38 (2010), 1492–1506.

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