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EXPONENTIAL BOUNDS OF RUIN PROBABILITIES FOR NON-HOMOGENEOUS RISK MODELS

BY

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Abstract. Lundberg-type inequalities for ruin probabilities of non-homogeneous risk models are presented in this paper. By employing martingale method, the upper bounds of ruin probabilities are obtained for the general risk models under weak assumptions. In addition, several risk models, including the newly defined united risk model and quasi-periodic risk model with interest rate, are studied.

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1. INTRODUCTION

Research on the ultimate ruin probability $\psi(u)$, i.e., the probability that the reserve ever drops below zero:

$$\psi(u) := \mathbf{P}[\inf_{t \geqslant 0} R(t) < 0 | R(0) = u],$$

where $R(t)$ is a risk reserve process with initial reserve $R(0) = u > 0$, is attracting increasing attention since classic works of Lundberg [20] and Cramér [9], [10]. After that, a substantial amount of works have been devoted to finding the ruin probabilities and the upper bounds of ruin probabilities, see, e.g., Dickson [11], Gerber [12], and Rolski et al. [22]. However, the most outstanding result about the

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behavior of $\psi(u)$ is the Lundberg inequality, which states that under appropriate assumptions (see [2] and [13] for more details)

$$\psi(u) \leqslant e^{-Lu} \qquad \text{for all} \quad u \geqslant 0. \tag{1.1}$$

The largest number L is unique and called the adjustment coefficient or Lundberg exponent. Further, the following Lundberg-type inequality is also studied

$$\forall u \geqslant 0, \qquad \psi(u) \leqslant Ce^{-Lu} \qquad \text{with} \quad C < \infty \qquad \text{and} \qquad L > 0. \tag{1.2}$$

In the classical works of homogeneous risk models, both inter-occurrence times and claim sizes are assumed to be i.i.d. random variables. However, in reality, both factors are influenced by the economic environment. For instance, inflation and interest rate can affect the evolution of the reserves of the company. Thus the assumptions of homogeneity of inter-occurrence times and claim amounts can be too restrictive for practical use. It is obvious that the non-homogeneous models better reflect the real insurance activities compared to the homogeneous risk models.

Recently, the attention on non-homogeneous risk models has considerably increased. In the non-homogeneous risk models the inter-occurrence times or (and) the claim sizes are independent but not necessarily identically distributed. In Blaževičius, et al. [7], Castañer, et al. [8], Lefèvre and Picard [19], and Vernic [25], they investigated the non-homogeneous risk models with independent, identically distributed inter-occurrence times but not necessarily identically distributed claims. For more details about the non-homogeneous risk models with claim sizes, which are independent and identically distributed random variables, and independent but not identically distributed inter-occurrence times one can refer to Bernackaitė and Šiaulys [6], Ignator and Kaishev [15] and Tuncel and Tank [24].

But we found only a few works in which estimates of the types (1.1) or (1.2) were obtained. In Andrulytė, et al. [1], Kievinaitė and Šiaulys [16], and Kizinevič and Šiaulys [18], the non-homogeneous renewal risk model, where claim sizes and inter-occurrence times are both independent but not necessarily identically distributed, is considered and the Lundberg type inequality of this model is investigated.

This paper considers the non-homogeneous compound Poisson risk model and studies its ruin probability $\psi(u)$. Our purpose is to obtain the exponential upper bound of ultimate ruin probability under general assumptions.

The rest of the paper is organized as follows. In Section 2, we present our main results. First, we study the non-homogeneous compound Poisson risk model without interest rate. By employing martingale method, the Lundberg inequality of this model is obtained. Second, a non-homogeneous risk model with interest rate is studied and the Lundberg inequality of its ruin probability is also obtained. In Section 3, the applications of our main results with a special attention to new possibilities of the behavior of probability $\psi(u)$ are considered. First, the periodic and quasi-periodic risk models with interest rate, which are also automatically non-homogeneous, are investigated. Second, a new non-homogeneous risk model, which is called united risk model, is defined. And the Lundberg inequality of its probability of ruin is also obtained. In the last section, we compare our results with the previous results in Andrulytė, et al. [1], Kievinaitė and Šiaulys [16], and Kizinevič and Šiaulys [18].

2. MAIN RESULTS

2.1. Non-homogeneous risk model without interest rate. We begin to consider a general risk model in which the distribution of each inter-arrival time $\theta_n := T_n - T_{n-1}$ may depend on time T_{n-1}.

Model A: Let $N(t)$ be a non-homogeneous Poisson process with non-decreasing accumulate intensity function $\Lambda(t) := \mathbf{E}N(t) < \infty$ where $N(0) = \Lambda(0) = 0$. Denote by $Z(t) \geqslant 0$ the claim size in case when it arrives at time $t > 0$ and for each j=0,1,2,..., let

$$T_j := \min\{t \geqslant 0 : N(t) \geqslant j\}.$$

Then it is natural to introduce risk reserve process $R(t)$ in the following way

$$R(t) = u + p(t) - \sum_{j=1}^{N(t)} Z(T_j), \qquad \forall t \geqslant 0,$$

where $u = R(0) > 0$ is the initial reserve, $p(t) \geqslant 0$ is the aggregate premium income over $[0, t]$, which is assumed to be non-decreasing and $\{Z(T_j)\}_{j\geqslant 1}$ are the claim sizes which are supposed to be independent but not identically distributed.

It is easy to see that model A is more general than the classical compound Poisson risk model which supposes that $N(t)$ is a homogeneous Poisson process, $\{Z(T_j)\}_{j\geqslant 1}$ are i.i.d. random variables and the premium rate is a constant.

For mathematical purpose, introduce now the claim surplus process

$$S(t) = \sum_{j=1}^{N(t)} Z(T_j) - p(t), \tag{2.1}$$

thus $R(t) = u - S(t)$. Then the ultimate ruin probability can be expressed as

$$\psi(u) = \mathbf{P}(\sup_{t\geqslant 0} S(t) > u).$$

We suppose that random processes $(N(\cdot), Z(\cdot))$ are independent. From Kingman [17] and Paulsen [21] we can see that for any real number $h \geqslant 0$ and any $t > 0$ if $Q(h, t) = \mathbf{E}e^{hZ(t)} - 1$ exists and is continuous in $t \in (0, \infty)$, then for the claim surplus process $S(t)$ defined in (2.1)

$$\mathbf{E}e^{hS(t)} = \exp\Big\{ \int_0^t Q(h,x)d\Lambda(x) - hp(t)\Big\}, \qquad \forall h \geqslant 0,\ t > 0, \tag{2.2}$$

where $\Lambda(t) := \mathbf{E}N(t)$ is the accumulated intensity function of $N(t)$ and $p(t)$ is the aggregate premium income over $[0,t]$. Note that the formula in (2.2) is valid for a larger class of non-homogeneous processes $S(t)$ with independent increments.

Before studying model A, we first state the following basic theorem

THEOREM 2.1. *Let $W(t)$ be a separable process with independent increments, then for any real numbers x and $T, h \geqslant 0$*

$$\mathbf{P}(\sup_{0\leqslant t\leqslant T} W(t) > x) \leqslant e^{-hx} \sup_{0\leqslant t\leqslant T} \mathbf{E}e^{hW(t)}. \tag{2.3}$$

In addition, for all x and $h \geqslant 0$

$$\mathbf{P}(\sup_{t\geqslant 0} W(t) > x) \leqslant e^{-hx} \sup_{t\geqslant 0} \mathbf{E}e^{hW(t)}. \tag{2.4}$$

Before proving Theorem 2.1 we first introduce a key lemma which comes from paper [26].

LEMMA 2.1. *If random variables $Y_1, Y_2, \dots$ are mutually independent, then for any $n \geqslant 1$ and any real number x and $h \geqslant 0$*

$$\mathbf{P}[\sup_{1\leqslant k\leqslant n} W_k > x] \leqslant e^{-hx} \sup_{1\leqslant k\leqslant n} \mathbf{E}e^{hW_k},$$

where $W_k := Y_1 + Y_2 + \ldots + Y_k$. In addition, for any real number x and $h \geqslant 0$

$$\mathbf{P}[\sup_{k\geqslant 1} W_k > x] \leqslant e^{-hx} \sup_{k\geqslant 1} \mathbf{E}e^{hW_k},$$

Proof of Theorem 2.1. Since $W(t)$ is a separable random process, then there exists a sequence $t_1, t_2, \ldots \in [0,T]$ such that

$$\max_{1\leqslant k\leqslant n} W(t_k) \uparrow \sup_{k\geqslant 1} W(t_k) = \sup_{0\leqslant t\leqslant T} W(t).$$

Thus

$$\mathbf{P}(\sup_{0\leqslant t\leqslant T} W(t) > x) = \lim_{n\to\infty} \mathbf{P}(\max_{1\leqslant k\leqslant n} W(t_k) > x).$$

Let $\{t_1, t_2, \ldots, t_n\} = \{t_1^* < t_2^* < \ldots < t_n^*\}$, then

$$\max_{1\leqslant k\leqslant n} W(t_k) = \max_{1\leqslant k\leqslant n} \sum_{j=1}^{k} [W(t_j^*) - W(t_{j-1}^*)],$$

where $t_0^* = 0,\ W(t_0^*) = 0$. Since $\{W(t_j^*) - W(t_{j-1}^*)\}_{j\geqslant 1}$ are independent, then from Lemma 2.1 we have

$$\mathbf{P}(\max_{1\leqslant k\leqslant n} W(t_k) > x) \leqslant e^{-hx} \max_{1\leqslant k\leqslant n} \mathbf{E}e^{hW(t_k)}. \tag{2.5}$$

Let $n \to \infty$ on both sides of (2.5), we have

$$\mathbf{P}(\sup_{0\leqslant t\leqslant T} W(t) > x) \leqslant e^{-hx} \sup_{0\leqslant t\leqslant T} \mathbf{E}e^{hW(t)}.$$

Let $T \to \infty$ in (2.3) then (2.4) is obtained. ∎

Note that Theorem 2.1 allows us to obtain the following results of the ruin probabilities of model A.

THEOREM 2.2. *Suppose the claim surplus process* $S(t)$ *defined in* (2.1) *is a separable process with independent increments. Then for any* $u > 0$ *and* $T, h \geqslant 0$

$$\psi(u,T) = \mathbf{P}(\sup_{0\leqslant t\leqslant T} S(t) > u) \leqslant e^{-hu} \sup_{0\leqslant t\leqslant T} \mathbf{E}e^{hS(t)}.$$

In addition, for any $u > 0$ *and* $h \geqslant 0$

$$\psi(u) = \mathbf{P}(\sup_{t\geqslant 0} S(t) > u) \leqslant e^{-hu} \sup_{t\geqslant 0} \mathbf{E}e^{hS(t)}.$$

Theorem 2.2 and (2.2) immediately imply the following Lundberg inequality of ruin probability of model A.

COROLLARY 2.1. *Under the conditions of Theorem 2.2, if* $S(t)$ *satisfies* (2.2) *then the Lundberg inequality* (1.1) *holds with* $L = \underline{L}$, *where*

$$\underline{L} = \sup\big\{h \geqslant 0 : \sup_{t\geqslant 0} \big[\int_0^t Q(h,x)d\Lambda(x) - hp(t)\big] \leqslant 0\big\}.$$

Homogeneous compound Poisson risk model is a special case of model A, if

$$\mathbf{E}e^{hS(t)} = \exp\big\{t\big[\lambda(\mathbf{E}e^{hZ} - 1) - hp\big]\big\}, \qquad \forall t > 0,\ \forall h \geqslant 0. \tag{2.6}$$

Then from Corollary 2.1 the classical Lundberg inequality (1.1) takes place with

$$L = \sup\big\{h \geqslant 0 : \lambda(\mathbf{E}e^{hZ} - 1) \leqslant hp\big\}. \tag{2.7}$$

REMARK 2.1. From (2.7) it can be seen that the Lundberg inequality obtained by our method is better than the famous Lundberg inequality in its classical form, which can be found in papers [2], [13], [22], because we do not exclude the cases when $\mathbf{E}e^{LS(t)} < 1$ and/or when the expectation $\mathbf{E}S(t) = -\infty$.

2.2. Non-homogeneous risk model with interest rate. As we all know, interest rate is one of the important factors that affect the evolution of the reserves of the company. And underline that all risk models under rates of interest are automatically non-homogeneous. This fact gives us additional motivations in our research of non-homogeneous risk models. Thus in this subsection we consider the non-homogeneous compound Poisson risk model with interest rate.

Here we consider the following model

Model B: Let $N(t)$, $Z(t)$ and $p(t)$ be as in model A, suppose that for any $t_2 > t_1 \geqslant 0$ risk reserve process $R(t)$ satisfies the following property

$$R(t_2) = (1+\alpha(t_1,t_2))R(t_1) + (1+\beta(t_1,t_2))[p(t_2)-p(t_1)] - [q(t_2)-q(t_1)], \tag{2.8}$$

where $(1+\alpha(t_1,t_2))R(t_1)$ is the accumulated value of $R(t_1)$ from t_1 to t_2 under interest rate $\alpha(t_1,t_2)$ and $(1+\beta(t_1,t_2))[p(t_2)-p(t_1)]$ is the accumulated value of premiums collected from t_1 to t_2 under interest rate $\beta(t_1,t_2)$.

Here $q(t)$ denotes the total claims over time interval $[0,t]$. Of course, in the first order we will consider the case when $q(t) = \sum_{j=1}^{N(t)} Z(T_j)$. We suppose that

$$\beta(t_1,t_2) \geqslant 0 \qquad \text{and} \qquad \alpha(t_1,t_2) \geqslant e^{r(t_2)-r(t_1)}, \qquad \forall t_2 > t_1 \geqslant 0, \tag{2.9}$$

where $r(t)$ is non-decreasing and $r(0)=0$. So we have from (2.8) and (2.9) that

$$R(t_2) \geqslant e^{r(t_2)-r(t_1)}R(t_1) - [S(t_2)-S(t_1)],$$

with $S(t) = q(t) - p(t)$. Hence for all $t_2 > t_1 \geqslant 0$

$$e^{-r(t_2)}R(t_2) \geqslant e^{-r(t_1)}R(t_1) - e^{-r(t_2)}[S(t_2)-S(t_1)]. \tag{2.10}$$

From the inequality (2.10) for any $0 = t_0 < t_1 < t_2 < \cdots < t_m = t$ we have

$$\begin{aligned} e^{-r(t)}R(t) - u &= \sum_{k=1}^{m}[e^{-r(t_k)}R(t_k) - e^{-r(t_{k-1})}R(t_{k-1})] \\ &\geqslant -\sum_{k=1}^{m} e^{-r(t_k)}[S(t_k)-S(t_{k-1})]. \end{aligned} \tag{2.11}$$

If $S(t)$ is a function of bounded variation, then standard arguments allow us, as $m \to \infty$, to change the sum in the right hand side of (2.11) by the integral

$$e^{-r(t)}R(t) \geqslant u - \int_0^t e^{-r(x)}dS(x) = u - Y(t), \qquad \forall t > 0. \tag{2.12}$$

In addition, from (2.12) we have

$$\psi(u) = \mathbf{P}(\inf_{t\geqslant 0} R(t) < 0) = \mathbf{P}(\inf_{t\geqslant 0} e^{-r(t)}R(t) < 0) \leqslant \mathbf{P}(\sup_{t\geqslant 0} Y(t) > u). \tag{2.13}$$

It is easy to see that if the integral in (2.12) has sense for some process $S(t)$ with independent increments and non-random $r(t)$, then $Y(t)$ is also a process with independent increments. This fact and (2.12) imply the following assertion.

THEOREM 2.3. *Suppose that* (2.12) *holds where $Y(t)$ is a separable process with independent increments. Then all the assertions of Theorem 2.2 take place with $Y(t)$ instead of $S(t)$.*

COROLLARY 2.2. *Under the assumptions of Theorem 2.3, also assume that*

$$\mathbf{E}e^{hY(t)} = e^{a(h,t)}, \tag{2.14}$$

where

$$a(h,t) := -h\int_0^t e^{-r(x)}dp(x) + \int_0^t Q(he^{-r(x)}, x)d\Lambda(x).$$

Then the Lundberg inequality (1.1) *holds with $L = L^*$, where*

$$L^* = \sup\big\{h \geqslant 0 : \sup_{t\geqslant 0} a(h,t) \leqslant 0\big\}.$$

This assertion is an analogue of Corollary 2.1 for models without rates of interest.

REMARK 2.2. Suppose the representation (2.2) takes place with a non-decreasing function $\Lambda(\cdot)$ and natural continuity assumptions on function $Q(\cdot,\cdot)$ in the domain where it is finite. It is not difficult to understand that if the integral in (2.12) makes sense for some process $S(t)$ with independent increments and non-random function $r(t)$ of bounded variation then $Y(t)$ is also a process with independent increments such that

$$\mathbf{E}e^{hY(t)} = e^{a(h,t)} \qquad \text{with} \quad a(h,t) := -h\int_0^t e^{-r(x)}dp(x) + \int_0^t Q(he^{-r(x)}, x)d\Lambda(x).$$

We can refer to Paulsen [21] for more details. Thus the assumption (2.14) in Corollary 2.2 is reasonable

REMARK 2.3. Note that Theorems 2.2 and 2.3 immediately come from Theorem 2.1.

3. APPLICATIONS OF MAIN RESULTS

In this section, we consider several applications of our general results with a special attention to new possibilities for the behavior of probability $\psi(u)$ in non-homogeneous cases. First, we investigate periodic and quasi-periodic risk models which are also automatically non-homogeneous. Second, we put forward a new non-homogeneous risk model and study its probability of ruin.

3.1. Periodic and quasi-periodic risk models. In this subsection, we study the risk models which take place in the periodic environment. For the classical periodic case, Asmussen and Rolski [3] has shown that the adjustment coefficient in the periodical risk model is the same as for the standard time-homogeneous compound Poisson risk process obtained by averaging the parameters over one period. Here we study the general periodic risk models with interest rates.

In the following, under the conditions of Theorem 2.3, we also assume that $\Lambda(t)$ has density $\Lambda^{'}(t)$ and the aggregate premium income $p(t)$ has density $p^{'}(t)$. Then (2.14) can be rewritten as

$$\mathbf{E}e^{hY(t)} = \exp\Big\{ -\int_0^t he^{-r(x)}p^{'}(x)dx + \int_0^t (\mathbf{E}e^{he^{-r(x)}Z(x)} - 1)\Lambda^{'}(x)dx\Big\}, \qquad \forall t > 0.$$

COROLLARY 3.1. *Suppose there exist real numbers* $l, t_0 > 0$ *and* $\widetilde{L} > 0$ *such that for any* $t \geqslant t_0$

$$\mathbf{E}e^{\widetilde{L}(Y(t+l)-Y(t))} \leqslant 1. \tag{3.1}$$

Then under the conditions of Theorem 2.3, the assertions in Theorem 2.3 hold for each $h \in [0, \widetilde{L}]$ *with*

$$\sup_{t\geqslant 0} \mathbf{E}e^{hY(t)} = \sup_{0\leqslant t<t_0+l} \mathbf{E}e^{hY(t)}. \tag{3.2}$$

In addition, for any $u > 0$

$$\psi(u) \leqslant \inf_{h\in[0,\widetilde{L}]} \Big\{e^{-hu} \sup_{0\leqslant t<t_0+l} \mathbf{E}e^{hY(t)}\Big\} \leqslant C_1 e^{-\widetilde{L}u},$$

where $C_1 := \sup_{0\leqslant t<t_0+l} \mathbf{E}e^{\widetilde{L}Y(t)}$.

P r o o f. For any $t \geqslant t_0 + l$, random variables $\Delta_{n,l} = Y(t) - Y(t-l)$ and $Y(t-l)$ are independent. Hence for each $h \in [0, \widetilde{L}]$,

$$\mathbf{E}e^{h\Delta_{n,l}} \leqslant (\mathbf{E}e^{\widetilde{L}\Delta_{n,l}})^{h/\widetilde{L}} \leqslant 1 \qquad \text{and} \qquad \mathbf{E}e^{hY(t)} = \mathbf{E}e^{h\Delta_{n,l}}\mathbf{E}e^{hY(t-l)} \leqslant \mathbf{E}e^{hY(t-l)}. \tag{3.3}$$

Using induction with respect to t it is not difficult to obtain from (3.3) that for any $t \geqslant t_0 + l$ and $h \in [0, \widetilde{L}]$

$$\mathbf{E}e^{hY(t)} \leqslant \sup_{t<t_0+l} \mathbf{E}e^{hY(t)}.$$

Hence

$$\sup_{0\leqslant t<t_0+l} \mathbf{E}e^{hY(t)} \leqslant \sup_{t\geqslant 0} \mathbf{E}e^{hY(t)} \leqslant \sup_{0\leqslant t<t_0+l} \mathbf{E}e^{hS(t)}, \qquad \forall h \in [0, \widetilde{L}],$$

i.e.,

$$\sup_{t\geqslant 0}\mathbf{E}e^{hS(t)}=\sup_{0\leqslant t<t_0+l}\mathbf{E}e^{hS(t)},\qquad \forall h\in[0,\widetilde{L}].$$

The rest of assertions come from Theorem 2.3 and Corollary 2.2. ∎

THEOREM 3.1. *Suppose there exists a real number $l>0$ such that for each $h\in[0,L(Y(l))]$ and any $t\geqslant 0$*

$$e^{r(t+l)}\big[\mathbf{E}e^{he^{-r(t+l)}Z(t+l)}-1\big]\leqslant e^{r(t)}\big[\mathbf{E}e^{he^{-r(t)}Z(t)}-1\big], \tag{3.4}$$

where

$$L(Y(l)):=\sup\big\{h\geqslant 0:\mathbf{E}e^{hY(l)}\leqslant 1\big\}. \tag{3.5}$$

Assume also that for any $t\geqslant 0$

$$e^{-r(t+l)}\Lambda^{'}(t+l)\leqslant e^{-r(t)}\Lambda^{'}(t)\qquad \textit{and}\qquad e^{-r(t+l)}p^{'}(t+l)\geqslant e^{-r(t)}p^{'}(t). \tag{3.6}$$

Then under the conditions of Theorem 2.3, all the assertions in Theorem 2.3 hold for each $h\in[0,L(Y(l))]$ with

$$\sup_{t\geqslant 0}\mathbf{E}e^{hY(t)}=\sup_{0\leqslant t\leqslant l}\mathbf{E}e^{hY(t)}. \tag{3.7}$$

In addition, for any $u>0$

$$\psi(u)\leqslant\inf_{h\in[0,L(Y(l))]}\{e^{-hu}\sup_{0\leqslant t\leqslant l}\mathbf{E}e^{hY(t)}\}\leqslant C_2e^{-L(Y(l))u},$$

where $C_2:=\sup_{0\leqslant t\leqslant l}\mathbf{E}e^{L(Y(l))Y(t)}$.

P r o o f. For any $t>l$, random variables $\Delta_{n,l}:=Y(t)-Y(t-l)$ and $Y(t-l)$ are independent, then

$$\mathbf{E}e^{hY(t)}=\mathbf{E}e^{hY(t-l)}\mathbf{E}e^{h\Delta_{n,l}}.$$

Let

$$F(t):=\int\limits_{t-l}^{t}G(x)dx,\qquad \text{where}\qquad G(x):=-he^{-r(x)}p^{'}(x)+(\mathbf{E}e^{he^{-r(x)}Z(x)}-1)\Lambda^{'}(x).$$

Then

$$\mathbf{E}e^{h\Delta_{n,l}}=e^{F(t)}.$$

Under the conditions of (3.4) and (3.6) we have

$$\begin{aligned}F^{'}(t) &= G(t) - G(t-l)\\ &= h(e^{-r(t-l)}p^{'}(t-l) - e^{-r(t)}p^{'}(t)) + (\mathbf{E}e^{he^{-r(t)}Z(t)} - 1)\Lambda^{'}(t)\\ &\quad -(\mathbf{E}e^{he^{-r(t-l)}Z(t-l)} - 1)\Lambda^{'}(t-l)\\ &= h(e^{-r(t-l)}p^{'}(t-l) - e^{-r(t)}p^{'}(t)) + e^{-r(t)}\Lambda^{'}(t)\cdot e^{r(t)}\big[\mathbf{E}e^{he^{-r(t)}Z(t)} - 1\big]\\ &\quad -e^{-r(t-l)}\Lambda^{'}(t-l)\cdot e^{r(t-l)}\big[\mathbf{E}e^{he^{-r(t-l)}Z(t-l)} - 1\big]\\ &\leqslant 0.\end{aligned}$$

Thus $F(t)$ is non-increasing such that

$$\mathbf{E}e^{h\Delta_{n,l}} = e^{F(t)} \leqslant e^{F(l)} = \mathbf{E}e^{hY(l)} \leqslant 1, \qquad \forall h \in [0, L(Y(l))].$$

Therefore, we obtain that

$$\mathbf{E}e^{hS(t)} \leqslant \mathbf{E}e^{hS(t-l)}, \qquad \forall h \in [0, L(Y(l))]. \tag{3.8}$$

Using induction with respect to t it is not difficult to obtain from (3.8) that for any $t > l$ and $h \in [0, L(Y(l))]$

$$\mathbf{E}e^{hS(t)} \leqslant \sup_{0\leqslant t\leqslant l} \mathbf{E}e^{hS(t)}.$$

Hence

$$\sup_{0\leqslant t\leqslant l} \mathbf{E}e^{hS(t)} \leqslant \sup_{t\geqslant 0} \mathbf{E}e^{hS(t)} \leqslant \sup_{0\leqslant t\leqslant l} \mathbf{E}e^{hS(t)}, \qquad \forall h \in [0, L(Y(l))]. \tag{3.9}$$

Then the results are proved from (3.9), Theorem 2.3 and Corollary 2.2. ∎

The following corollary is a special case of Theorem 3.1.

COROLLARY 3.2. *Suppose there exists a real number* $l > 0$, *such that for all* $t \geqslant 0$ *claim sizes* $Z(t+l)$ *and* $Z(t)$ *are identically distributed. And also assume that* (3.6) *holds. Then the assertions of Theorem 3.1 still hold.*

Model satisfying conditions of Corollary 3.1 or Theorem 3.1 will be called quasi-periodic. Model satisfying conditions of Corollary 3.2 with

$$\forall t \geqslant 0, \qquad e^{-r(t+l)}\Lambda^{'}(t+l) = e^{-r(t)}\Lambda^{'}(t) \qquad \text{and} \qquad e^{-r(t+l)}p^{'}(t+l) = e^{-r(t)}p^{'}(t),$$

is called periodic (or even purely periodic).

REMARK 3.1. Let $r(t) \equiv 0$ in Corollary 3.1 or Theorem 3.1 then it is not difficult to see that the similar results also hold for periodic risk models without interest rates.

REMARK 3.2. From Theorem 3.1 we find several evident advantages of our results comparing with the results in Asmussen and Rolski [3]. Firstly, in Theorem 3.1 if we let $r(t) = 0$ and, let (3.4) and (3.6) be equalities, then we can obtain the results which were presented in [3]. Secondly, from Remark 2.1 and (3.5) it is easy to see that our estimate is better. Thirdly, we do not use the relatively difficult method by a change of measure which needs technical details. Our proof is based on a martingale approach with less assumptions.

EXAMPLE 3.1. Suppose the period $l = 2$ and the claim sizes $Z(t)$ have exponential distribution $f(x) = e^{-x}$, $x > 0$. We also assume that $p^{'}(t) = 4t$ and $\Lambda^{'}(t) = 1$, then they satisfy all the conditions of Theorem 3.1 with $r(t) = 0$.

We can calculate that for $h < 1$

$$\mathbf{E}e^{hZ(t)} = \int_0^\infty e^{hx}e^{-x}dx = 1/(1-h).$$

Then for any $t > 0$

$$\begin{aligned}\mathbf{E}e^{hS(t)} &= \exp\big\{-h\int_0^t 4xdx + \int_0^t h/(1-h)dx\big\}\\ &= \exp\big\{-2ht^2 + th/(1-h)\big\}.\end{aligned}$$

It is easy to see that

$$L(S(2)) = \sup\big\{h \geqslant 0 : \mathbf{E}e^{hS(2)} \leqslant 1\big\} = 3/4.$$

Then by Theorem 3.1, for any $u > 0$ we have

$$\psi(u) \leqslant \inf_{h\in[0,3/4]}\big\{e^{-hu}\sup_{0\leqslant t\leqslant 2}\mathbf{E}e^{hS(t)}\big\} \leqslant (3/2)e^{-(3/4)u}.$$

3.2. United risk model. Insurance companies can reduce the probability of ruin by investing their assets in risk-free assets or risky assets. In addition, some insurance companies may transfer part of the claims to the reinsurance company to reduce the probability of ruin. More details about the investment and reinsurance can be found in Schmidli [23] and, Bai and Guo [4].

Here we consider an interesting risk model in which a number of claim surplus processes are incorporated such that the ith claim surplus process is added at time $t^{(i)}$, which means that at time $t^{(i)}$ the insurance company will get another premium income with new premium rate and may need to pay new claims. Here we call these claim surplus processes as a number of branches of the insurance company such that branch i begins to work at time $t^{(i)}$.

Let $S^{(0)}(t)$, $S^{(1)}(t), S^{(2)}(t), \ldots$ be a sequence of independent claim surplus processes such that for any $t > 0$

$$S^{(i)}(t) = \sum_{k=1}^{N^{(i)}(t)} Z^{(i)}(T_k^{(i)}) - p^{(i)}t, \qquad i = 0, 1, \ldots, \qquad \text{with} \quad S^{(i)}(0) = 0,$$

where $N^{(i)}(t)$ is the Poisson process with intensity $\lambda^{(i)}$, $p^{(i)}$ is the non-random premium rate and $\{Z^{(i)}(T_k^{(i)}), k = 0, 1, 2, \dots\}$ are i.i.d. claim sizes which are independent of $N^{(i)}(t)$ where

$$T_k^{(i)} := \min\{t \geqslant 0 : N^{(i)}(t) \geqslant k\}, \qquad k = 0, 1, \dots.$$

From (2.6) that for each $i = 0, 1, 2, \dots$ and any $h \geqslant 0$

$$\mathbf{E}e^{hS^{(i)}(t)} = e^{ta^{(i)}(h)} \qquad \text{with} \quad a^{(i)}(h) = \lambda^{(i)}(\mathbf{E}e^{hZ^{(i)}} - 1) - hp^{(i)}. \tag{3.10}$$

And let $0 = t^{(0)} < t^{(1)} < \dots$ be a sequence of non-random times. Define a new claim surplus process

$$S(t) := \sum_{i=0}^{\infty} S^{(i)}((t - t^{(i)})^+). \tag{3.11}$$

We may interpret $S^{(i)}(t)$ as a process describing the *ith* class of business of the insurance company which begins to work at time $t^{(i)}$.

Thus we consider the following surplus process

$$R(t) = u - S(t), \qquad \forall t \geqslant 0, \tag{3.12}$$

with the initial surplus $R(0) = u > 0$ and $S(t)$ defined in (3.11). In this case, we say that the insurer's surplus $R(t)$ defined in (3.12) varies according to the united risk model.

We can see that this model can also be studied by applying model A.

For any $t > 0$ and $h \geqslant 0$ introduce notations

(3.13)
$$a(h,t) = \sum_{i=0}^{\infty} (t - t^{(i)})^+ a^{(i)}(h), \qquad a_k(h) = \sum_{i=1}^{k} (t^{(k)} - t^{(i)}) a^{(i)}(h), \qquad a(h) = \sup_{k \geqslant 0} a_k(h).$$

And also introduce notations

$$\overline{L} := \sup\big\{h \geqslant 0 : a(h) \leqslant 0\big\}, \qquad L^{(i)} := \sup\big\{h \geqslant 0 : a^{(i)}(h) \leqslant 0\big\}. \tag{3.14}$$

COROLLARY 3.3. *Under the above statements, suppose $S(t)$ defined in* (3.11) *is a separable process with independent increments then the Lundberg inequality in* (1.1) *holds with $L := \overline{L}$ where*

$$L^{(0)} \geqslant \overline{L} \geqslant \inf_{i \geqslant 0} L^{(i)}. \tag{3.15}$$

P r o o f. First, it is clear that (3.15) holds. By (3.10), (3.11) and the property of independence, we get that

$$\begin{aligned}\mathbf{E}e^{hS(t)} &= \prod_{i=0}^{\infty}\mathbf{E}e^{hS^{(i)}((t-t^{(i)})^+)}\\ &= \prod_{i=0}^{\infty}e^{(t-t^{(i)})^+a^{(i)}(h)}\\ &= e^{\sum\limits_{i=0}^{\infty}(t-t^{(i)})^+a^{(i)}(h)}\\ &= e^{a(h,t)}.\end{aligned}$$

By (3.13) it is obvious that

$$\sup_{t\geqslant 0}a(h,t)=\sup_{k\geqslant 0}a_k(h),$$

since $a(h,\cdot)$ is a linear piecewise function.

The rest of the proof comes from Theorem 2.2, we omit it here. ■

Next we are going to present an example such that $\overline{L}=L^{(0)}$.

EXAMPLE 3.2. Suppose that $L^{(0)}>0$ and

$$a^{(i)}(L^{(0)})>0,\qquad i=1,3,5,7,\dots,$$

$$a^{(i-1)}(L^{(0)})+a^{(i)}(L^{(0)})\leqslant 0,\qquad i=1,3,5,7,\dots. \tag{3.16}$$

It may be interpreted that claim surplus processes with numbers $1,3,5,7,\dots$ have negative income. But before opening "bad" branch we open "good" branch with property (3.16). In this case $\overline{L}=L^{(0)}$.

REMARK 3.3. If $i=0$ in the united risk model then it degenerates into a homogeneous compound Poisson risk model with

$$\mathbf{E}e^{hS(t)}=exp\big\{t[\lambda^{(0)}(\mathbf{E}e^{hZ^{(0)}}-1)-hp^{(0)}]\big\}.$$

4. COMPARISONS WITH THE EXISTING RESULTS

In this section, we are going to continue studying the result of non-homogeneous renewal risk model from Zhou, Sakhanenko ang Guo [26] and comparing with the results in Andrulytė et al. [1], Kievinaitė and Šiaulys [16], and Kizinevič and Šiaulys [18].

Consider a class of risk processes $R(t)$ with the following properties:

(1) Process $R(t)$ may have positive jumps only at random or non-random times $T_1, T_2, \ldots$ such that

$$\forall k = 1, 2, \ldots, \qquad T_{k+1} > T_k > T_0 := 0 \qquad \text{and} \qquad T_n \to \infty \; a.s.$$

(2) Process $R(t)$ is monotone on each interval $[T_{k-1}, T_k),\ k = 1, 2, \ldots,$ and $R(0) = u > 0$.

Model R: Assume the k-th claim Z_k occurs at time T_k, i.e.,

$$-Z_k := R(T_k) - R(T_k - 0) \leqslant 0, \qquad \theta_k := T_k - T_{k-1} > 0, \qquad k = 1, 2, \ldots.$$

Suppose that on each interval $[T_{k-1}, T_k)$ the premium rate is p_k, i.e.,

$$\forall t \in [T_{k-1}, T_k), \qquad R(t) - R(T_{k-1}) = p_k(t - T_{k-1}), \qquad k = 1, 2, \ldots.$$

Assume also that random vectors

$$(p_k, Z_k, \theta_k), \qquad k = 1, 2, \ldots$$

are mutually independent.

Then conditions (1) and (2) hold and the random variables

$$Y_k = R(T_{k-1}) - R(T_k) = Z_k - p_k\theta_k = Z_k - X_k, \qquad k = 1, 2, \ldots \tag{4.1}$$

are also mutually independent in which $X_k := p_k\theta_k$ is the nonnegative accumulated premium over a period of time θ_k. Model R is also called non-homogeneous renewal risk model which is different from the model A introduced in section 2.1. The ruin probability of model R is that for any $u > 0$

$$\psi(u) = \mathbf{P}(\sup_{k\geqslant 1} S_k > u),$$

where $S_k = Y_1 + Y_2 + \ldots + Y_k$. Theorem 1 of [26] states that for any $u > 0$ and $h \geqslant 0$

$$\psi(u) \leqslant e^{-hu} \sup_{n\geqslant 1} \mathbf{E}e^{hS_n}. \tag{4.2}$$

Then, we are going to estimate $\mathbf{E}e^{hS_n}$ in (4.2) to obtain a further explicit upper bound of ruin probability of non-homogeneous renewal risk model and compare our estimation with the results in Andrulytė et al. [1], Kievinaitė and Šiaulys [16], and Kizinevič and Šiaulys [18].

The following simple assertion will be useful below.

LEMMA 4.1. *Let random variables $Y_1, Y_2, \ldots$ have finite expectations. Then for any $h \geqslant 0$ and $m \geqslant 1$*

$$M_m(h) := \max_{1\leqslant k\leqslant m} \mathbf{E}e^{hS_k} \leqslant e^{hC_m}\mathbf{E}e^{hS_m}, \tag{4.3}$$

where

$$0 \leqslant C_m := \max_{1\leqslant k\leqslant m} \mathbf{E}S_k - \mathbf{E}S_m < \infty. \tag{4.4}$$

P r o o f. For any $1 \leqslant k \leqslant m$ and $h \geqslant 0$, by the Jensen's inequality in probability space we have

(4.5)
$$\mathbf{E}e^{hS_n} = \mathbf{E}e^{hS_k} \cdot \mathbf{E}e^{h(S_m - S_k)} \geqslant \mathbf{E}e^{hS_k} \cdot e^{h(\mathbf{E}S_m - \mathbf{E}S_k)} \geqslant \mathbf{E}e^{hS_k} \cdot e^{-hC_m}.$$

From (4.5) we immediately have

$$\mathbf{E}e^{hS_k} \leqslant e^{hC_m} \mathbf{E}e^{hS_m}. \tag{4.6}$$

By taking maximum on both sides of (4.6) over $1 \leqslant k \leqslant m$ the result is obtained. ■

The following inequality is evident and useful for us: for any $h \geqslant 0$ and $1 \leqslant m \leqslant n$

$$M(h) := \sup_{n \geqslant 1} \mathbf{E}e^{hS_n} \leqslant M_m(h) \cdot \big(\sup_{n > m} \mathbf{E}e^{h(S_n - S_m)} \vee 1 \big). \tag{4.7}$$

In fact, for any $h \geqslant 0$ and $1 \leqslant m \leqslant n$

$$\begin{aligned} M(h): &= \sup_{n \geqslant 1} \mathbf{E}e^{hS_n} \\ &= (\sup_{n \leqslant m} \mathbf{E}e^{hS_n}) \vee (\sup_{n > m} \mathbf{E}e^{hS_n}) \\ &= M_m(h) \vee \{\sup_{n > m} \mathbf{E}e^{h(S_n - S_m)} \cdot e^{hS_m}\} \\ &= M_m(h) \vee \{\mathbf{E}e^{hS_m} \cdot \sup_{n > m} \mathbf{E}e^{h(S_n - S_m)}\} \\ &\leqslant M_m(h) \cdot \{1 \vee \sup_{n > m} \mathbf{E}e^{h(S_n - S_m)}\}, \end{aligned}$$

since $\mathbf{E}e^{hS_m} \leqslant M_m(h)$.

The estimations of $M(h)$ and $M_n(h)$ are the central technical problems in applications of (4.2). Many ideas are known in this direction. The most famous ones are connected with papers of Bennett [5] and Hoeffding [14]. In this section, we are going to recall and use the ideas from [5] and [14]. First, by using the important limits of mathematical analysis such that $\lim_{n\to\infty}(1 + 1/n)^n = e$ we have

$$\mathbf{E}e^{hS_n} = \prod_{k=1}^{n} \mathbf{E}e^{hY_k} \leqslant \Big(\frac{1}{n} \sum_{k=1}^{n} \mathbf{E}e^{hY_k}\Big)^n \leqslant \exp\Big\{ \sum_{k=1}^{n} (\mathbf{E}e^{hY_k} - 1)\Big\}.$$

Next, introduce non-negative function $g(x) = e^x - 1 - x \geqslant 0$ and note that

$$\mathbf{E}e^{hY_k} = 1 + h\mathbf{E}Y_k + \mathbf{E}g(hY_k).$$

From [5] and [14] we see that function $g(x)$ has many useful properties. We are going to use them here. As we all know for each $k = 1, 2, \dots$

$$Y_k = Z_k - X_k \qquad \text{with} \quad X_k, Z_k \geqslant 0,$$

then

$$\begin{aligned} \mathbf{E}g(hY_k) &= \mathbf{E}[g(hY_k) : Y_k \geqslant 0] + \mathbf{E}[g(hY_k) : Y_k < 0] \\ &= \mathbf{E}[g(hY_k^+)] + \mathbf{E}[g(-hY_k^-)] \leqslant \mathbf{E}g(hZ_k) + \mathbf{E}g(-hX_k). \end{aligned}$$

It is easy to see that for any $h, C \geqslant 0$

$$\mathbf{E}g(-hX_k) \leqslant \frac{h^2}{2}\mathbf{E}\big[X_k^2 : 0 < X_k \leqslant C\big] + h\mathbf{E}\big[X_k : X_k > C\big].$$

But the most remarkable for our purpose is the following property: if $H > 0$ by using the error term of Taylor expansion of e^x at $x = 0$ such that $\delta(x) = e^x - 1 - x = x^2 \int_0^1 (1-t)e^{tx}dt$ and the definition of $g(x)$ with $x = hZ_k$ we have

$$\begin{aligned} \forall h \in (0, H], \qquad h^{-2}\mathbf{E}g(hZ_k) &= \mathbf{E}\int_0^1 Z_k^2(1-t)e^{thZ_k}dt \\ &\leqslant \mathbf{E}\int_0^1 Z_k^2(1-t)e^{tHZ_k}dt = H^{-2}\mathbf{E}g(HZ_k). \end{aligned} \tag{4.8}$$

For $H, C > 0$ introduce notations

$$A_n(C) = \sum_{k=1}^n \mathbf{E}\big[X_k : X_k > C\big], \qquad B_n(H, C) = \frac{1}{H^2}\sum_{k=1}^n \mathbf{E}g(HZ_k) + \frac{1}{2}\sum_{k=1}^n \mathbf{E}\big[X_k^2 : X_k \leqslant C\big]. \tag{4.9}$$

From the above statements we can see that

$$\mathbf{E}e^{hS_n} \leqslant \exp\Big\{\sum_{k=1}^n \big[h\mathbf{E}Y_k + \mathbf{E}g(hY_k)\big]\Big\} \leqslant \exp\big\{h\mathbf{E}S_n + hA_n(C) + h^2B_n(h, C)\big\}. \tag{4.10}$$

Then from (4.10) we have the following assertion.

COROLLARY 4.1. *Suppose that for some integer* $m > 0$ *and real numbers* $h, C > 0$,

$$A_n(C) + hB_n(h, C) \leqslant -\mathbf{E}S_n, \qquad \forall n \geqslant m. \tag{4.11}$$

Then

$$M(h) \leqslant e^{hC_m}, \tag{4.12}$$

where C_m *is defined in* (4.4)*. In particular, for any* $u > 0$

$$\psi(u) \leqslant e^{h(C_m - u)}. \tag{4.13}$$

P r o o f. From (4.10) and (4.11), it is easy to see that

$$\mathbf{E}e^{hS_n} \leqslant \exp\{h\mathbf{E}S_n + h(-\mathbf{E}S_n)\} = 1, \qquad \forall n \geqslant m. \tag{4.14}$$

Then by inequalities of (4.3), (4.7) and (4.14) we obtain that

$$\begin{aligned} M(h) &\leqslant M_m(h)(\sup_{n>m} \mathbf{E}e^{h(S_n-S_m)} \vee 1) \\ &\leqslant e^{hC_m}\mathbf{E}e^{hS_m}(\sup_{n>m} \mathbf{E}e^{h(S_n-S_m)} \vee 1) \\ &= e^{hC_m}(\mathbf{E}e^{hS_m} \vee \sup_{n>m} \mathbf{E}e^{hS_n}) \\ &= e^{hC_m} \cdot \sup_{n\geqslant m} \mathbf{E}e^{hS_n} \\ &\leqslant e^{hC_m}. \end{aligned}$$

(4.13) is directly obtained from (4.2) and (4.12). ∎

If (4.11) does not hold, the following fact will be useful.

COROLLARY 4.2. *Suppose that for some integer $m > 0$*

$$A_n(C) \leqslant -c^*\mathbf{E}S_n \qquad \textit{with} \quad c^* \in (0,1), \qquad \forall n \geqslant m, \tag{4.15}$$

and

$$B_n(H,C) \leqslant C^*(-\mathbf{E}S_n) \qquad \textit{with} \quad 0 < H, C^* < \infty, \qquad \forall n \geqslant m. \tag{4.16}$$

Then all assertions of Corollary 4.1 hold with

$$0 < h \leqslant \min\{H, (1-c^*)/C^*\}. \tag{4.17}$$

P r o o f. It follows from (4.8) and (4.9) that $B_n(h,C) \leqslant B_n(H,C)$ for $h \in (0,H]$. From this fact and (4.16), for any $h \in (0,H]$ we have

$$hB_n(h,C) \leqslant hB_n(H,C) \leqslant hC^*(-\mathbf{E}S_n). \tag{4.18}$$

Thus, inequality (4.18), with h in (4.17), yields that $hB_n(h,C) \leqslant (1-c^*)(-\mathbf{E}S_n)$ and, hence, (4.11) is satisfied in view of (4.15). ∎

COROLLARY 4.3. *Suppose that*

$$\limsup_{n\to\infty} \mathbf{E}S_n < 0, \qquad \lim_{C\to\infty} \limsup_{n\to\infty} \frac{A_n(C)}{-\mathbf{E}S_n} < 1, \tag{4.19}$$

and that for some $H > 0$ and any $C > 0$

$$\limsup_{n\to\infty} \frac{B_n(H,C)}{-\mathbf{E}S_n} < \infty. \tag{4.20}$$

Then there exists $h > 0$ for which (4.12) *and* (4.13) *hold.*

P r o o f. It follows from (4.19) that there exists an integer $m > 0$ such that $\sup_{n\geqslant m} \mathbf{E}S_n < 0$ and that (4.15) holds with some $c^* \in (0,1)$. After that we have from (4.20) that (4.16) is evidently true with some $C^* < \infty$. So, Corollary 4.3 follows from Corollary 4.2. ∎

REMARK 4.1. It is clear that

$$\mathbf{E}g(HZ_k) \leqslant \mathbf{E}e^{HZ_k} - 1 \leqslant \mathbf{E}e^{HZ_k},$$

since $H > 0$ and Z_k is nonnegative random variable. And for all $C > 0$

$$\mathbf{E}[X_k^2 : X_k \leqslant C] \leqslant C^2\mathbf{P}[X_k > 0] \leqslant C^2. \tag{4.21}$$

It is not difficult to verify that from these facts and Corollaries 4.2 and 4.3 all assertions in [1], [16], [18] can be obtained.

Moreover, in this way we may obtain better constants than those in [1], [16], [18], since in these three papers the authors use

$$\mathbf{E}g(HZ_k/2) \leqslant \frac{1}{2}(\frac{H}{2})^2\mathbf{E}\big[Z_k^2 e^{HZ_k/2}\big] \leqslant 2e^{-2}\mathbf{E}e^{HZ_k}$$

instead of (4.8).

Note that we may always put $Z_k = Y_k^+$ and $X_k = Y_k^-$. In this case (4.21) is not trivial with $\mathbf{P}[X_k > 0] = \mathbf{P}[Y_k < 0]$.

EXAMPLE 4.1. Let independent random variables $Y_1, Y_2, \ldots$ have normal distributions such that

$$B_n := \mathbf{Var}S_n \to \infty,$$
$$\forall k = 1, 2, \ldots, \qquad \mathbf{E}Y_k := -(27/64)(B_k + B_{k-1})\mathbf{Var}Y_k.$$

(We may, for simplicity, take $B_k := k$.) In this case $\mathbf{E}S_n = -(27/64)B_n^2$ and

$$\mathbf{E}e^{hS_n} = e^{h\mathbf{E}S_n + h^2 \operatorname{var} S_n/2} = e^{-(27/64)hB_n^2 + h^2B_n/2} = e^{f(B_n,h)}.$$

It is easy to see that

$$\sup_{n\geqslant 1} \mathbf{E}e^{hS_n} = \sup_{n\geqslant 1} e^{f(B_n,h)} \leqslant \sup_{x\geqslant 0} e^{f(x,h)} = e^{f(16h/27,h)} = e^{4h^3/27}.$$

Hence

$$e^{-hu}\sup_{n\geqslant 1} \mathbf{E}e^{hS_n} \leqslant e^{-hu+4h^3/27} = e^{g(h,u)}.$$

Then for any $u > 0$

$$\psi(u) \leqslant \inf_{h\geqslant 0} e^{-hu}\sup_{n\geqslant 1} \mathbf{E}e^{hS_n} \leqslant \inf_{h\geqslant 0} e^{g(h,u)} = e^{g((3/2)\sqrt{u},u)} = e^{-u^{3/2}}.$$

Thus, it is possible in non-homogeneous case that we use $h = h(u) \to \infty$ as $u \to \infty$ to obtain better bound than $\psi(u) = e^{-O(u)}$. Such situation is impossible in the case of homogeneity.

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