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ON A CONJECTURE ABOUT THE COMPARABILITY OF PARALLEL SYSTEMS WITH RESPECT TO THE CONVEX TRANSFORM ORDER*

BY

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Abstract. We study the comparability of the lifetimes of heterogeneous parallel systems with independent exponentially distributed components. It is known that the order statistics of systems composed of two types of components may be comparable with respect to the star transform order. On what concerns the stronger convex transform order results have been obtained only for the sample maxima assuming that one of the systems is homogeneous. We prove, under the same assumptions as for the star transform ordering, that the lifetime of heterogeneous parallel systems are not comparable with respect to the convex transform order.

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1. INTRODUCTION

Deciding about the ageing properties of systems whose lifetime is random requires an appropriate meaning of the comparison criterion. The literature is abundant in alternative definitions of ageing properties and the corresponding orderings. Generally speaking, the approach starts by the definition of a way to measure a relevant risk, that will be considered as the failure rate. The characterization of the monotonicity of the failure rate function for a lifetime distribution is an important aspect and has been studied, among others, by Barlow and Proschan [4], Patel [16], Sengupta [17] or El-Bassiouny [7].

Once we define a risk measure, we may become interested in ordering lifetime distributions with respect to the considered risk. This may be viewed as deciding

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which distribution is ageing faster. Order relations, mostly falling in some family of transform orders, have been studied extensively by various authors (see for example, Desphande et al. [6], Kochar and Wiens [10], Singh [19], Fagiuoli and Pellerey [8] or Shaked and Shanthikumar [18]). We refer the reader to section 4.B in Shaked and Shanthikumar [18] for general definitions and properties of transform order relations. It is interesting to mention that the convex transform order that we will be considering later, has a geometric interpretation, as described by van Zwet [21], providing a way to compare the skeweness of lifetime distributions.

The order properties of the lifetime distributions of parallel systems has been thoroughly studied in the literature, with some recent results being found in Zhang et al. [23], Kayal [9] or Wu et al. [22], where the role of different lifetime distribution of the components is studied, in Cai et al. [5] or Li and Li [13] dealing with the effect of dependence of components lifetimes. The present note responds to a question raised in Kochar and Xu [11]. These authors were interested in comparing the ageing performance of parallel systems with respect to the convex transform order when the components have exponential and independent lifetime distributions. In their Theorem 3.1, Kochar and Xu [11] proved that, for systems with the same number of components, a parallel system with homogeneous components ages faster when compared to a parallel system with heterogeneous components. In Remark 3.2 in Kochar and Xu [11], it is conjectured that the same ageing behaviour holds when comparing two heterogeneous systems based on components that have exponentially distributed lifetimes with hazard rates that can be ordered in a suitable way (see Definition 3.1 below for details). Their conjecture is based on the intuitive extension of their Theorem 3.1 and on numerical evidence they collected. Although Kochar and Xu [12] proved that the ordering relationship holds with respect to the star transform order under an alternative formulation of the assumptions on the parameters, the corresponding result for the convex transform order remained open.

In this paper we give a simpler proof of the star ordering proved in [12], based on quite different arguments. Our approach to the proof of the star transform ordering provides a method allowing to show that the Kochar and Xu conjecture referred to the convex transform order is not valid. Furthermore, it is important to highlight that the contribution of this work isn't only the disproval of the conjecture (this could have been proven via a counterexample) but the fact that our proof gives a general method to obtain regions where the required relationship defining the convex order fails. These regions seem to be rather narrow, so some insight about their location is significant to decide about the ordering.

2. PRELIMINARIES

Let X be a nonnegative random variable with density function f_X, distribution function F_X, and tail function $\overline{F}_X = 1 - F_X$. Moreover, for each $x \geqslant 0$ the failure

rate function of X is given by $r_X = \frac{f_X}{\overline{F}_X}$. Two of the most simple and common ageing notions are defined in terms of the failure rate function.

DEFINITION 2.1. Let X be a nonnegative valued random variable.

1. X is said IFR (resp., DFR) if r_X is increasing (resp., decreasing) for $x \geqslant 0$.

2. X is said IFRA (resp., DFRA) if $\frac{1}{x}\int_0^x r_{X,s}(t)\,dt$ is increasing (resp., decreasing) for $x > 0$.

The above definitions refer to monotonicity properties of the distribution. In the following, we introduce criteria to compare distribution functions.

DEFINITION 2.2. Let $\mathcal{F}$ denote the family of distribution functions such that $F(0) = 0$, and X and Y be nonnegative random variables with distribution functions $F_X, F_Y \in \mathcal{F}$.

1. The random variable X (or its distribution F_X) is said to be smaller than Y (or its distribution F_Y) in convex transform order, and we write $X \leqslant_{\mathrm{c}} Y$, or equivalently, $F_X \leqslant_{\mathrm{c}} F_Y$, if $\overline{F}_Y^{-1}(\overline{F}_X(x))$ is convex.

2. The random variable X (or its distribution F_X) is said to be smaller than Y (or its distribution F_Y) in star transform order, and we write $X \leqslant_* Y$, or equivalently, $F_X \leqslant_* F_Y$, if $\frac{\overline{F}_Y^{-1}(\overline{F}_X(x))}{x}$ is increasing (this is also known as $\overline{F}_Y^{-1}(\overline{F}_X(x))$ being star-shaped).

The definitions above fall in the family of iterated IFR and IFRA orders, respectively, introduced and initially studied in Nanda et al. [15], Arab and Oliveira [1], [2] or Arab et al. [3], considering their iteration parameter to be 1. It is particularly useful to highlight at this point that the IFR and IFRA orders are scale invariant (see Nanda et al. [15]). Consequently, in case of families of distributions that have scale parameters, this allows to choose them in the most convenient way.

A general characterization of the transform order relations is given below.

THEOREM 2.1 (Propositions 3.1 and 4.1 in Nanda et al. [15]). *Let X and Y be random variables with distribution functions $F_X, F_Y \in \mathcal{F}$.*

1. *$X \leqslant_* Y$ if and only if for any real number a, $\overline{F}_Y(x) - \overline{F}_X(ax)$ changes sign at most once, and if the change of signs occurs, it is in the order "$-, +$", as x traverses from 0 to $+\infty$.*

2. *$X \leqslant_{\mathrm{c}} Y$ if and only if for any real numbers a and b, $\overline{F}_Y(x) - \overline{F}_X(ax+b)$ changes sign at most twice, and if the change of signs occurs twice, it is in the order "$+, -, +$", as x traverses from 0 to $+\infty$.*

REMARK 2.1. *As mentioned in Remark 25 in Arab and Oliveira [1], it is enough to verify the above characterizations for $a > 0$. Moreover, when describing a sign variation of any given function, we will always be considering that x goes from 0 to $+\infty$.*

The characterization given by Theorem 2.1 requires explicit expressions of the

tails of the distributions, which are often not available. Computationally tractable characterizations were studied in Arab and Oliveira [1], [2] and Arab et al. [3] (see Theorems 2.3 and 2.4 in the latter reference). As one may verify in the proofs given in [1] the control of the sign variation is usually more complex when considering the case $b < 0$. However, a prior verification of the star transform ordering will help circumvent this difficulty.

THEOREM 2.2 (Theorem 29 in Arab et al. [3]). *Let X and Y be random variables with distribution functions $F_X, F_Y \in \mathcal{F}$, respectively. If $X \leqslant_* Y$ and the criterion in 2. from Theorem 2.1 is verified for $b \geqslant 0$, then $X \leqslant_{\mathrm{c}} Y$.*

The lifetime of parallel systems is expressed as the maximum of the lifetimes of each component. When these are exponentially distributed, the distribution function of the system lifetime is expressed as a linear combination of exponential terms. Later, we will be interested in counting and localizing the roots of such expressions. The following result will play an important role on this aspect.

THEOREM 2.3 (Tossavainen [20]). *Let $n \geqslant 0$, $p_0 > p_1 > \cdots > p_n > 0$, and $\alpha_j \neq 0$, $j = 0, 1, \ldots, n$, be real numbers. Then the function $f(t) = \sum_{j=0}^{n} \alpha_j p_j^t$ has no real zeros if $n = 0$, and for $n \geqslant 1$ has at most as many real zeros as there are sign changes in the sequence of coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$.*

3. MAIN RESULTS

We begin this section by quoting a result by Kochar and Xu [11] that suggested the conjecture we will be discussing in the sequel and therefore it is presented here for the sake of completeness.

THEOREM 3.1 (Theorem 3.1 in Kochar and Xu [11]). *Let $X_1, \ldots, X_n$ be independent and exponentially distributed random variables with common hazard rate λ. Similarly, let $Y_1, \ldots, Y_n$ be independent and exponentially distributed random variables with hazard rates θ_i, $i = 1, \ldots, n$. Then $\max(X_1, \ldots, X_n) \leqslant_{\mathrm{c}} \max(Y_1, \ldots, Y_n)$.*

For the remainder of this section, we will be interested in characterizing the order relationship between parallel systems of heterogeneous components with exponential lifetime distributions. We recall an order relation between $\mathbb{R}^n$ vectors introduced in Definition A.1 in Marshall and Olkin [14].

DEFINITION 3.1. Let $(\lambda_1, \ldots, \lambda_n)$ and $(\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$. Denote by $\lambda_{(1)} \leqslant \cdots \leqslant \lambda_{(n)}$ the ordered coordinates of the first vector, and likewise for the coordinates of the second vector. We say that $(\lambda_1, \ldots, \lambda_n) \prec (\theta_1, \ldots, \theta_n)$ if

$$\sum_{i=1}^{k} \lambda_{(i)} \geqslant \sum_{i=1}^{k} \theta_{(i)} \quad \text{for} \quad k = 1, \ldots, n-1, \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_{(i)} = \sum_{i=1}^{n} \theta_{(i)}.$$

REMARK 3.1. *Without loss of generality, we can assume the components of the hazard rate vectors* $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ *to be arranged increasingly, therefore in the sequel, we will be assuming our hazard rate vectors to be like so, that is* $\lambda_k \leqslant \lambda_m$ *when* $k \leqslant m$.

Based on their proof and some numerical evidence, Kochar and Xu [11] conjectured that the conclusion of Theorem 3.1 would still hold if the X_i have hazard rate λ_i and the Y_i have hazard θ_i such that $(\lambda_1, \ldots, \lambda_n) \prec (\theta_1, \ldots, \theta_n)$.

3.1. The star transform ordering. Kochar and Xu [12] proved a stronger ordering result but with respect to the weaker star transform order.

THEOREM 3.2 (Theorem 3.1 in Kochar and Xu [12]). *Let* $X_1, \ldots, X_n$, *be independent and exponentially distributed where* $X_1, \ldots, X_p$ *have hazard rate* λ_1 *and* $X_{p+1}, \ldots, X_n$ *have hazard rate* λ_2, *and* $Y_1, \ldots, Y_n$ *satisfy similar conditions with hazard rates* θ_1 *and* θ_2. *If* $\frac{\lambda_2}{\lambda_1} \geqslant \frac{\theta_2}{\theta_1}$, *then the* k^{th} *order statistics are ordered with respect to the star transform order, that is,* $X_{k:n} \leqslant_* Y_{k:n}$.

REMARK 3.2. *The star transform order is insensitive to multiplication of the random variables by constants. Taking this into account, it is easily seen that an equivalent formulation of Theorem 3.2 is obtained by assuming that the hazard rates satisfy* $(\lambda_1, \lambda_2) \prec (\theta_1, \theta_2)$.

In Subsection 3.2, we will compare the lifetimes of parallel systems with respect to the convex transform order. Our approach to the comparison is based on the characterization of the sign variation of suitable functions, as expressed by Theorem 2.1. Therefore, taking into account Theorem 2.2, the following reduced version of Theorem 3.2 provides a useful first step for the proof of Theorem 3.4, and provides some calculatory details.

THEOREM 3.3. *Let* X_1 *and* X_2 *be independent and exponentially distributed with hazard rates* λ_1 *and* λ_2, *respectively. Analogously, let* Y_1 *and* Y_2 *be independent and exponentially distributed with hazard rates* θ_1 *and* θ_2, *respectively. If* $(\lambda_1, \lambda_2) \prec (\theta_1, \theta_2)$, *then* $X = \max(X_1, X_2) \leqslant_* Y = \max(Y_1, Y_2)$.

Proof. If $\lambda_1 = \theta_1$, Definition 3.1 implies that $(\lambda_1, \lambda_2) = (\theta_1, \theta_2)$, X and Y are equivalent. Suppose now that $\lambda_1 > \theta_1$. Let $\overline{F}_X$ and $\overline{F}_Y$ be the survival functions of X and Y, respectively. Then we have

$$\overline{F}_X(x) = e^{-\lambda_1 x} + e^{-\lambda_2 x} - e^{-(\lambda_1+\lambda_2)x}$$

and

$$\overline{F}_Y(x) = e^{-\theta_1 x} + e^{-\theta_2 x} - e^{-(\theta_1+\theta_2)x}.$$

Taking into account Theorem 2.1 and Remark 2.1, it is sufficient to prove that $\overline{V}(x) = \overline{F}_Y(x) - \overline{F}_X(ax)$ changes sign at most once, and if the sign change occurs, it is in the order "$-, +$", for every real number $a > 0$. We will consider three

separate cases, depending on the value of a. First, note that the assumptions on the hazard rates imply that $\theta_1 < \lambda_1 < \lambda_2 < \theta_2$ and $\lambda_1 + \lambda_2 = \theta_1 + \theta_2$, hence, none of the two systems has homogeneous components.

Case 1. $a = 1$: We have $\overline{V}(x) = e^{-\theta_1 x} + e^{-\theta_2 x} - (e^{-\lambda_1 x} + e^{-\lambda_2 x})$. Reordering the exponential terms so that they appear in decreasing order of their basis, the sign pattern of the coefficients is "$+, -, -, +$". Hence, according to Theorem 2.3, $\overline{V}$ has at most two real roots. Moreover, $\lim_{x\to-\infty} \overline{V}(x) = +\infty$, while $\lim_{x\to+\infty} \overline{V}(x) = 0^+$. Furthemore, taking into account that $\overline{V}(0) = 0$, $\overline{V}'(0) = 0$ and $\overline{V}''(0) = -\lambda_1^2 + \theta_1^2 - \lambda_2^2 + \theta_2^2 = -(\theta_2 - \lambda_2)(\lambda_1 - \lambda_2 + \theta_1 - \theta_2) > 0$, it follows that $\overline{V}(x) \geqslant 0$, which means that $\overline{F}_Y(x) \geqslant \overline{F}_X(x)$, for every $x \in \mathbb{R}$, thus no sign changes occur.

Case 2. $a > 1$: As $\overline{F}_X$ is decreasing, it follows that, for $x \geqslant 0$, $\overline{V}(x) \geqslant \overline{F}_Y(x) - \overline{F}_X(x) \geqslant 0$ so, again, no sign changes occur.

Case 3. $0 < a < 1$: To analyse the sign pattern of the coefficients in $\overline{V}$, we distinguish two cases:

$\theta_1 < a\lambda_1$: After reordering the exponentials in $\overline{V}$ according the their basis, the sign pattern of the coefficients is "$+, -, -, +, +, -$". Thus, according to Theorem 2.3, $\overline{V}$ has at most three real roots. The sign pattern of the coefficients implies that $\lim_{x\to-\infty} \overline{V}(x) = -\infty$, while $\lim_{x\to+\infty} \overline{V}(x) = 0^+$. Finally, taking into account that $\overline{V}(0) = \overline{V}'(0) = 0$, the possible sign changes for $\overline{V}$ are either "$+$" or "$-, +$".

$\theta_1 \geqslant a\lambda_1$: As $\overline{F}_X$ is decreasing and $a \leqslant \frac{\theta_1}{\lambda_1}$, it follows that, for $x \geqslant 0$,

$$\begin{aligned} \overline{V}(x) &\leqslant \overline{H}(x) = \overline{F}_Y(x) - \overline{F}_X(\tfrac{\theta_1}{\lambda_1}x) \\ &= e^{-\theta_2 x} - e^{-(\theta_1+\theta_2)x} - \left(e^{-\frac{\theta_1\lambda_2}{\lambda_1}x} - e^{-\frac{\theta_1}{\lambda_1}(\lambda_1+\lambda_2)x}\right). \end{aligned}$$

After reordering the exponentials in $\overline{H}$ according to their basis, the sign pattern of the coefficients in $\overline{H}$ is "$-, +, +, -$", implying that, according to Theorem 2.3, $\overline{H}$ has at most two real roots. The sign of the coefficients of $\overline{H}$ also implies that $\lim_{x\to+\infty} \overline{H}(x) = 0^-$ which, together with the fact that $\overline{H}(0) = \overline{H}'(0) = 0$ and $\overline{H}''(0) = 2\theta_1\left(\frac{\theta_1}{\lambda_1}\lambda_2 - \theta_2\right) < 0$, further imply that $\overline{H}(x) \leqslant 0$, so consequently $\overline{V}(x) \leqslant 0$, that is, no sign changes occur.

So, finally, $\overline{V}$ has at most one sign change when x goes from 0 to $+\infty$ and, if the change occurs, it is in the order "$-, +$". Thus, according to Theorem 2.1, $X \leqslant_* Y$. ■

Recalling Remark 3.2, the following alternative equivalent formulation is immediate.

COROLLARY 3.1. *Let X_1 and X_2 be independent and exponentially distributed with hazard rates λ_1 and λ_2, respectively. Analogously, let Y_1 and Y_2 be independent and exponentially distributed with hazard rates θ_1 and θ_2, respectively. If $\frac{\lambda_2}{\lambda_1} \geqslant \frac{\theta_2}{\theta_1}$, then $X = \max(X_1, X_2) \leqslant_* Y = \max(Y_1, Y_2)$.*

REMARK 3.3. *Note that, although Theorem 3.2 allows for an arbitrary number of components, each system is only allowed to have two different types of components. On the other hand, Theorem 3.3 is obtained for lifetimes which may all have different hazard rates, but its proof is limited to systems with only two components. It remains an open problem proving the star transform order between systems with more than three types of components.*

3.2. The convex transform ordering. We may now give a partial answer to the conjecture announced in Kochar and Xu [11], about the comparability of the lifetime distribution of parallel systems with respect to the convex transform order.

THEOREM 3.4. *Let X and Y be as in Theorem 3.3. Then X and Y are not comparable with respect to the convex transform order.*

Proof. As before if $\lambda_1 = \theta_1$, it follows that X and Y are equivalent. Suppose now that $\lambda_1 > \theta_1$. We start by noticing that the sign variation analysis of $\overline{V}(x) = \overline{F}_Y(x) - \overline{F}_X(ax + b)$ is inconclusive. Note that, unlike when studying the star transform order case (Theorem 3.3), we have that $\overline{V}(0) = 1 - \overline{F}_X(b) > 0$.

The favourable cases. Taking into account Theorems 2.1 and 2.2, and Remark 2.1 we need to describe the sign variation of $\overline{V}$ for $a > 0$ and $b > 0$. In each case, this control is obtained by the identification of the possible number of real roots with Theorem 2.3, and coupling this with the behaviour of $\overline{V}$ when $x \longrightarrow \pm\infty$.

Case 1. $a \geqslant 1$: As $b > 0$ and $x \geqslant 0$, we have $ax + b \geqslant x$, hence $\overline{V}(x) \geqslant \overline{F}_Y(x) - \overline{F}_X(x) \geqslant 0$.

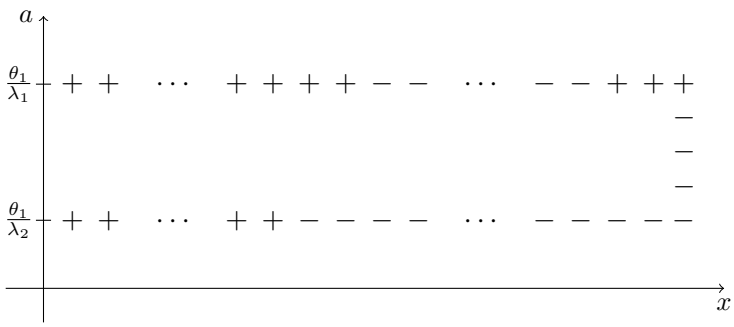
Case 2. $\frac{\theta_1}{\lambda_1} \leqslant a < 1$: Reordering appropriately the terms in $\overline{V}$ the sign pattern of its coefficients is "$+, -, -, +, +, -$" (or "$+, -, +, +, -$" if $a = \frac{\theta_1}{\lambda_1}$), hence, according to Theorem 2.3, $\overline{V}$ has at most three real roots. Moreover, it is easily seen that $\lim_{x\to+\infty} \overline{V}(x) = 0^+$. As $\overline{V}(0) > 0$, this means there are at most two nonnegative real roots. Thus, the sign pattern can only be either "$+$" or "$+, -, +$".

Case 3. $0 < a \leqslant \frac{\theta_1}{\lambda_2} < 1$: Reordering again the terms in $\overline{V}$ to apply Theorem 2.3, we find the sign pattern for its coefficients "$-, -, +, +, +, -$" (collapsing to '$-, +, +, +, -$" if $a = \frac{\theta_1}{\lambda_2}$), so $\overline{V}$ has at most two real

roots. At infinity, we find that $\lim_{x\to+\infty}\overline{V}(x)=0^-$. Hence, as $\overline{V}(0)>0$, $\overline{V}$ has one nonnegative real root and the sign pattern of $\overline{V}$ is "$+,-$".

The violating case. $\frac{\theta_1}{\lambda_2}<a<\frac{\theta_1}{\lambda_1}$: The sign pattern of the ordered coefficients in $\overline{V}$ is "$-,+,-,+,+,-$". Thus, using Theorem 2.3, $\overline{V}$ may have up to four real roots. The fact that $\lim_{x\to-\infty}\overline{V}(x)=-\infty$, together with $\overline{V}(0)>0$, implies that one of the roots is negative. Moreover, we have that $\lim_{x\to+\infty}\overline{V}(x)=0^-$, so this is compatible with the sign variations, when x traverses from 0 to $+\infty$, "$+,-$" or "$+,-,+,-$". That is, the usage of Theorem 2.3 is not conclusive...

We need to show that the sign variation "$+,-,+,-$" is indeed achieved for an appropriate choice of the parameters $a>0$ and $b>0$, hence violating the comparison criterion. From the previous analysis, we know that if $a=\frac{\theta_1}{\lambda_2}$ the sign variation of $\overline{V}$ is "$+,-$". Likewise, if $a=\frac{\theta_1}{\lambda_1}$ the sign variation of $\overline{V}$ is either "$+$" or "$+,-,+$". Let us choose $b_0>0$ such that the sign variation of $\overline{V}(x)=\overline{F}_Y(x)-\overline{F}_X\left(\frac{\theta_1}{\lambda_1}x+b_0\right)$ is "$+,-,+$", and keep this choice fixed for the sequel of the proof. Furthermore, remember that when $\frac{\theta_1}{\lambda_2}<a<\frac{\theta_1}{\lambda_1}$, we have verified that $\lim_{x\to+\infty}\overline{V}(x)=0^-$. Hence we have the following graphical description of the sign of $\overline{V}$, depending on x and a:



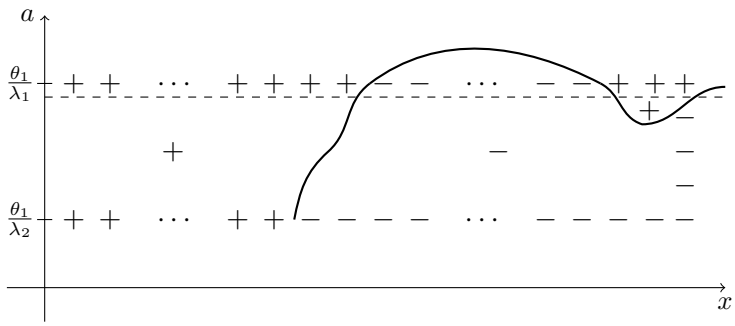
Keeping b_0 fixed, we may look at $\overline{V}$ as a function of x and a. Thus, we may differentiate with respect to a, to find

$$\frac{\partial\overline{V}}{\partial a}(x,a,b)=xf_X(ax+b),$$

where f_X is the density function of X, implying that for every possible choice for $b>0$, in particular for $b=b_0$, $\frac{\partial\overline{V}}{\partial a}(x,a,b)>0$. Hence, as a function of a alone, $\overline{V}$ is increasing. Thus, when a increases from $\frac{\theta_1}{\lambda_2}$ to $\frac{\theta_1}{\lambda_1}$ the value for $\overline{V}$ is also increasing, therefore once it becomes positive it may no longer get back to negative values. For the particular choice $b=b_0$,

that produces the lines of signs "$+$" and "$-$" above, the increasingness of $\overline{V}$ with respect to a explains why the initial sequence of "$+$" signs for $a_0 = \frac{\theta_1}{\lambda_1}$ is longer than the corresponding initial sequence when $a = \frac{\theta_1}{\lambda_2}$. It remains to justify that the choice for b_0 does exist. First, note that we proved in Case 3. of the proof of Theorem 3.3 that $\overline{F}_Y(x) - \overline{F}_X(a_0 x) \leqslant 0$, for every $x \geqslant 0$, and the inequality is strict for every $x > 0$. Now, choosing some $x_0 > 0$, we have that $\overline{F}_X^{-1}(\overline{F}_Y(x_0)) > a_0 x_0$, so we may find b_0 (depending on x_0) such that $\overline{F}_X^{-1}(\overline{F}_Y(x_0)) > a_0 x_0 + b_0$, which implies that $\overline{F}_Y(x_0) < \overline{F}_X(a_0 x_0 + b_0)$. As the functions are continuous, this inequality will hold on some neighbourhood of x_0, so the sign pattern represented above always happens.

Getting back to the graphical representation above, we now locate the set of points (x, a) such that $\overline{V}(x, a, b_0) = 0$, that is, we may take $a = h(x)$. As $\overline{V}$ is continuous, this function h is also continuous, and we will find the behaviour of h described by the thick line below, where we also identify the sign of $\overline{V}$ in each region:



The position of the horizontal dashed line identifies a value for the parameter a for which the sign variation of $\overline{V}$, with $b = b_0$, is actually "$+, -, +, -$", so the random variables are not comparable with respect to the convex transform order. ■

REMARK 3.4. *Although a counterexample would have been enough, the proof of Theorem 3.4 goes one step further, as it shows the way to identify a set of parameters for which the convex transform order comparability fails, that Kochar and Xu [11] could not locate, since without prior indication of where to look for, it is easy to miss the appropriate choice for a and b. The construction above depends critically on an appropriate choice for b, producing the "$+, -, +$" sign pattern on the top line. Numerical experiments confirm the arguments produced in the course of the proof, indicating that b must be chosen close to 0. Moreover, numerical experiments suggest that, even with a convenient choice for b the top-right region with the "$+$" signs may be relatively narrow.*

The original motivation for the conjecture stems from the characterization of skewness of the densities, in the sense introduced by van Zwet [21], which was the problem studied by Kochar and Xu [11]. The link between the convexity approach used by Kochar and Xu [11] and our sign variation approach follows from the characterization of convexity by means of sign variation, described in Theorem 20 in Arab and Oliveira [1], [2]. The narrow region for the choice of a *and* b *violating the sign variation pattern, means that the function* $\overline{F}_Y^{-1}(\overline{F}_X(x))$ *is concave in almost the whole of its domain and convex in a small interval.*

An explicit example and choice of the parameters violating the $\leqslant_{\mathrm{c}}$-comparability may be obtained taking, for example, $\lambda_1 = 2$, $\lambda_2 = 3$, $\theta_1 = 1.5$, $\theta_2 = 3.5$, $a = 0.749$ and $b = 0.0125$. Note that the choice of hazard rates and parameters is meant to identify where the sign variation condition is violated, not directly the convexity of $\overline{F}_Y^{-1}(\overline{F}_X(x))$ itself. The graphical representation of $\overline{F}_Y^{-1}(\overline{F}_X(x))$ is not very illustrative, as this function behaves almost as a straight line. Moreover, the function $\overline{F}_Y$ is obviously not explicitly invertible, so simple closed representations of the derivatives seems to be out of reach. However, proceeding numerically, one may obtain the approximations $\overline{F}_Y^{-1}(\overline{F}_X)''(1.5) \approx 0.20344$ and $\overline{F}_Y^{-1}(\overline{F}_X)''(2.5) \approx -0.06557$, hence the function is neither convex nor concave.

Let us go back to the framework of Theorems 3.3 and 3.4 and describe the sign pattern of $\overline{V}(x,a,b) = \overline{F}_Y(x) - \overline{F}_X(ax+b)$, considering the effect of the three variables x, a and b. It is easily seen that the intersections of the surface defined by $\overline{V}(x,a,b) = 0$ with the plane $b = 0$ is described by $a = \frac{\overline{F}_X^{-1}(\overline{F}_Y(x))}{x}$, and the intersection with the plane $a = 0$ by $b = \overline{F}_X^{-1}(\overline{F}_Y(x))$. Therefore, the intersection with the plane $b = 0$ defines, as a function of x, a decreasing curve. An inspection of the proof of Theorem 3.3 implies that this function is always larger or equal than $\frac{\theta_1}{\lambda_1}$. Indeed, the proof of Theorem 3.3 shows that for $(x,a,0)$ with $x \geqslant 0$ and $a \leqslant \frac{\theta_1}{\lambda_1}$ the sign of $\overline{V}$ is negative, hence this region does not intersect the surface $\overline{V}(x,a,b) = 0$. On the other hand, when $x \geqslant 0$ and $a > \frac{\theta_1}{\lambda_1}$, the sign pattern of $\overline{V}(x,a,0)$ may be "$-,+$", therefore an intersection with $\overline{V}(x,a,b) = 0$ may occur. These considerations imply that the curve $a(x) = \frac{\overline{F}_X^{-1}(\overline{F}_Y(x))}{x}$ is decreasing and has a lower bound. As this function is obviously nonnegative, $\lim_{x\to+\infty} a(x)$ is finite and is larger or equal than $\frac{\theta_1}{\lambda_1}$. Moreover, taking into consideration that the sign variation of $\overline{F}_X^{-1}(\overline{F}_Y(x)) - (\frac{\theta_1}{\lambda_1}x + b)$ is either "$-$" or "$-,+,-$", it follows that $\lim_{x\to+\infty} \overline{F}_X^{-1}(\overline{F}_Y(x)) - \frac{\theta_1}{\lambda_1}x = 0$. Therefore, for the particular example mentioned above, where $(\lambda_1,\lambda_2) = (2,3)$ and $(\theta_1,\theta_2) = (1.5,3.5)$, we have that $\lim_{x\to+\infty} \overline{F}_X^{-1}(\overline{F}_Y(x)) - \frac{3}{4}x = 0$.

We now address an extension of Theorem 3.4 to a generalised version of the framework of Theorem 3.2. The observation above that $\overline{F}_Y^{-1}(\overline{F}_X(x))$ behaves almost like a straight line suggests a criterion for the non-comparability, that we

apply to lifetimes of parallel systems.

THEOREM 3.5. *Let $X_1, \ldots, X_n$, be independent and exponentially distributed random variables where $X_1, \ldots, X_p$ have hazard rate λ_1 and $X_{p+1}, \ldots, X_n$ have hazard rate λ_2, and $Y_1, \ldots, Y_n$ satisfy similar conditions with hazard rates θ_1 and θ_2. If $(\lambda_1, \lambda_2) \prec (\theta_1, \theta_2)$, then $X = \max(X_1, \ldots, X_n)$ and $Y = \max(Y_1, \ldots, Y_n)$ are not comparable with respect to the convex transform order.*

Proof. We seek to conclude that $\overline{F}_Y^{-1}(\overline{F}_X(x))$ is neither convex nor concave. It is easily seen that this function is increasing and $\overline{F}_Y^{-1}(\overline{F}_X(0)) = 0$. We shall prove that, as $x \longrightarrow +\infty$, $\overline{F}_Y^{-1}(\overline{F}_X(x))$ approaches a straight line that starts at the origin. Hence, the conclusion about the convexity follows immediately. Before proceeding, notice that $\overline{F}_Y^{-1}(\overline{F}_X(x)) = F_Y^{-1}(F_X(x))$, $F_Y(x) = F_1^p(x)F_2^{n-p}(x)$ and $F_X(x) = G_1^p(x)G_2^{n-p}(x)$, where $F_i(x) = 1 - e^{-\theta_i x}$ and $G_i(x) = 1 - e^{-\lambda_i x}$. Taking into account the comments above, choose $c_1 = \frac{\lambda_1}{\theta_1} > 1$ and consider $H(x) = F_X(x) - F_Y(c_1 x)$. After expansion, $H(x)$ is represented as linear combination of exponentials. Moreover, when $x \longrightarrow +\infty$, the dominant terms correspond to the smallest values of the multiplicative constants appearing in the exponents. That is, considering the choice for c_1, the dominant terms are the ones that involve $e^{-\lambda_1 x}$. Consider now the perturbation $H_1(x) = F_X(x) - F_Y(c_1 x - \varepsilon)$, where $\varepsilon > 0$. It is obvious that $\lim_{x\to+\infty} H_1(x) = 0$ and $H_1'(x) = f_X(x) - c_1 f_Y(c_1 x - \varepsilon)$. The dominant term, when $x \longrightarrow +\infty$, in $H_1'(x)$ is $\lambda_1 e^{-\lambda_1 x}(1 - e^{\theta_1 \varepsilon}) < 0$. Hence, for large values of x, it follows that H_1 is decreasing towards 0, that is, that $F_X(x) \geqslant F_Y(c_1 x - \varepsilon)$. Likewise, we may conclude that, for x large enough, $F_X(x) \leqslant F_Y(c_1 x + \varepsilon)$. As F_Y is increasing, these two inequalities imply that, when $x \longrightarrow +\infty$, $\left|F_Y^{-1}(F_X(x)) - c_1 x\right| \leqslant \varepsilon$, therefore, as $\varepsilon > 0$ is arbitrary, $\lim_{x\to+\infty} \left|F_Y^{-1}(F_X(x)) - c_1 x\right| = 0$, so the proof is concluded. ■

REMARK 3.5. *Theorems 3.4 and 3.5 were proved under the assumption $(\lambda_1, \lambda_2) \prec (\theta_1, \theta_2)$ on the hazard rates. Similar to what was mentioned with respect to Corollary 3.1, the conclusion of all these results still holds under the alternative assumptions $\lambda_1 < \lambda_2$ and $\frac{\lambda_2}{\lambda_1} > \frac{\theta_2}{\theta_1}$.*

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