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URBANIK TYPE SUBCLASSES OF FREE - INFINITELY DIVISIBLE TRANSFORMS.

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Abstract. For the class of free-infinitely divisible transforms are introduced three families of increasing Urbanik type subclasses of those transforms. They begin with the class of free-normal transforms and end up with the whole class of free-infinitely divisible transforms. Those subclasses are derived from the ones of classical infinitely divisible measures for which are known their random integral representations. Special functions like Hurwitz-Lerch, polygamma and hypergeometric appear in kernels of the corresponding integral representations.

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The limit distribution theory is one of the main topics in the probability theory. Historically, it began with the central limit theorem which says that properly normalized partial sums of *independent and identically distributed* (i.i.d) random variables with finite second moment, converge in distribution to *the standard normal (Gaussian)* variable. When one drops the assumption about moments but still assumes i.i.d. variables, at the limit we get the class of *stable distributions* (variables). Still further, if we assume that observations (variables) are only stochastically independent but after some normalization by positive constants the corresponding triangular is *uniformly infinitesimal* then at the limit we obtain the class of *selfdecomposable distributions* (Lévy class L). Finally, limits of sums of arbitrary infinitesimal and row-wise independent triangular arrays coincide with the class of *infinitely divisible distributions*; see Feller [4], Chapter XVII or Gnedenko and Kolmogorov [5], Sect. 17-19, 29-30 and 33 or Loeve [23], Sect. 23. Thus we have

$$(\textit{Gaussian}) \subset ... \subset (\textit{selfdecomposable}) \subset ... \subset (\textit{infinitely divisible}) \ (\star)$$

[Here it might be worthy to notice that the class of selfdecomposable measures can be obtained from *strongly mixing sequences* not necessarily stochastically independent; cf. Bradley and Jurek [3]. That possible direction of studies is not continued in this article.]

Urbanik [24], [25] refined the left hand side inclusion (below class L) and Jurek [13], [14] the right hand side inclusion by introducing new subclasses of some limit distributions.
Later on, all the introduced subclasses were generalized to distributions on infinite dimensional spaces and then they were described as distributions of some random integrals on arbitrary Banach spaces. In Jurek [10], the normalization of partial sums of random variables was done by linear bounded operators on a Banach space. Those and other multidimensional set-ups might be of some use in a generalization to multidimensional free-probability theory. [For the random integral representation conjecture see the link given below the reference Jurek [12].]
In this paper we give characterizations of the above mentioned results (i.e., the refinements of inclusions in $(\star)$) for the additive free-independence (free-additive convolution $\boxplus$). More precisely, we describe their corresponding free-independent (Voiculescu) transforms. Those transforms are considered only on the imaginary axis which is enough for the identification of the corresponding measure; see Jurek [16], Jankowski and Jurek [7].
In Theorem 1 and auxiliary Lemma 1 we treat the general random integrals mappings that lead to subsets of free-independent transforms. Propositions 1-4 provide applications of Theorem 1 to some specified mappings. A complete filtration of the class of all free-ininitely divisible transforms is given in Corollary 2. Finally, Theorem 2 shows an intrinsic relation between two classes of free-infinitely divisible transforms: one derived from *linear* scalings and the other from *non-linear* scalings.

0. INTRODUCTION AND NOTATIONS.

We will introduce Urbanik type subclasses of free-infinitely divisible Voiculescu transforms in a such way that

$$(0.1)\quad (Gaussian,\boxplus) \subset (stable,\boxplus) \subset ... \subset (\mathcal{U}^{<k+1>},\boxplus) \subset (\mathcal{U}^{<k>},\boxplus) \subset ... \subset (\mathcal{U}^{<2>},\boxplus) \subset (\mathcal{U}^{<1>},\boxplus) \equiv (\mathcal{U},\boxplus) \equiv (\mathbb{U}_1,\boxplus) \subset ... \subset (\mathbb{U}_k,\boxplus) \subset (\mathbb{U}_{k+1},\boxplus) \subset ... \subset \overline{\cup_{k=1}^{\infty}(\mathbb{U}_k,\boxplus)} \equiv (ID,\boxplus),$$

where the closure is in the point-wise convergence of Voiculescu transforms (the topology of weak convergence of measures) and $\boxplus$ is the free-additive convolution.

Classes $(\mathcal{U}^{<k>},\boxplus)$ and $(\mathbb{U}_k,\boxplus)$ are the free-probability counterparts of the classical probability classes $(\mathcal{U}^{<k>},*)$ and $(\mathbb{U}_k,*)$ in $(ID,*)$. For each of these classes we have their characterizations in terms of the random integrals; for a general conjecture see the link below reference Jurek [12].

For the above classes we have the following integral representations

(0.2) $(i)\ \mu \in (\mathcal{U}^{<k>}, *)$ iff $\mu = \mathcal{L}\big(\int\limits_{(0,1]} t dY(\tau_k(t))\big)$,

$$\tau_k(t) := \frac{1}{(k-1)!} \int\limits_0^t (-\log v)^{k-1} dv;\ 0 < t \leqslant 1;\ \ k = 1, 2, ...$$

$(ii)\ \nu \in (\mathbb{U}_k, *)$ iff $\mu = \mathcal{L}\big(\int\limits_{(0,1]} t dY(r_k(t))\big),\ r_k(t) := t^k,\ 0 \leqslant t \leqslant 1,$

and $(Y(t), t \geqslant 0)$ is a cadlag Lévy process and $\mathcal{L}(Z)$ denotes a probability distribution (a law) of a random variable Z.
[Although $\int (-\log x)^k dx = \Gamma(k, -\log x) + const$, (the incomplete Euler gamma function), we do not use that identity here.]

The above, (i) and (ii), are particular examples of random integrals

$$\rho = I^{h,r}_{(a,b]}(\mu) := \mathcal{L}\big(\int\limits_{(a,b]} h(t) dY(r(t))\big),\ \ \mathcal{L}(Y(1)) = \mu \in \mathcal{D}^{h,r}_{(a,b]} \tag{0.3}$$

where h is a real function, r (a time change) is a monotone, nonnegative function and $\mathcal{D}^{h,r}_{(a,b]}$ denotes the domain of a random integral $I^{h,r}_{(a,b]}$; for details see, for instance, Jurek [13], [14], [15], [17], [19]. To Y (to μ) we refer as the *background driving Lévy process* (BDLP) (*the background driving probability distribution*(BDPD)) of the measure ρ.

The identification (the isomorphism) between classical infinitely divisible characteristic functions $\phi_\mu(t)$ and their counter part Voiculescu free-infinitely divisible transforms $V_{\tilde{\mu}}(it)$ (or measures) is given as follows:

$$(ID, *) \ni \mu \to V_{\tilde{\mu}}(it) = it^2 \int\limits_0^\infty \log \phi_\mu(-u) e^{-tu} du,\ \ t > 0; \tag{0.4}$$

see Jurek [17], Corollary 6 and the random integral mapping $\mathcal{K}^{(e)}$ which was the origin for the identity (0.4). The need for a such identification arises when one wants to use Bercovici-Pata isomorphism but we do not have parameters a and m, in the Lévy-Khintichine or Bervovici-Voiculescu formula, for the respective classical and free independence. That those two approaches do coincide was shown in Jurek [17], [18], [20][Theorem 2.1]. Moreover, in Jurek [20], on page 350, is given a diagram how one may connect any two abstract semigroups.

From the above mapping (0.4) we infer the properties

$$V_{\widetilde{\mu * \nu}}(it) = V_{\tilde{\mu}}(it) + V_{\tilde{\nu}}(it) = V_{\tilde{\mu} \boxplus \tilde{\nu}}(it);\ \ \text{for } c > 0,\ V_{\widetilde{T_c\mu}}(it) = cV_{\tilde{\mu}}(it/c),$$

and the last property is in the sharp contrast with $\phi_{T_c\mu}(t) = \phi_\mu(ct)$, for the characteristic functions.

The fundamental Lévy-Khintchine characterization says that

$$\text{(0.5)}\quad \mu \in ID \text{ iff } \phi_\mu(t) = \exp\big[ita + \int_{\mathbb{R}} (e^{itx} - 1 - \frac{itx}{1+x^2})\frac{1+x^2}{x^2} m(dx)\big]$$
$$= \exp\big[ita - \frac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{(0)\}} (e^{itx} - 1 - \frac{itx}{1+x^2}) M(dx)\big], \quad t \in \mathbb{R},$$

where parameters: a real number a, a finite Borel measure m, $m(0) = \sigma^2$ and a Lévy spectral measure $M(dx) = \frac{1+x^2}{x^2} m(dx)$ are uniquely determined. In the latter case, we will simply write $\mu = [a, \sigma^2, M]$.

For the free-infinite divisibility here is the Voiculescu analogue of the Lévy-Khintchine formula:

$$\text{(0.6)}\qquad \nu \in (ID, \boxplus) \text{ iff } V_\nu(it) = a + \int_{\mathbb{R}} \frac{1+itx}{it-x} m(dx),\ t \neq 0,$$

where the parameters: a real number a and a finite Borel measure m, are uniquely determined; for details see Voiculescu [26], Bercovici and Voiculescu [2], Barndorff-Nielsen and S. Thorbjorsen [1] and Jurek [16], [17].

Uniqueness of parameters a and m in the formulas (0.5) and (0.6) give an natural identification between classical and free-infinitely divisible transforms as it was already mentioned above. On the other hand, if one knows that $\phi_\mu \in (ID, *)$ or $V_\nu(it) \in (ID, \boxplus)$ finding their corresponding parameters a and m might be, in general, quite difficult. In that case, for $(ID, *)$ and $(ID, \boxplus)$ we use the identification given by (0.4).

1. A BASIC THEOREM.

Here is a basic result which will allow us to introduce effectively new subclasses of free-infinitely divisible transforms by specifying their corresponding random integral representations $I^{h,r}_{(a,b]}(\mu)$; see (0.3), above.

THEOREM 1. For deterministic functions h and r on an interval $(a, b]$, let us define the constants **c**, **d** and the function $\mathbf{g}_\pm$ depending on them (i.e., on h and r) as follows:

$$\mathbf{c} := \int_a^b h(s)dr(s),\ \mathbf{d} := \int_a^b h^2(s)dr(s),\ \ \mathbf{g}_\pm(z) := \int_a^b \frac{h(s)}{z\,h(s) \pm 1} dr(s),\ z \in \mathbb{C}.$$

Then for $\mu = [a, \sigma^2, M] \in \mathcal{D}^{h,r}_{(a,b]}$ and $\rho := I^{h,r}_{(a,b]}(\mu)$ there exists a counter part measure $\tilde{\rho} \in (ID, \boxplus)$ such that

$$\text{(1.1)}\quad V_{\tilde{\rho}}(it) = a\,\mathbf{c} + (\pm)\frac{\sigma^2}{it}\,\mathbf{d} + \int_{\mathbb{R}\setminus(0)} (\pm)x[\mathbf{g}_\pm(\frac{ix}{t}) - \frac{(\pm)\mathbf{c}}{1+x^2}] M(dx), \quad t > 0,$$

where, in the pair $(\pm)$, the upper sign is for non decreasing r and the lower sign for non increasing r, respectively.

Equivalently,by putting $m(dx) := \frac{x^2}{1+x^2}M(dx)$ on $\mathbb{R}\setminus\{0\}$ and $m(\{0\}) := \sigma^2$ we get a finite measure m such that

$$V_{\tilde{\rho}}(it) = a\,\mathbf{c} + \int_{\mathbb{R}} (\pm)\big[\mathbf{g}_{\pm}(\frac{ix}{t}) - \frac{(\pm)\mathbf{c}}{1+x^2}\big]\frac{1+x^2}{x}m(dx), \;\; t>0, \tag{1.2}$$

where the integrand in (1.2) at zero is equal $(\pm)\mathbf{d}(it)^{-1}$.

Moreover, if $h(s) > 0$, we have the following relation between $\tilde{\rho}$ and $\tilde{\mu}$, the free-probability counterparts of ρ and its background driving measure μ, respectively.

$$V_{\tilde{\rho}}(it) = \int_a^b h(s)V_{\tilde{\mu}}(it/h(s))dr(s) = \int_a^b V_{\widetilde{T_{h(s)}\mu}}(it)dr(s), \; t>0, \tag{1.3}$$

where $(T_c(\mu))(B) := \mu(c^{-1}B), c > 0$, for Borel sets B.

P r o o f. The isomorphism (0.4) gives one-to-one correspondence between the classical $\phi_\mu(t)$ and the free-infinity divisible $V_{\tilde{\mu}}(it)$ transforms. The law (of the random integral) $\rho = I^{h,r}_{(a,b]}(\mu)$ has the characteristic function

$$\log\phi_\rho(v) = \int_a^b \log\phi_\mu((\pm)h(s)v)(\pm)dr(s), \tag{1.4}$$

by (0.3), where $(\pm)$ is for a non-decreasing and a non- increasing function r, respectively.

Since by (0.5), $\log\phi_\mu(t) = ita - \frac{\sigma^2}{2}t^2 + \int_{\mathbb{R}\setminus\{(0)\}}\big(e^{itx} - 1 - \frac{itx}{1+x^2}\big)M(dx)$, therefore, using the Fubini Theorem and (0.4), we get

$$
\begin{aligned}
V_{\tilde{\rho}}(it) &= it^2 \int_0^\infty \log \phi_\rho(-u) e^{-tu} du \\
&= it^2 \int_0^\infty \int_a^b \log \phi_\mu(-(\pm)h(s)u)(\pm)dr(s)\ e^{-tu} du \\
&= \int_a^b it^2 \int_0^\infty [\log \phi_\mu(\mp h(s)u)\, e^{-tu} du]\, (\pm)dr(s) \\
&= \int_a^b ia(\mp h(s)) it^2 \int_0^\infty u e^{-tu} du (\pm) dr(s) \\
&- \int_a^b (\mp h(s))^2 \frac{1}{2}\sigma^2 it^2 \int_0^\infty u^2 e^{-tu} du (\pm) dr(s)
\end{aligned}
$$

$$
+ \int_a^b \int_{\mathbb{R}\setminus\{(0)\}} it^2 \int_0^\infty \big(e^{(\mp) ih(s)xu} - 1 - \frac{x}{1+x^2}((\mp) ih(s)u)\big) e^{-tu} du M(dx)(\pm)dr(s)
$$

$$
\begin{aligned}
&= \int_a^b ia((\mp)h(s)) it^2 \frac{1}{t^2}(\pm)dr(s) - \int_a^b (h(s))^2 \frac{1}{2}\sigma^2 it^2 \frac{2}{t^3}(\pm)dr(s) \\
&+ \int_a^b \int_{\mathbb{R}\setminus\{(0)\}} it^2 \big(\frac{1}{t \pm ih(s)x} - \frac{1}{t} - \frac{i(\mp)h(s)x}{1+x^2}\frac{1}{t^2}\big) M(dx)(\pm)dr(s)
\end{aligned}
$$

$$
= a\,\mathbf{c} + \frac{\sigma^2}{it}(\pm)\mathbf{d} + \int_{\mathbb{R}\setminus\{(0)\}} \int_a^b [\frac{(\pm)t\, h(s)x}{t \pm ixh(s)} + \frac{x}{1+x^2}(\mp)h(s)](\pm)dr(s) M(dx)
$$

$$
= a\,\mathbf{c} + \frac{\sigma^2}{it}(\pm)\mathbf{d} + \int_{\mathbb{R}\setminus\{(0)\}} (\pm)x \int_a^b [\frac{h(s)}{1 \pm ixh(s)/t} + \frac{1}{1+x^2}(\mp)h(s)] dr(s) M(dx)
$$

$$
= a\mathbf{c} + \frac{\sigma^2}{it}(\pm)\mathbf{d} + \int_{\mathbb{R}} (\pm)x\,[\mathbf{g}_\pm(ix/t) - \frac{(\pm)\mathbf{c}}{1+x^2}]\, M(dx),
$$

which proves the formula (1.1).

To get (1.2), let us note that $g_\pm(0) = \pm\mathbf{c}$ and

$$
\lim_{x\to 0}(\pm)[\frac{\mathbf{g}_\pm(\frac{ix}{t}) - \frac{(\pm)\mathbf{c}}{1+x^2}}{x}] = (\pm)\lim_{x\to 0}\int_a^b \frac{(h(s))^2}{(ixh(s)/t \pm 1)^2}\ \frac{-i}{t} dr(s) = \frac{1}{it}(\pm)\mathbf{d}.
$$

Finally, similarly as above, we have

$$V_{\tilde{\rho}}(it) = \int_a^b it^2 \int_0^\infty [\log\phi_\mu(-h(s)u)\, e^{-tu} du]\, dr(s)$$

$$= \int_a^b it^2 \int_0^\infty [\log\phi_\mu(-w)\, e^{-tw/h(s)} \frac{dw}{h(s)}]\, dr(s) = \int_a^b h(s) V_{\tilde{\mu}}(it/h(s) dr(s)$$

$$= \int_a^b V_{\widetilde{T_{h(s)}\mu}}(it) dr(s), \text{ as by (0.4), } V_{\widetilde{T_c\mu}}(it) = cV_{\tilde{\mu}}(it/c),\ c > 0;$$

which completes a proof. ∎

The kernel $\mathbf{g}(z)$ from Theorem 1 admits the following representation:

LEMMA 1. For non-decreasing r, functions $\mathbf{g}(z) := \int_a^b \frac{h(s)}{zh(s)+1} dr(s)$ map upper half-plane of $\mathbb{C}^+$ into lower half-plane $\mathbb{C}^-$ are analytic ones with the Pick - Nevanlinna representation

$$\mathbf{g}(z) = \int_a^b \frac{h(s)}{1+h^2(s)} dr(s) + \int_{\mathbb{R}} \frac{1+zx}{z-x} (\int_a^b \frac{h^2(s)}{1+h^2(s)} \delta_{-1/h(s)}(dx) dr(s))$$

P r o o f. First, note that $\Im(\mathbf{g}(z)) = -\Im(z) \int_a^b \frac{h^2(s)}{|1+zh(s)|^2} dr(s)$, where the integral is positive as r is non decreasing. This means that $\mathbf{g} : \mathbb{C}^+ \to \mathbb{C}^-$ and is an analytic function with

$$\frac{d^n}{dz^n} \mathbf{g}(z) = (-1)^n n! \int_a^b \big(\frac{h(s)}{1+zh(s)}\big)^{n+1} dr(s), n = 0, 1, 2, ..$$

Second, for $b \in \mathbb{R}$ we have explicit Pick-Nevanlinna representation

$$\frac{1}{z+b} = u_b + \int_{\mathbb{R}} \frac{1+zx}{z-x} m_b(dx),\ \ u_b := \frac{b}{1+b^2},\ \ m_b(A) := \frac{1}{1+b^2} \delta_{-b}(A),$$

and hence taking $b := 1/h(s)$ and integrating the above with respect to $dr(s)$ we get

$$\mathbf{g}(z) = u + \int_{\mathbb{R}} \frac{1+zx}{z-x} m(dx), \text{ where } u := \int_a^b \frac{h(s)}{1+h^2(s)} dr(s),$$

is the shift parameter and the measure m is

$$m(A) := \int_a^b \frac{h^2(s)}{1+h^2(s)} \delta_A(-\frac{1}{h(s)}) dr(s) = \int_a^b \frac{h^2(s)}{1+h^2(s)} \delta_{-\frac{1}{h(s)}}(A) dr(s),$$

is a mixture of the point-mass Dirac measures, $k(s)\delta_{f(s)}(A)$ which gives the proof of Lemma 1. ∎

2. CLASSES $(\mathcal{U}^{<k>}, \boxplus)$ OF FREE-INFINITELY DIVISIBLE TRANSFORMS (MEASURES) FOR $k = 1, 2, \ldots$

For the one-parameter semigroup $(U_r, r > 0)$ of non-linear *shrinking operations* (in short: *s-operations*) defined as follows

$$U_r : \mathbb{R} \to \mathbb{R} \text{ as } U_r(0) := 0, \;\; U_r(x) := \max\{|x| - r, 0\}\frac{x}{|x|}, \;\; x \neq 0; r > 0,$$

in Jurek [8], [9] was introduced the class $\mathcal{U}$ of limiting distributions of sequences

$$U_{r_n}(X_1) + U_{r_n}(X_1) + \ldots + U_{r_n}(X_n) + x_n,$$

where terms $U_{r_n}(X_j),\ 1 \leqslant j \leqslant n$, are uniformly infinitesimal and random variables $X_n, n = 1, 2, \ldots$ are stochastically independent. Measures $\mu \in \mathcal{U}$ were termed as *s-selfdecomposable measures.*

REMARK 1. *Note that nowadays, in the mathematical finance, for $X > 0$, the s-operation $U_r(X) = (X - r)_+$ is called* the European call option *on a stock X with an exercise price r.*

In Jurek [15] were introduced and characterized the following subclasses of the class $(ID, *)$ of the classical infinitely divisible measures:

$$\ldots \subset \mathcal{U}^{<k+1>} \subset \mathcal{U}^{<k>} \subset \ldots \subset \mathcal{U}^{<1>} \equiv \mathcal{U} \subset ID,$$

and the measures $\mu \in \mathcal{U}^{<k>}$ were called *k-times s-selfdecomposable measures*. Furthermore, as mentioned in the Introduction, taking the time change

$$\tau_k(t) := \frac{1}{(k-1)!}\int_0^t (-\log v)^{k-1} dv, \text{ we get } (\mathcal{U}^{<k>}, *) = I_{(0,1]}^{t,\tau_k(t)}(ID), \tag{2.1}$$

and $I_{(0,1]}^{t,\tau_k(t)}(\nu) = I_{(0,1]}^{t,t}(I_{(0,1]}^{t,t}(\ldots(I_{(0,1]}^{t,t}(\nu)))$, (k-times); see Jurek [15], Proposition 4 and Corollary 2 and for more general theory of compositions of random integrals see Jurek [19].

Here are the free-infinity divisible counterparts of k-times s-selfdecomposable probability measures:

PROPOSITION 1. For $k = 1, \ldots$, a measure $\tilde{\nu}$ is a free-probability counterpart of $\nu = [a, \sigma^2, M] \in (\mathcal{U}^{<k>}, *)$, that is $\tilde{\nu} \in (\mathcal{U}^{<k>}, \boxplus)$, if and only if

$$V_{\tilde{\nu}}(it) = \frac{a}{2^k} + \frac{\sigma^2}{3^k}\frac{1}{it} + \int_{\mathbb{R}\setminus(0)} x[\Phi(\frac{x}{it}, k, 2) - \frac{1}{1+x^2}\frac{1}{2^k}]M(dx). \tag{2.2}$$

Equivalently,

$$V_{\tilde{\nu}}(it) = \frac{a}{2^k} + \int_{\mathbb{R}} [\Phi(\frac{x}{it}, k, 2) - \frac{1}{1+x^2}\frac{1}{2^k}]\frac{1+x^2}{x} m(dx), \tag{2.3}$$

where $a \in \mathbb{R}$, $m(dx) := \frac{x^2}{1+x^2}M(dx)$ on $\mathbb{R} \setminus \{0\}$ and $m(0) := \sigma^2$, is finite Borel measure m and $\Phi(z, s, v) := \sum_{n=0}^{\infty} \frac{z^n}{(v+n)^s}, \quad |z| < 1, v \neq 0, -1, -2, \ldots$ is the Hurwitz-Lerch function. Finally, the integrand in (2.3) at zero is equal to $(3^k it)^{-1}$.

P r o o f. Taking into account (2.1) and Theorem 1, we get the constants $\mathbf{c} = 2^{-k}$ and $\mathbf{d} = 3^{-k}$. Furthermore, to find $\mathbf{g}(z)$ we quote from Gradshteyn and Ryzhik [6], formula (**9.556**) that Hurwitz-Lerch function admits the integral representations:

if $\Re v > 0, or\, |z| \leqslant 1, z \neq 1, \Re s > 0, or\; z = 1, \Re s > 1$ then

$$\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-vt}}{1 - ze^{-t}} dt.$$

Hence

$$\mathbf{g}(z) = \frac{1}{(k-1)!} \int_0^1 \frac{s(-\log s)^{k-1}}{1 + zs} ds$$
$$= \frac{1}{(k-1)!} \int_0^\infty \frac{w^{k-1}e^{-2w}}{1 + zw} dw = \Phi(-z, k, 2),$$

which completes the proof of Proposition 1. ■

REMARK 2. *(i). For the class $(\mathcal{U}^{<1>}, \boxplus)$ of free s-selfdecomposable measures we may use the identity $\Phi(-ix/t, 1, 2) = it(-x - it\log(1 + ix/t))$. The characterization of the class $(\mathcal{U}^{<1>}\boxplus) \equiv (\mathcal{U}, \boxplus)$ was earlier given in Jurek [18], Proposition 1(b). Note a misprint there: it should be t^2, not $(it)^2$ in the part (b).*

(ii). Putting $k = 0$ in Proposition 1, we get that $(\mathcal{U}^{<0>}, \boxplus) \equiv (ID, \boxplus)$, because of the formula (0.6).

Here are relations between consecutive classes $(\mathcal{U}^{<k>}, \boxplus)$:

COROLLARY 1. Let us define a differential operator $\mathbb{D}f(t) := 2f(t) - t\frac{d}{dt}f(t)$. Then for $k \geqslant 1$

$$\mathbb{D} : (\mathcal{U}^{<k>)}, \boxplus) \to (\mathcal{U}^{<k-1>}, \boxplus), \text{ where } (\mathcal{U}^{<0>}, \boxplus) := (ID, \boxplus).$$

Hence $\mathbb{D}^k : (\mathcal{U}^{<k>}, \boxplus) \to (ID, \boxplus)$.

P r o o f. Let $V_{\tilde{\nu}}(it) = \frac{a}{2^k} + \frac{1}{3^k}\frac{\sigma^2}{it} \in (\mathcal{U}^{<k>}, \boxplus)$. Then

$$\mathbb{D}(V_{\tilde{\nu}}(it)) = \frac{a}{2^{k-1}} + \frac{2\sigma^2}{3^k}\frac{1}{it} - t\frac{\sigma^2}{3^k}(-1)i(it)^{-2} = \frac{a}{2^{k-1}} + \frac{1}{3^{k-1}}\frac{\sigma^2}{it} \in (\mathcal{U}^{<k-1>}, \boxplus).$$

Since by Wolframalpha.com

$$\frac{d}{dt}[\Phi(\frac{x}{it}, k, 2)] = -t^{-1}\big(\Phi(\frac{x}{it}, k-1, 2) - 2\Phi(\frac{x}{it}, k, 2)\big)$$

therefore for the Poisson part in (2.2) we have

$$\mathbb{D}[\Phi(\frac{x}{it},k,2)-\frac{1}{2^k}\frac{1}{1+x^2}]=2\Phi(\frac{x}{it},k,2)-\frac{1}{2^{k-1}}\frac{1}{1+x^2}$$
$$-t\frac{d}{dt}(\Phi(\frac{x}{it},k,2))=2\Phi(\frac{x}{it},k,2)-\frac{1}{2^{k-1}}\frac{1}{1+x^2}+\Phi(\frac{x}{it},k-1,2)$$
$$-2\Phi(\frac{x}{it},k,2)=\Phi(\frac{x}{it},k-1,2)-\frac{1}{2^{k-1}}\frac{1}{1+x^2},$$

which is the kernel in (2.2) corresponding to the free-infinitely divisible measure in $(\mathcal{U}^{<k-1>},\boxplus)$. This completes a proof of Corollary 1. ■

3. CLASSES $(\mathbb{U}_k,\boxplus)$ OF FREE-INFINITELY DIVISIBLE TRANSFORMS FOR $k=0,1,2,...$.

For a fixed k, a probability measure μ is in $(\mathbb{U}_k,*)$, if there exists a sequence $\nu_n\in(ID,*), n=1,2,...$ such that

(3.1) $$\rho_n:=T_{\frac{1}{n}}(\nu_1*\nu_2*...*\nu_n)^{*n^{-k}}\Rightarrow\mu,\ \text{ as }\ n\to\infty;$$

see Jurek [13], Theorem 1.1 and Corollary 1.1. Take there, an operator $Q=I$, Borel measures ν_k on the real line and a parameter $\beta=k$.

A class of all possible limits in (3.1) is denoted by $(\mathbb{U}_k,*)$ and measures $\mu\in(\mathbb{U}_k,*)$ are referred to as *k-times s-selfdecomposable measures*. Note that for $k=1$ we get the class $(\mathbb{U},*)$ of s-selfdecomposable measures.

Furthermore, subclasses $(\mathbb{U}_k,*)$ form an increasing filtration of whole class of infinitely divisible measures, and all subclasses admit random integral representations (see (3.2) below). Namely,

(3.2) if $0\leqslant k\leqslant l$ then $(\mathbb{U}_0,*)\subset(\mathbb{U}_k,*)\subset(\mathbb{U}_l,*)\subset(ID,*)$; in particular,
$(\mathbb{U}_0,*)\equiv(L_0,*)$ (selfdecomposable measures; see Section 4, below);
$(\mathbb{U}_1,*)\equiv(\mathcal{U}^{<1>},*)$; (s-selfdecomposable measures; see Section 2, above);
$$(\mathbb{U}_k,*)=I^{t,t^k}_{(0,1]}(ID);\ \text{ and }\ \overline{\cup_{k=1}^{\infty}(\mathbb{U}_k,*)}=(ID,*);$$

Here are transforms of free-infinitely divisible counterparts of measures from classes $(\mathbb{U}_k,*)$:

PROPOSITION 2. For $k\geqslant 1$, a measure $\tilde{\nu}$ is a free-probability counterpart of

$\nu = [a, \sigma^2, M] \in (\mathbb{U}_k, *)$, that is $\tilde{\nu} \in (\mathbb{U}_k, \boxplus)$, if and only if for $t > 0$

(3.3)
$$V_{\tilde{\nu}}(it) = \frac{k}{k+1}\,a + \frac{k}{k+2}\frac{\sigma^2}{it} + \int_{\mathbb{R}\setminus\{0\}} [k\,it\Phi(\frac{x}{it},1,k) - it - \frac{k}{k+1}\frac{x}{1+x^2}]M(dx)$$
$$= \frac{k}{k+1}\,a + \int_{\mathbb{R}} [k\,it\big(\Phi(\frac{x}{it},1,k) - k^{-1}\big) - \frac{k}{k+1}\frac{x}{1+x^2}]\frac{1+x^2}{x^2}m(dx)$$

where M is arbitrary Lévy measure and a measure m, defined as $m(dx) := \frac{x^2}{1+x^2}M(dx)$ on $\mathbb{R}\setminus\{0\}$, and $m(\{0\}) := \sigma^2$, is a finite measure and the integrand in (3.3) at zero is $\frac{k}{k+2}\frac{1}{it}$.

[$\Phi(z,s,v)$ is the Hurwitz-Lerch function.]

P r o o f. Since $\mathbb{U}_k = I^{t,t^k}_{(0,1]}(ID)$ therefore we take $a = 0, b = 1, h(t) = t$ and $r(t) = t^k$ in Theorem 1. Thus $\mathbf{c} = k/(k+1)$, $\mathbf{d} = k/(k+2)$ and

$$\mathbf{g}_+(z) = k\int_0^1 \frac{s^k}{1+zs}ds = \frac{k}{k+1}\,{}_2F_1(1,k+1;k+2;-z); \text{ by } \mathbf{3.194}(5),$$

in Gradshteyn and Ryzhik [6],($|arg(1+z)| < \pi$) and ${}_2F_1$ denotes *the hypergeometric function*. It is defined as

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\ \ (x)_n := x(x+1)...(x+n-1),\ c \neq -\mathbb{N};$$

where $(x)_n$ is *the Pochhammer symbol* with the convention $(x)_0 := 1$.

Consequently, for the kernel $\mathbf{g}_+(z)$ we have

$$\mathbf{g}_+(z) = \frac{k}{k+1}\,{}_2F_1(1,k+1;k+2;-z)$$
$$= k(k+1)^{-1}\sum_{n=0}^{\infty}\frac{(1)_n(k+1)_n}{(k+2)_n}\frac{(-z)^n}{n!} = k\sum_{n=0}^{\infty}\frac{(-z)^n}{k+n+1} = k(-z)^{-1}\sum_{j=1}^{\infty}\frac{(-z)^j}{k+j}$$
$$= k(-z)^{-1}[\sum_{n=0}^{\infty}\frac{(-z)^n}{k+n} - \frac{1}{k}] = k(-z)^{-1}[\Phi(-z,1,k) - k^{-1}];$$

and $\Phi(z,s,a)$ is the Hurwitz-Lerch function.
Finally, we have $x\mathbf{g}(\frac{ix}{t}) = itk(\Phi(x/(it),1,k) - k^{-1})$ and this completes a proof of Proposition 2. ∎

COROLLARY 2. If $\tilde{\nu_k} \in (\mathbb{U}_k, \boxplus)$ then

$$\lim_{k\to\infty} V_{\tilde{\nu_k}}(it) = a + \int_{\mathbb{R}} \frac{1+itx}{it-x}m(dx) = V(it) \in (ID,\boxplus),\ \text{ for } t>0.$$

In other words, $\overline{\cup_{k=1}^{\infty}(\mathbb{U}_k,\boxplus)} = (ID,\boxplus)$.

To this end, note that as $k \to \infty$ then

$k\Phi(\frac{x}{it}, 1, k) = k\sum_{n=0}^{\infty}(\frac{x}{it})^n \frac{1}{k+n} = \sum_{n=0}^{\infty}(\frac{x}{it})^n \frac{1}{1+n/k} \to \sum_{n=0}^{\infty}(\frac{x}{it})^n = \frac{it}{it-x}$,

and

$[kit\Phi(\frac{x}{it}, 1, k) - it - \frac{k}{k+1}\frac{x}{1+x^2}]\frac{1+x^2}{x^2} \to [it\frac{it}{it-x} - it - \frac{x}{1+x^2}]\frac{1+x^2}{x^2} = \frac{1+itx}{it-x}$,

which proves Corollary 2.

REMARK 3. *For any $\beta \geqslant -2$, classes $(\mathbb{U}_\beta, \boxplus)$ are well defined by* (3.1)*; cf. Jurek [13], [14]. Here we have restricted indices to the natural numbers to have the sequence of the inclusions as was announced in* (0.1). *Proposition 2 holds true when one replaces $k \geqslant 1$ by $\beta > 0$. Furthermore, for $\beta = 0$ we get the selfdecomposable distributions as they are discussed below.*

4. URBANIK TYPE CLASSES $(L_k, \boxplus)$ OF FREE-INFINITELY DIVISIBLE TRANSFORMS FOR $k = 0, 1, 2, \ldots$

Urbanik [24], [25] introduced a filtration of convolution semigroups of *selfdecomposable measures* (Lévy class L_0) in a such way that

(4.1) $\quad$ *(Gaussian)* $\subset$ *(stable)* $\subset L_\infty \subset \ldots \subset L_{k+1} \subset L_k \subset \ldots \subset L_0 \subset \ldots \subset ID$;

Then using the extreme points method he found their descriptions in terms of characteristic functions. Measures $\mu \in (L_k, *)$ are called *k-times selfdecomposable*; for a link to Urbanik [25] see:

`www.math.uni.wroc.pl/~zjjurek/urb-limitLawsOhio1973.pdf`.

Later on, all above classes were described in terms of random integrals. Namely, taking

$$r_k(t) := t^{k+1}/(k+1)!,\ t \in (0,\infty),\ \text{ (a time change) and } h(t) := e^{-t},$$

we have the following representations:

$$L_k = I_{(0,\infty)}^{e^{-t}, r_k(t)}(ID_{\log^{k+1}}),\ ID_{\log^{k+1}} := \{\nu \in ID : \int_{\mathbb{R}} \log^{k+1}(1+|x|)\nu(dx) < \infty\}.$$

Furthermore, from the integral representations one easily gets their characteristic functions in the same form as in Urbanik [24], [25]; see Jurek [11], Corollary 2.11 and Theorem 3.1.

Here are the free-infinitely divisible analogues of Urbanik classes $(L_k, *)$:

PROPOSITION 3. For $k = 0, 1, \ldots$, a measure $\tilde{\nu}$ is a free-probability counterpart of $\nu = [a, \sigma^2, M] \in (L_k, *)$, that is $\tilde{\nu} \in (L_k, \boxplus)$, if and only if

$$(4.2) \quad V_{\tilde{\nu}}(it) = a + \frac{1}{2^{k+1}}\frac{\sigma^2}{it} + \int_{\mathbb{R}\setminus(0)} \big(itLi_{k+1}(\frac{x}{it}) - \frac{x}{1+x^2}\big)M(dx),\ \ t > 0,$$

where a Lévy measure M has log-moment $\int_{(|x|>1)} \log^{k+1}(1+|x|)M(dx) < \infty$. Equivalently,

$$(4.3) \quad V_{\tilde{\nu}}(it) = a + \int_{\mathbb{R}} \big[itLi_{k+1}(\frac{x}{it}) - \frac{x}{1+x^2}\big] \frac{m(dx)}{\log^{k+1}(1+|x|^{2/(k+1)})}, \;\; t>0,$$

where m is a finite Borel measure such that $m(\{0\}) = \sigma^2$. The integrand in (4.3) at zero is equal to $\frac{1}{2^{k+1}}\frac{1}{it}$.

[Here $Li_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \; |z|<1,$ *(and analytically continued on* $\mathbb{C}$*) is* the polylogarithmic function.*]*

P r o o f. For $h(s) = e^{-s}, r_k(s) = s^{k+1}/(k+1)!$ and $(a,b] = (0,\infty)$, using Theorem 1, we get $\mathbf{c} = 1, \mathbf{d} = 2^{-k-1}$ and

$$\mathbf{g}(z) = \int_0^\infty \frac{e^{-s}}{1+ze^{-s}} \frac{s^k}{k!} ds = -z^{-1} Li_{k+1}(-z), \text{ by Wolframalpha.com}$$

Hence $\mathbf{g}(ix/t) = -t/(ix)Li_{k+1}(-ix/t) = it/xL_{k+1}(x/it)$. Inserting this, together with $\mathbf{c} = 1$ and $\mathbf{d} = 2^{-k-1}$, into (1.1) in Theorem 1 we get

$$(4.4) \quad V_{\tilde{\rho}}(it) = a + \frac{\sigma^2}{it} 2^{-k-1} + \int_{\mathbb{R}\setminus(0)} [itLi_{k+1}(x/it) - \frac{x}{1+x^2}]M(dx), \;\; t>0.$$

Since $m(dx) := \log^{k+1}(1+|x|^{2/(k+1)})M(dx)$ is a finite measure on $\mathbb{R}\setminus\{0\}$, (see Jurek and Mason [21], Proposition 1.8.13.) and adding an atom $m(\{0\}) := \sigma^2$, we complete the proof of Proposition 3. ■

REMARK 4. *Since* $Li_1(\frac{x}{it}) \equiv PolyLog[1, \frac{x}{it}] = -\log(1-\frac{x}{it})$ *then taking* $k = 0$ *in Proposition 3 above, we retrieve Proposition 2 from Jurek [18].*

Here is a relation between the consecutive classes $(L_k, \boxplus)$.

COROLLARY 3. Let define the differential operator $Df(t) := f(t) - t\frac{d}{dt}f(t)$. Then for $k \geqslant 0$,

$$D : (L_k, \boxplus) \to (L_{k-1}, \boxplus), \quad \text{where } \; (L_{-1}, \boxplus) \equiv (ID, \boxplus).$$

Hence $D^{k+1} : (L_k, \boxplus) \to (ID, \boxplus)$.

P r o o f. Let $V_{\tilde{\nu}}(it) = a + \frac{1}{2^{k+1}}\frac{\sigma^2}{it} \in (L_k, \boxplus)$. Then

$$D(V_{\tilde{\nu}}(it)) = a + \frac{1}{2^{k+1}}\frac{\sigma^2}{it} - t\frac{\sigma^2}{2^{k+1}}(-1)i(it)^{-2} = a + \frac{1}{2^k}\frac{\sigma^2}{it} \in (L_{k-1}, \boxplus).$$

Keeping in mind that $d/dzLi_{k+1}(z) = z^{-1}Li_k(z)$ for the Poissonian part of (4.2) we have

$$D[\int_{\mathbb{R}\setminus(0)} \big(itLi_{k+1}(\frac{x}{it}) - \frac{x}{1+x^2}\big)M(dx)]$$

$$= \int_{\mathbb{R}\setminus(0)} [\big(itLi_{k+1}(\frac{x}{it}) - \frac{x}{1+x^2}\big) - t\frac{d}{dt}(itLi_{k+1}(\frac{x}{it}))]M(dx)$$

$$= \int_{\mathbb{R}\setminus(0)} [-\frac{x}{1+x^2} - t(it\frac{d}{dt}(Li_{k+1}(\frac{x}{it}))]M(dx)$$

$$= \int_{\mathbb{R}\setminus(0)} [-\frac{x}{1+x^2} - it^2(\frac{it}{x}Li_k(\frac{x}{it})(-ix)(it)^{-2}]M(dx)$$

$$= \int_{\mathbb{R}\setminus(0)} [-\frac{x}{1+x^2} + itLi_k(\frac{x}{it})]M(dx) \in (L_{k-1}, \boxplus),$$

which completes the proof of Corollary 3. ■

5. RELATIONS BETWEEN CLASSES $(L_k, \boxplus)$ AND $(\mathcal{U}^{<k>}, \boxplus)$.

Since $(L_k, *) \subset (\mathcal{U}^{<k>}, *)$, for $k = 0, 1, ...$ (see Jurek [15], Corollaries 2 and 7), therefore the injection (0.4) between the classical and the free infinite divisibility implies that

$$(L_k, \boxplus) \subset (\mathcal{U}^{<k>}, \boxplus), \quad k = 0, 1, 2, ... \tag{5.1}$$

Although the classes $(L_k, *)$ and $(\mathcal{U}^{<k>}, *)$ were introduced via the linear and the non-linear scaling, respectively, here is another relation, besides (5.1), between their free-probability counterparts.

For the notational simplicity, as is in Jurek [12], let us introduce

$$\mathcal{I}(\nu) \equiv I^{e^{-t},t}_{(0,\infty)}(\nu), \quad \nu \in ID_{\log}; \qquad \text{and } \mathcal{J}(\rho) \equiv I^{t,t}_{(0,1]}(\rho), \ \ \rho \in ID.$$

Then we have

$$(L_0, *) = \mathcal{I}(ID_{log}), \ L_{k+1} = \mathcal{I}(I^{e^{-t},r_k(t)}_{(0,\infty)}(ID_{\log^{k+2}}), \ r_k(t) = \frac{1}{(k+1)!}t^{k+1};$$

$$\mathcal{U}^{<0>} = \mathcal{J}(ID), \quad \mathcal{U}^{<k+1>} = \mathcal{J}(I^{t,\tau_k(t)}_{(0,1]}(ID)), \tau_k(t) = \int_0^t (-logx)^{k-1}dx;$$

that is, those classes correspond to the compositions of $k+1$ mappings $\mathcal{I}$ and $\mathcal{J}$, respectively; see Jurek [19] for the general theory of compositions of the random integral mappings.

THEOREM 2. For $k = 0, 1, ...,$ a measure $\tilde{\rho} \in \widetilde{(\mathcal{U}^{<k>}}, \boxplus)$ is in $(L_k, \boxplus)$ if and only if there exists $\omega \in (ID_{\log^{k+1}}, *)$ such that $\tilde{\rho} = \widetilde{\mathcal{I}(\omega)} \boxplus \tilde{\omega}$.

P r o o f. Let $k = 0$ and $\tilde{\rho}$ be the free-counterpart of $\rho \in (\mathcal{U}^{<0>}, *) \cap (L_0, *)$. Therefore there exist $\nu \in ID$ and $\mu \in ID_{\log}$ such that $\rho = \mathcal{J}(\nu) = \mathcal{I}(\mu)$. However, to have such equality it is necessary and sufficient that $\nu = \mu * \mathcal{I}(\mu)$; see Theorem 4.5 in Jurek [12]. Equivalently $\rho = \mathcal{J}(\nu) = \mathcal{J}(\mu) * \mathcal{I}\big(\mathcal{J}(\mu)\big)$. Taking $\omega := \mathcal{J}(\mu)$ we have that $\omega \in ID_{\log}$ as $\mu \in ID_{\log}$ and finally $\rho = \omega * \mathcal{I}(\omega)$. Hence $\tilde{\rho} = \widetilde{\big(\mathcal{I}(\omega) * \omega\big)} = \widetilde{\mathcal{I}(\omega)} \boxplus \tilde{\omega}$, which proves the Theorem 2 for $k = 0$.

Assume that the theorem is true for the classes with indices $0 \leqslant j \leqslant k$ and let $\tilde{\rho} \in \mathcal{U}^{<k+1>}, \boxplus) \cap (L_{k+1}, \boxplus)$ be the counterpart of $\rho \in \mathcal{U}^{<k+1>}, *) \cap (L_{k+1}, *)$. Then there exist $\nu \in ID$ and $\mu \in ID_{\log^{k+2}}$ such that

$$\rho = I_{(0,1)}^{t,\tau_{k+1}(t)}(\nu) = \mathcal{J}(I_{(0,1)}^{t,\tau_k(t)}(\nu)) \text{ and } \rho = I_{(0,\infty)}^{e^{-t},r_{k+1}(t)}(\mu) = \mathcal{I}(I_{(0,\infty)}^{e^{-t},r_k(t)}(\mu)),$$

and by putting $\nu_1 := I_{(0,1)}^{t,\tau_k(t)}(\nu) \in \mathcal{U}^{<k>}$ and $\mu_1 := \mathcal{I}(I_{(0,\infty)}^{e^{-t},r_k(t)}(\mu) \in L_k$, from the above line, we have $\rho = \mathcal{J}(\nu_1) = \mathcal{I}(\mu_1)$. From this (as in the case $k = 0$) we get $\nu_1 = \mu_1 * \mathcal{I}(\mu_1)$ and $\rho = \mathcal{J}(\mu_1) * \mathcal{I}(\mathcal{J}(\mu_1))$. Taking $\omega := \mathcal{J}(\mu_1) \in ID_{\log^{k+1}}$ we get $\tilde{\rho} = \tilde{\omega} \boxplus \widetilde{\mathcal{I}(\omega)}$, which completes a proof. ■

Since there is no random integral representation for the class $(L_\infty, *)$, so there is no direct application of the basic Theorem 1. Nevertheless we have

PROPOSITION 4. (i) A measure $\tilde{\nu}$ is a free-probability counterpart if $\nu \in (L_\infty, *)$, that is $\tilde{\nu} \in (L_\infty, \boxplus)$, if and only if

$$V_{\tilde{\nu}}(it) = c - \int\limits_{(-2,2]\setminus\{0\}} \frac{\Gamma(|x|+1)ie^{i\pi x/2} + x}{t^{|x|-1}\,(1-|x|)}\, G(dx), \tag{5.2}$$

where $c \in \mathbb{R}$, G is a finite Borel measure and the integrand at ± 1 is equal to $i\pi/2 \mp \gamma$, respectively.

(ii) A measure $\tilde{\nu}$ is a free-probability counterpart if $\nu \in (\mathcal{U}^{<\infty>}, *)$, that is, $\tilde{\nu} \in (\mathcal{U}^{<\infty>}, \boxplus)$, if and only if $V_{\tilde{\nu}}(it)$ is of the form (5.2) above.

P r o o f. From Urbanik [24] Theorem 2 or Urbanik [25], Theorem 2 (or `www.math.uni.wroc.pl/~zjjurek/urb-limitLawsOhio1973.pdf`)

we know that $\nu \in (L_\infty, *)$ iff

$$\phi_\nu(t) = \exp\big(iat - \int\limits_{(-2,2]\setminus\{0\}} \big[|t|^{|x|}(\cos(\frac{\pi x}{2}) - i\frac{t}{|t|}\sin(\frac{\pi x}{2})) + itx\big]\frac{G(dx)}{1-|x|}\big)$$

where $a \in \mathbb{R}$ and G is a finite Borel measure on $(-2,0) \cup (0,2]$.

Let use the identification (0.4). Then

$$V_{\tilde{\nu}}(it) = it^2 \int_0^\infty \log \phi_\nu(-u) e^{-tu} du =$$

$$it^2 \int_0^\infty \big(-iau - \int_{(-2,2]\setminus\{0\}} \big[|u|^{|x|}(\cos(\frac{\pi x}{2}) + i\frac{u}{|u|}\sin(\frac{\pi x}{2})) - iux\big] \frac{G(dx)}{1-|x|}\big) e^{-tu} du$$

$$= a - \int_{(-2,2]\setminus\{0\}} it^2 \int_0^\infty \big[u^{|x|}(\cos(\frac{\pi x}{2}) + i\frac{u}{|u|}\sin(\frac{\pi x}{2})) - iux\big] e^{-tu} du \, \frac{G(dx)}{1-|x|}$$

$$= a - \int_{(-2,2]\setminus\{0\}} [it^2\, \Gamma(1+|x|)t^{-(1+|x|)}(\cos(\frac{\pi x}{2}) + i\sin(\frac{\pi x}{2})) + x\big] \frac{G(dx)}{1-|x|}$$

$$= a - \int_{(-2,2]\setminus\{0\}} [\Gamma(1+|x|)t^{1-|x|}\, ie^{i\pi x/2} + x\big] \frac{G(dx)}{1-|x|}$$

where

$$\lim_{x\to 1} \frac{\Gamma(|x|+1)ie^{i\pi x/2} + x}{1-|x|} = i\pi/2 - \gamma; \ \lim_{x\to -1} \frac{\Gamma(|x|+1)ie^{i\pi x/2} + x}{1-|x|} = i\pi/2 + \gamma,$$

as for $x > 0$

$$\lim_{x\to 1} \frac{d}{dx}\Gamma(x+1) = \lim_{x\to 1} \frac{d}{dx} \int_0^\infty u^x e^{-u} du = \lim_{x\to 1} \int_0^\infty \log(u) u^x e^{-u} du$$

$$= \int_0^\infty u \log(u) e^{-u} du = 1 - \gamma; \quad \text{(Euler's constant)};$$

which gives part (i) of Proposition 4. Part (ii) follows from the identity $(L_\infty, *) = (\mathcal{U}^{<\infty>}, *)$; see Jurek [15], Corollary 7. ∎

Because of the special role of ± 1, in Proposition 4, let us consider the following example:

EXAMPLE 1. *Let take $G(dx) := 1/2\delta_{-1}(dx) + 1/2\delta_1(dx)$ (Rademacher distribution) in Proposition 4. Then*

$$V_{\tilde{\nu}}(it) = c - i\pi/2 = c + 1/2 \int_{\mathbb{R}} \frac{1+itx}{it-x} \frac{dx}{1+x^2}, \ t > 0,$$

which is the classical example of Pick function; (Voiculescu representation of a free-infinitely divisible $\tilde{\nu}$); ($\int_{\mathbb{R}} \frac{1+itx}{it-x}\frac{dx}{1+x^2} = -i\pi$).

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