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ON THE SEMI-MITTAG-LEFFLER DISTRIBUTIONS

BY

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Abstract. The semi-Mittag-Leffler (SML) distribution arises as the marginal of a stationary Markovian process, and is a generalization of the well-known Mittag-Leffler (ML) or positive Linnik distribution. Unlike the ML distribution, which has been well established, few properties of the SML distribution are developed in the literature. In this paper, we derive some more characterizations of the SML and related distributions. By using stochastic inequalities, we further extend some characterizations, including Pitman and Yor's (2003) result about the hyperbolic sine distribution.

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1. INTRODUCTION

The semi-Mittag-Leffler distribution arises as the marginal of a stationary Markovian process, and generalizes the well-known Mittag-Leffler (or positive Linnik) distribution. The latter is well established in the literature, while few properties of the former are known. We start with the definition of the distribution.

Let F be the distribution function of a nonnegative random variable X, denoted $X \sim F_X = F$. For a systematic approach, we say that F is a semi-Mittag-

Leffler (SML) distribution function with exponent $\alpha \in (0,1]$ and order $b \in (0,1)$, if its Laplace–Stieltjes transform (LS-transform) $\hat{F}$ is of the form

$$\hat{F}(s) = \mathsf{E}[\exp(-sX)] = (1+\eta(s))^{-1}, \ \ s \in \mathbb{R}_+ := [0,\infty), \tag{1.1}$$

with $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ increasing (nondecreasing) and satisfying the equation:

$$\eta(s) = b^{-\alpha}\eta(bs), \ \ s \in \mathbb{R}_+ \tag{1.2}$$

(Pillai [27]; Jayakumar and Pillai [11]). It is clear that $\eta(0^+) = \lim_{s\to 0^+}\eta(s) = 0$ (since $\hat{F}(0^+) = 1$) and that η is infinitely differentiable on $(0,\infty)$, denoted $\eta \in \mathcal{C}^\infty(0,\infty)$. For convenience, we also say that F, X and $\hat{F}$ are SML(α, b) distribution function, random variable, and (LS-)transform, respectively.

We have some observations. (i) The degenerate distribution at zero is SML with $\eta = 0$. (ii) If X is SML(α, b), then so is cX for any $c > 0$ (the property of positive scale invariant). (iii) If the function η in (1.1) satisfies Eq. (1.2), then for each constant $c > 1$, both $(1+\eta(s))^{-1}$ and $(1+c\eta(s))^{-1}$ are SML(α, b) transforms. (iv) To understand better Eq. (1.2), define the function $h(x) = \eta(\mathrm{e}^x)/\mathrm{e}^{\alpha x}, \ x \in \mathbb{R} := (-\infty,\infty)$. Then $\eta(s) = s^\alpha h(\log s)$ and we can rewrite (1.1) as

$$\hat{F}(s) = (1 + s^\alpha h(\log s))^{-1}, \ \ s > 0, \tag{1.3}$$

where the nonnegative function h has a period, $-\log b$, due to the scaling property (1.2). Clearly, h belongs to $\mathcal{C}^\infty(\mathbb{R})$.

On the other hand, the order of an SML distribution defined above for convenience is not unique. For example, an SML(α, b) distribution is also SML(α, b^2). But if it is an SML(α, b_1) as well as SML(α, b_2) distribution with the quotient $\log b_1/\log b_2$ irrational, then the h in (1.3) will be a constant function, say $h(x) = \lambda \geqslant 0, \ x \in \mathbb{R}$. In this case, the SML$(\alpha, b)$ distribution reduces to a Mittag-Leffler (ML) distribution with LS-transform

$$\hat{F}(s) = (1 + \lambda s^\alpha)^{-1}, \ \ s \in \mathbb{R}_+, \tag{1.4}$$

which was first investigated by Gnedenko [5]. The $\hat{F}$ in (1.4) is simply called an ML transform with exponent α, denoted ML(α) transform.

The ML distribution possesses the following properties (for simplicity, consider here the case $\lambda = 1$ in Eq. (1.4)):

1. Each ML distribution function has (a) an explicit form and (b) a completely monotone derivative (Pillai [28]; Lin [19]). Precisely, the ML distribution function F with LS-transform $\hat{F}(s) = (1 + s^{\alpha})^{-1}, \ \ s \geqslant 0$, is of the form

$$F(x) = 1 - E_{\alpha}(-x^{\alpha}), \ \ x \geqslant 0, \tag{1.5}$$

and has the density

$$f(x) = x^{\alpha-1}E_{\alpha,\alpha}(-x^{\alpha}), \ \ x > 0.$$

Here, the one-parameter and two-parameter Mittag-Leffler functions are defined as follows (we consider only real-valued functions with positive parameters α, β):

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \ \ t \in \mathbb{R}. \tag{1.6}$$

2. The tail of the ML distribution function F in (1.5) satisfies $\overline{F}(x) \sim x^{\alpha}/(1 + x^{\alpha})$ as $x \to \infty$ (Lin [19]).

3. Any ML(α) distribution is geometrically infinitely divisible (GID) and hence infinitely divisible (ID), so we can consider the ML(α) process (with LS-transform $(1 + s^{\alpha})^{-t}$ at time $t > 0$) (Pillai [29]; Lin [19]).

4. Any ML(α) distribution is a generalized Gamma convolution (GGC), and it has a hyperbolically completely monotone (HCM) density function if and only if $\alpha \leqslant 1/2$ (Jedidi and Simon [13], p. 1835). In this regard, see Bondesson [2].

For more properties of the ML distribution, see the survey paper by Jayakumar and Suresh [12] and the references therein.

Recently, Kataria and Vellaisamy [15] considered the convolution of Mittag-Leffler distributions and its applications to (state dependent) fractional point processes, while Jose et al. [14] pointed out the possible applications of generalized

Mittag-Leffler distributions and processes to astrophysics and time series modeling. The multivariate ML distributions are investigated by Albrecher et al. [1]. On the other hand, the ML function E_α in (1.6) is one of the important special functions related to the Fractional Calculus. In particular, it is a Fox H-function (and hence a transcendental function) and the function $g(t) = E_\alpha(\lambda t^\alpha),\ t > 0$, satisfies the fractional-order integral equation:

$$\lambda(J_{0+}^\alpha g)(t) = g(t) - 1, \ \ t > 0, \ \lambda \in \mathbb{R}.$$

Here, the Riemann–Liouville (left-sided) fractional integral $(J_{0+}^\alpha g)(t)$ (of order α) of a measurable function g is defined by

$$(J_{0+}^\alpha g)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(\tau)}{(t-\tau)^{1-\alpha}}\, \mathrm{d}\tau, \ \ t > 0,$$

provided the integral exists. For more generalizations of the Mittag-Leffler function and their applications to physical and applied sciences as well as information and communication problems, see the monograph by Gorenflo et al. [6] and the survey papers by Gorenflo et al. [7], Haubold et al. [8], Mainardi [22, 23] and Mathai [24].

On the contrast, as we mentioned above, few properties of the SML distribution are available in the literature although it is a natural extension of the ML distribution and is closely related to the well-established positive semi-stable (SS) distribution (Maejma [21]). More precisely, the SML distribution is the compound-exponential of a positive SS distribution (and hence is ID) as described below. Rewrite the SML(α, b) transform (1.1) as the following:

$$(1.7) \quad \hat{F}(s) = \frac{1}{1+\eta(s)} = \frac{1}{1-\log \hat{G}(s)} = \int_0^\infty (\hat{G}(s))^y \mathrm{e}^{-y} \mathrm{d}y, \ \ s \geqslant 0,$$

where $\eta(s)$ satisfies (1.2) and $\hat{G}(s) = \exp(-\eta(s)),\ s \geqslant 0$, is the LS-transform of a positive SS distribution with exponent α and order b. In practice, an SML(α, b) random variable generates a stationary-independent-increments stochastic (composition) process, which is the subordination of the positive SS(α, b) process to the standard gamma process. This constitutes a one-to-one correspondence between the positive SS and SML distributions (see, e.g., Kozubowski [17] as well as Steutel and van Harn [31], Chapters I and II). The latter relation can help us construct some explicit SML transforms. Besides, each SML(α, b) distribution is in fact a *positive* semi-α-Laplace distribution and it belongs to the domain of partial attraction of a corresponding positive SS distribution (see the explanation in Remark 2.1 below). A possible application to the household income problem was given by Pillai [26]. These facts together suggest us studying the SML distribution.

When $\alpha = 1$, the SML(α, b) distribution reduces to the exponential one with mean $\mu = \eta'(0^+) \geqslant 0$ (including the degenerate case); in other words, the η in (1.1) is a linear function: $\eta(s) = \mu s,\ s \geqslant 0$ (see Hu and Lin [9], p. 145).

The SML distribution can also arise as a solution to the stability problem in geometric compounding model (see Theorem 3.1 below or Hu and Lin [9], Theorem 2). In the next section, we first give a decomposition characterization of the SML distribution (Theorem 2.1) and then apply it to elaborate on two stationary stochastic processes of which the SML distribution is a solution (marginal). In Section 3, we give some more characterizations of the SML and related distributions (Theorems 3.1 – 3.3). Finally, by using stochastic inequalities, we further extend in Section 4 some characterizations (Theorems 4.1 – 4.3), including Pitman and Yor's [30] result about the hyperbolic sine distribution.

2. SML DISTRIBUTION AS MARGINALS OF STATIONARY STOCHASTIC PROCESSES

We first present a decomposition characterization of the SML distribution and then apply it to the stochastic processes in question.

THEOREM 2.1. *Let* $\alpha \in (0, 1],\ b \in (0, 1)$ *be two constants and let* X, X_1, X_2 *be three nonnegative random variables having the same distribution function* F. *Assume further that* B *is a Bernoulli random variable with* $\Pr(B = 1) = b^\alpha$ *and* *that X_1, X_2, B are independent. Then the distributional equation*

$$X \stackrel{\mathrm{d}}{=} bX_1 + (1 - B)X_2 \tag{2.1}$$

holds if and only if F is an SML(α, b) distribution function.

Proof. Let $\hat{F}$ be the LS-transform of X and let $\eta(s) = 1/\hat{F}(s) - 1,\ s \geqslant 0.$ Then rewrite Eq. (2.1) in terms of LS-transforms as

$$\hat{F}(s) = \hat{F}(bs)[b^\alpha + (1 - b^\alpha)\hat{F}(s)], \quad s \geqslant 0. \tag{2.2}$$

Consequently, we have

$$\frac{1}{\hat{F}(bs)} = (1 - b^\alpha) + \frac{b^\alpha}{\hat{F}(s)}, \quad s \geqslant 0,$$

or, equivalently, $\eta(bs) = b^\alpha \eta(s),\ s \geqslant 0$. This proves the theorem. ■

REMARK 2.1. *In Theorem* 2.1, *if we consider general random variables X, X_1, X_2 taking values in the real line $\mathbb{R}$ (instead of $\mathbb{R}_+$), then all the solutions to Eq.* (2.1) *are the so-called semi-α-Laplace distributions (Pillai [26], Theorem* 1*) and hence they are GID distributions defined by Klebanov et al. [16] (see, e.g., Mohan et al. [25], pp. 174–175). Therefore, we may say that each SML(α, b) distribution is a positive semi-α-Laplace distribution. Our proof here for the nonnegative case is much simpler. Moreover, the SML(α, b) distribution belongs to the (strict-sense) domain of partial attraction of a corresponding positive SS distribution (with the same exponent α and order b) along the similar lines of Pillai's [26] Theorem* 2. *For some properties and applications of the semi-α-Laplace distribution and its multivariate cases, see Divanji [3] and Yeh [32].*

REMARK 2.2. *Note that in the decomposition Eq.* (2.1), *both X and bX_1 are SML(α, b), while $Y := (1 - B)X_2$ has a mixture distribution of two SML(α, b) ones. All the three random variables are GID. When $\alpha = 1$, the distributional equation* (2.1) *characterizes the exponential distribution. In this regard, see Lin and Hu [20], Theorem* 3, *for a further extension.*

We are ready to clarify and elaborate on two stochastic processes considered in Jayakumar and Pillai [11]. Both processes have the SML solution (marginal) under the stationarity. We need some notations. Let

$$\{\varepsilon_n\}_{n=-\infty}^{\infty} = \{\varepsilon_n : n = 0, \pm 1, \pm 2, \ldots\}$$

be a set of independent and identically distributed (i.i.d.) nonnegative random variables with the same distribution function F_ε and LS-transform $\hat{F}_\varepsilon$, and let $\{B_n\}_{n=-\infty}^{\infty}$, independent of $\{\varepsilon_n\}_{n=-\infty}^{\infty}$, be a set of i.i.d. Bernoulli random variables with $\Pr(B_n = 1) = b^\alpha$, where $\alpha \in (0, 1]$ and $b \in (0, 1)$ are two constants. We use "$\stackrel{\mathrm{d}}{=}$" for equality in distribution. The next result is an immediate consequence of Theorem 2.1 (compare Eqs. (2.1) – (2.3)).

COROLLARY 2.1. (Stationary Markovian Process.) *Under the above setting, consider the Markovian process starting from* X_0 :

(2.3) $$X_0 = \varepsilon_0, \quad X_n = bX_{n-1} + (1 - B_n)\varepsilon_n, \;\; n \in \mathbb{N} := \{1, 2, 3, \ldots\}.$$

Suppose that $\{X_n\}_{n=0}^{\infty}$ *is stationary, namely, all* X_n *have the same distribution and hence* $X_n \stackrel{\mathrm{d}}{=} \varepsilon_n$ *for all* $n \geqslant 0$. *Then each* X_n *is an SML*(α, b) *random variable.*

We next turn to the following first-order autoregressive process (AR(1) model, Eq. (2.4)), in which X_0 depends on $B_n, \varepsilon_n, n = 0, -1, -2, \ldots$; explicitly,

$$X_0 = \sum_{n=0}^{\infty} b^n (1 - B_{-n})\varepsilon_{-n}.$$

Note that $X_0 \neq \varepsilon_0$ in general; see Gaver and Lewis [4] for examples of this kind of AR(1) Models. Applying again Eqs. (2.1) and (2.2), we have the following.

COROLLARY 2.2. (AR(1) Model.) *Under the above setting, consider the process:*

(2.4) $$X_n = bX_{n-1} + (1 - B_n)\varepsilon_n, \;\; n = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

(i) If $\{X_n\}_{n=-\infty}^{\infty}$ is stationary, then the common LS-transform $\hat{F}$ of X_n satisfies

$$\frac{\hat{F}(s)}{\hat{F}(bs)} = b^{\alpha} + (1-b^{\alpha})\hat{F}_{\varepsilon}(s), \quad s \geqslant 0.$$

(ii) If, in addition, $X_n \stackrel{\mathrm{d}}{=} \varepsilon_n$, then each X_n is an SML(α, b) random variable.

3. CHARACTERIZATIONS OF THE SML AND RELATED DISTRIBUTIONS

Let $X, X_1, X_2, \ldots, X_n, \ldots$ be a sequence of i.i.d. nonnegative random variables with distribution function F. For each $p \in (0,1)$, let $N_1(p)$ be a geometric random variable, independent of $\{X_n\}_{n=1}^{\infty}$, with $\Pr(N_1(p) = n) = p(1-p)^{n-1}$, $n \in \mathbb{N}$. Consider the random summation (geometric compounding model)

$$S_{N_1(p)} := \sum_{n=1}^{N_1(p)} X_n. \tag{3.1}$$

Then, in addition to Theorem 2.1, we have the following characterization result.

THEOREM 3.1. (Hu and Lin [9].) *Let $p \in (0,1)$ and $\alpha \in (0,1]$ be two constants. Then under the above setting, the distributional equation*

$$X \stackrel{\mathrm{d}}{=} p^{1/\alpha} S_{N_1(p)}$$

holds if and only if X has an SML(α, b) distribution with $b = p^{1/\alpha}$.

There are two kinds of geometric random variables: one takes values in $\mathbb{N} = \{1, 2, 3, \ldots\}$, the other in $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. In view of Theorem 3.1, it is natural to ask the question: What will happen if in the random summation $S_{N_1(p)}$ (defined in Eq. (3.1)) we use instead a geometric random variable taking values in $\mathbb{N}_0$? We now consider the geometric random variable N_0 defined by

$$\Pr(N_0 = n) = \frac{p}{1+p}\Big(\frac{1}{1+p}\Big)^{n}, \quad n \in \mathbb{N}_0, \tag{3.2}$$

and have instead the following characteristic property of the GID mixture of SML distributions, where $S_0 := 0$. The other characterization result follows.

THEOREM 3.2. *Let $p \in (0,1)$ and $\alpha \in (0,1]$ be two constants, and let $N_0 = N_0(p/(1+p))$ be the geometric random variable defined in Eq.* (3.2). *Further, assume that $\{X_n\}_{n=1}^{\infty}$ is a sequence of independent copies of $0 \leqslant X \sim F$ and it is independent of N_0. Then the distributional equation*

$$X \stackrel{\mathrm{d}}{=} p^{1/\alpha} S_{N_0} := p^{1/\alpha} \sum_{n=1}^{N_0} X_n \tag{3.3}$$

holds if and only if X has the GID mixture distribution function

$$F(x) = p + (1-p) F_{\alpha,b}(x), \quad x \geqslant 0, \tag{3.4}$$

where $F_{\alpha,b}$ is an SML(α, b) distribution function with $b = p^{1/\alpha}$.

P r o o f. It is clear that each of Eqs. (3.3) and (3.4) implies $\chi := \Pr(X = 0) > 0$. If X is degenerate at zero, i.e., $\chi = 1$, then both Eqs. (3.3) and (3.4) hold, and we have $\hat{F}_{\alpha,b}(s) = [1+\eta(s)]^{-1}$ with $\eta(s) = 0, \ s \geqslant 0$. Therefore, it remains to consider the case $\chi \neq 1$, i.e., $\chi < 1$.

Using LS-transform, Eqs. (3.3) and (3.4) are equivalent to the following Eqs. (3.5) and (3.6), respectively (recall that $S_0 = 0$):

$$\begin{aligned} \hat{F}(s) &= \sum_{n=0}^{\infty} \frac{p}{1+p} \Big(\frac{1}{1+p}\Big)^n (\hat{F}(p^{1/\alpha} s))^n \\ &= \Big[1 + \frac{1 - \hat{F}(p^{1/\alpha} s)}{p}\Big]^{-1}, \quad s \geqslant 0; \end{aligned} \tag{3.5}$$

$$\hat{F}(s) = p + (1-p) \hat{F}_{\alpha,b}(s), \quad s \geqslant 0. \tag{3.6}$$

We will prove the equivalence of Eqs. (3.5) and (3.6) with $b = p^{1/\alpha}$.

(Sufficiency) Suppose that Eq. (3.6) holds with $b = p^{1/\alpha}$. Namely,

$$\hat{F}(s) = p + (1-p) \frac{1}{1+\eta(s)}, \quad s \geqslant 0, \tag{3.7}$$

where $\eta(s) = b^{-\alpha} \eta(bs) = p^{-1} \eta(p^{1/\alpha} s)$. We want to prove that $\hat{F}$ satisfies Eq. (3.5). This can be done by just plugging Eq. (3.7) in the RHS of Eq. (3.5) and carrying out the calculations, in which we apply the identity:

$$1 - \hat{F}(p^{1/\alpha} s) = \frac{(1-p) p\, \eta(s)}{1 + p\, \eta(s)}, \quad s \geqslant 0.$$

(Necessity) Suppose that Eq. (3.5) holds true. Then we want to prove Eq. (3.6) with $b = p^{1/\alpha}$. Recall that $\lim_{s\to\infty} \hat{F}(s) = \Pr(X = 0) = \chi > 0$. Letting $s \to \infty$

in Eq. (3.5) yields $(\chi - 1)(\chi - p) = 0$. This implies $\chi = p$ because X is not degenerate at zero by our assumption above. Since the LS-transform $\hat{F}$ is completely monotone on $(0, \infty)$, we have $\chi = p < \hat{F}(s) \leqslant 1,\ \ s \geqslant 0$. Write

$$\hat{F}(s) = p + (1-p)\frac{1}{1+g(s)}, \ \ s \geqslant 0, \tag{3.8}$$

or, equivalently,

$$g(s) = \frac{1-\hat{F}(s)}{\hat{F}(s) - p}, \ \ s \geqslant 0,$$

which is well-defined. Then plugging Eq. (3.8) in (3.5) and carrying out the calculations, we finally have $g(s) = p^{-1}g(p^{1/\alpha}s),\ \ s \geqslant 0$. That is, $\phi(s) = [1 + g(s)]^{-1}$, $s \geqslant 0$, is an SML$(\alpha, p^{1/\alpha})$ transform. This, together with Eq. (3.8), implies that Eq. (3.6) holds true with $b = p^{1/\alpha}$. The proof of the theorem is complete. ■

THEOREM 3.3. *Let* $p \in (0, 1)$ *and* $\alpha \in (0, 1]$ *be two constants. Suppose that* B *is a Bernoulli random variable, independent of* X, X_1, X_2 *defined above, and* $\Pr(B = 1) = 1/(1 + p)$. *Then the distributional equation*

$$X \stackrel{\mathrm{d}}{=} B \cdot (X_1 + p^{1/\alpha} X_2) \tag{3.9}$$

holds if and only if X *has the GID mixture distribution function*

$$F(x) = p + (1-p)F_{\alpha,b}(x), \ \ x \geqslant 0,$$

where $F_{\alpha,b}$ *is an SML*(α, b) *distribution function with* $b = p^{1/\alpha}$.

P r o o f. Using LS-transform, Eq. (3.9) is equivalent to Eq. (3.5). The conclusion follows immediately from Theorem 3.2. ■

Clearly, if $X \sim F$ is degenerate at zero, i.e., $X = 0\ \ a.s.$ (almost surely), then the stability condition (3.3) is satisfied by all $p \in (0, 1)$. Conversely, if Eq. (3.3) holds for any two distinct real numbers $p \in (0, 1)$, then $X = 0\ \ a.s.$ This is an application of Eq. (3.5) as shown below. A similar result for Eq. (3.9) holds true.

COROLLARY 3.1. *Let* p_1 *and* p_2 *be two distinct real numbers in* $(0, 1)$. *In addition to the assumptions in Theorem* 3.2, *if the stability condition* (3.3) *is satisfied by* $p = p_1, p_2$, *then* $X = 0\ \ a.s.$

P r o o f. By the assumptions, Eq. (3.5) holds for $p = p_1, p_2$, and hence

$$\frac{1-\hat{F}(p_1^{1/\alpha}s)}{p_1} = \frac{1-\hat{F}(p_2^{1/\alpha}s)}{p_2}, \quad s \geqslant 0.$$

Letting $s \to \infty$ yields

$$\frac{1-\Pr(X=0)}{p_1} = \frac{1-\Pr(X=0)}{p_2}. \tag{3.10}$$

Suppose, on the contrary, that X is not degenerate at zero, then we have $\Pr(X = 0) < 1$ and by (3.10), $p_1 = p_2$, a contradiction. Therefore, $X = 0$ $a.s.$ ■

4. EXTENSIONS

We improve some characterization results by using stochastic inequalities. For two random variables X and Y, we say that X is smaller than or equal to Y in the stochastic order, denoted $X \leqslant_{st} Y$, if $\Pr(X > x) \leqslant \Pr(Y > x)$ for all $x \in \mathbb{R}$. The first result extends the case $\alpha = 1$ of Theorem 3.2. For simplicity, the proofs of theorems and the needed lemmas are given in the appendix below.

THEOREM 4.1. *Let* $0 \leqslant X \sim F,\ p \in (0,1),\ N_0 = N_0(p/(1+p))$ *and the sequence* $\{X_n\}_{n=1}^{\infty}$ *be the same as in Theorem* 3.2. *Then the stochastic inequality*

$$pS_{N_0} = p\sum_{n=1}^{N_0} X_n \ \leqslant_{st} X \tag{4.1}$$

holds if and only if X *has the GID mixture distribution function*

$$F(x) = p + (1-p)(1-\exp(-\lambda x)), \ \ x \geqslant 0, \ \ for\ some\ \ \lambda \geqslant 0. \tag{4.2}$$

Similarly, we can extend the special case ($\alpha = 1$) of Theorem 3.3 to the following. The proof is omitted.

THEOREM 4.2. *Under the assumptions of Theorem* 3.3 *with* $\alpha = 1$, *the stochastic inequality* $B\,(X_1 + pX_2) \leqslant_{st} X$ *holds if and only if* X *has the GID mixture distribution function in Eq.* (4.2).

Moreover, we improve a result of Pitman and Yor [30] below.

THEOREM 4.3. *Let* $0 \leqslant X \sim F_X = F$ *have mean* $\mu = \mathsf{E}[X] \in (0, \infty)$ *and let* $T \geqslant 0$ *have distribution function* $F_T(x) = 2\sqrt{x} - x,\ 0 \leqslant x \leqslant 1$. *Assume further that the random variable* $Z \geqslant 0$ *has the length-biased distribution function induced by* F*:*

$$F_Z(z) = \frac{1}{\mu} \int_0^z x \, \mathrm{d}F(x), \quad z \geqslant 0,$$

and all random variables X, T, Z *are independent. Then we have the following:*
(i) If

$$Z \ \leqslant_{st}\ X + T\,Z, \tag{4.3}$$

then $\mathsf{E}[X^2] < \infty$ *and the coefficient of variation* $\mathsf{CV}(X) = \sqrt{\mathsf{Var}(X)}/\mu \leqslant 1/\sqrt{5}$.
(ii) If, in addition, $\mathsf{CV}(X) = 1/\sqrt{5}$, *then the LS-transform of* X *is*

$$\hat{F}(s) = \left(\sqrt{3\mu s}/\sinh\sqrt{3\mu s}\right)^2, \quad s \geqslant 0.$$

REMARK 4.1. *Under the assumptions in Theorem* 4.3 *with* $\mu = 2/3$, *Pitman and Yor [30], Proposition* 12, *proved that the distributional equation* $Z \overset{\mathrm{d}}{=} X + TZ$ *holds if and only if* $\hat{F}(s) = \left(\sqrt{2s}/\sinh\sqrt{2s}\right)^2,\ s \geqslant 0$.

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Appendix to Section 4

We finally provide the needed lemmas and the proofs of Theorems 4.1 and 4.3. Here, Lemmas 4.1 and 4.2 are well-known (see, e.g., Lin and Hu [20], p. 94, and Lin [18]), while Lemma 4.3 is a novel powerful tool (compare with Lemma 2.1(a) in Hu and Lin [10]). Lemma 4.4 extends Lemma 4.3 if the underlying distribution function F has finite mean.

LEMMA 4.1. *Let X, Y be two nonnegative random variables.*

(i) If $X \leqslant_{st} Y$, their LS-transforms satisfy the inequality: $\hat{F}_Y(s) \leqslant \hat{F}_X(s),\ s \geqslant 0$.

(ii) If $X \leqslant_{st} Y$ and $\mathsf{E}[X] = \mathsf{E}[Y] < \infty$, then $X \stackrel{\mathrm{d}}{=} Y$.

LEMMA 4.2. *Let $0 \leqslant X \sim F$. Then for each integer $n \geqslant 1$, the nth moment $\mathsf{E}[X^n] = \lim_{s\to 0^+}(-1)^n \hat{F}^{(n)}(s) = (-1)^n \hat{F}^{(n)}(0^+)$ (finite or infinite).*

LEMMA 4.3. *Let $p \in (0,1)$. Suppose that $0 \leqslant X \sim F$ satisfies the inequality*

$$\hat{F}(s) \leqslant \Big[1 + \frac{1 - \hat{F}(ps)}{p}\Big]^{-1},\ \ 0 \leqslant s \leqslant s_0,\ \ for\ some\ \ s_0 > 0. \tag{4.4}$$

Then we have the following:

(i) The mean $\mu = \mathsf{E}[X]$ is finite.

(ii) There exists an $s_1 \in (0, s_0)$ such that $\hat{F}(s) \leqslant p + (1-p)[1 + \mu s/(1-p)]^{-1}$, $s \in [0, s_1]$, and hence $\mathsf{E}[X^2] \leqslant 2\mu^2/(1-p) < \infty$.

P r o o f. (i) We first prove that the mean $\mathsf{E}[X]$ is finite. Since the LS-transform $\hat{F}$ is right continuous at zero and $\hat{F}(0) = 1$, there exists a constant $s_1 \in (0, s_0)$ such that $p < \hat{F}(s) \leqslant 1,\ \ s \in [0, s_1]$. Hence we can define

$$g(s) = \frac{1 - \hat{F}(s)}{\hat{F}(s) - p},\ \ s \in [0, s_1],$$

or, equivalently,

$$\hat{F}(s) = p + (1-p)\frac{1}{1 + g(s)},\ \ s \in [0, s_1]. \tag{4.5}$$

Moreover, $\hat{F}$ is completely monotone on $(0, \infty)$, so that by (4.5) and Lemma 4.2,

$$\hat{F}'(0^+) = -(1-p)g'(0^+). \tag{4.6}$$

On the other hand, it follows from (4.4) that

$$\frac{g(s)}{s} \geqslant \frac{g(ps)}{ps},\ \ s \in (0, s_1].$$

By iteration, we further get that for fixed $s \in (0, s_1]$,

$$\frac{g(s)}{s} \geqslant \frac{g(ps)}{ps} \geqslant \frac{g(p^2 s)}{p^2 s} \geqslant \cdots \geqslant \frac{g(p^n s)}{p^n s},\ \ \forall\, n \geqslant 1. \tag{4.7}$$

Letting $n \to \infty$ in (4.7) yields that

$$\infty > \frac{g(s)}{s} \geqslant g'(0^+), \quad s \in (0, s_1]. \tag{4.8}$$

This, together with (4.6), implies that $\mathsf{E}[X] = -\hat{F}'(0^+) = (1-p)g'(0^+) < \infty$.

(ii) We next prove part (ii). It follows from (4.6) and (4.8) that

$$\frac{g(s)}{s} \geqslant \frac{\mu}{1-p}, \quad s \in (0, s_1],$$

and hence, $\hat{F}(s) \leqslant p + (1-p)[1 + \mu s/(1-p)]^{-1}, \;\; s \in [0, s_1]$. Equivalently,

$$\frac{\hat{F}(s) - 1 + \mu\, s}{s^2} \leqslant \frac{\mu^2}{1 - p + \mu\, s}, \quad s \in (0, s_1].$$

Then, letting $s \to 0^+$ and using Lemma 4.2, we conclude that $\frac{1}{2}\mathsf{E}[X^2] = \frac{1}{2}\hat{F}''(0^+)$ $\leqslant \mu^2/(1-p) < \infty$. This completes the proof of the lemma. ∎

LEMMA 4.4. *Let $X \sim F$ be a nonnegative random variable with LS-transform $\hat{F}$ and mean $\mu = \mathsf{E}[X] \in (0, \infty)$. Let $0 \leqslant T \leqslant 1$ be a random variable with distribution function F_T and mean $\mathsf{E}[T] \in [0, 1)$. Define the Bernstein function*

$$\sigma(s) = \int_0^1 \frac{1 - \hat{F}(ts)}{t}\, \mathrm{d}F_T(t), \quad s \geqslant 0,$$

where the integrand $(1 - \hat{F}(ts))/t$ is defined for $t = 0$ by continuity to be equal to μs. Suppose that for some constant $s_0 > 0$,

$$\hat{F}(s) \leqslant [1 + \sigma(s)]^{-1}, \quad s \in [0, s_0). \tag{4.9}$$

Then for any number $p \in [0, 1)$ such that $F_T(p) \in (0, 1]$, the LS-transform $\hat{F}$ satisfies the inequality:

$$\hat{F}(s) \leqslant 1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda}\frac{1}{1 + \lambda s}, \quad s \in [0, s_0), \tag{4.10}$$

where $\lambda = \frac{\mu}{F_T(p)(1-p)} \geqslant \mu$, and hence $\mathsf{E}[X^2] \leqslant 2\lambda\mu < \infty$.

P r o o f. Assume that the random variable $X_* \sim F_*$ has the equilibrium distribution function induced by F, namely,

$$F_*(x) = \frac{1}{\mu}\int_0^x \bar{F}(t)\,\mathrm{d}t, \;\; x \geqslant 0,$$

where $\bar{F}(t) = \mathsf{P}[X > t] = 1 - F(t),\; t \geqslant 0$. Recall the relations

$$\hat{F}_*(0) = \hat{F}_{X_*}(0) = 1, \quad \hat{F}_*(s) = \hat{F}_{X_*}(s) = \frac{1-\hat{F}(s)}{\mu\, s} > 0, \;\; s > 0.$$

Then it follows from Eq. (4.9) that

$$\frac{1}{\hat{F}_*(s)} \leqslant \mu s + \frac{1}{\int_0^1 \hat{F}_*(ts)\,\mathrm{d}F_T(t)}, \;\; s \in [0, s_0). \tag{4.11}$$

Using Cauchy–Schwartz inequality:

$$\int_0^1 \hat{F}_*(ts)\,\mathrm{d}F_T(t) \cdot \int_0^1 \frac{1}{\hat{F}_*(ts)}\,\mathrm{d}F_T(t) \;\geqslant\; 1, \;\; s \geqslant 0,$$

we further have, by Eq. (4.11), that for $s \in [0, s_0)$,

$$\begin{aligned}\frac{1}{\hat{F}_*(s)} \;&\leqslant\; \mu s + \int_0^1 \frac{1}{\hat{F}_*(ts)}\,\mathrm{d}F_T(t) \;=\; \mu s + \int_0^p \frac{1}{\hat{F}_*(ts)}\,\mathrm{d}F_T(t) + \int_p^1 \frac{1}{\hat{F}_*(ts)}\,\mathrm{d}F_T(t)\\ &\leqslant\; \mu s + F_T(p)\frac{1}{\hat{F}_*(ps)} + (1-F_T(p))\frac{1}{\hat{F}_*(s)}.\end{aligned}$$

Consequently,

$$\frac{1}{\hat{F}_*(s)} \leqslant \frac{\mu s}{F_T(p)} + \frac{1}{\hat{F}_*(ps)}, \;\; s \in [0, s_0). \tag{4.12}$$

Setting $\beta = \mu/F_T(p)$ and by iteration on Eq. (4.12), we have a chain of inequalities: for fixed $s \in [0, s_0)$,

$$\begin{aligned}\frac{1}{\hat{F}_*(s)} \;&\leqslant\; \beta s + \frac{1}{\hat{F}_*(ps)} \;\leqslant\; \beta s + \beta p s + \frac{1}{\hat{F}_*(p^2 s)}\\ &\leqslant\; \cdots\cdots\cdots\cdots\cdots\cdots\\ &\leqslant\; \beta s(1 + p + p^2 + \cdots + p^{n-1}) + \frac{1}{\hat{F}_*(p^n s)}\\ &=\; \frac{\beta s(1-p^n)}{1-p} + \frac{1}{\hat{F}_*(p^n s)}, \;\; \forall\, n \geqslant 1.\end{aligned}$$

Letting $n \to \infty$ yields

$$\frac{1}{\hat{F}_*(s)} \leqslant \frac{\beta s}{1-p} + 1, \quad s \in [0, s_0),$$

or, equivalently,

$$\frac{\mu s}{1-\hat{F}(s)} \leqslant \lambda s + 1, \quad s \in (0, s_0),$$

where $\lambda = \beta/(1-p) = \mu/[F_T(p)(1-p)]$. Therefore,

$$\hat{F}(s) \leqslant 1 - \frac{\mu s}{1+\lambda s} = 1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda}\frac{1}{1+\lambda s}, \quad s \in [0, s_0).$$

This proves Eq. (4.10). The last claim follows from Eq. (4.10) and the facts that

$$\lim_{s\to 0^+} \frac{\hat{F}(s) - 1 + \mu s}{s^2} = \frac{1}{2}\hat{F}''(0^+) = \frac{1}{2}\mathsf{E}[X^2],$$

$$\lim_{s\to 0^+} \frac{1}{s^2}\left(-\frac{\mu}{\lambda} + \frac{\mu}{\lambda}\frac{1}{1+\lambda s} + \mu s\right) = \lambda\mu.$$

The proof of the lemma is complete. ∎

REMARK 4.2. *When* $T = p \in (0,1)$ *a.s., Eq.* (4.9) *reduces to Eq.* (4.4). *If* $T = 0$ *a.s., then, by definition, Eq.* (4.9) *becomes* $\hat{F}(s) \leqslant (1+\mu s)^{-1}, \ s \in [0, s_0)$. *On the other hand,* $F_T(p) = 1$ *for all* $p \in [0,1)$, *and the RHS of Eq.* (4.10) *(as a function of* p*) has the minimum* $(1+\mu s)^{-1}, \ s \in [0, s_0)$. *Therefore, the functional upper bound of* $\hat{F}(s)$ *in Eq.* (4.10) *is sharp in this sense.*

PROOF OF THEOREM 4.1. The sufficiency part is a consequence of Theorem 3.2 with $\alpha = 1$, because the SML(α, b) distribution function in Eq. (3.4) reduces to an exponential one in Eq. (4.2). To prove the necessity part, suppose $X \sim F$ satisfies Eq. (4.1). Then we have, by Lemma 4.1(i) above, that

$$\hat{F}(s) \leqslant \Big[1 + \frac{1-\hat{F}(ps)}{p}\Big]^{-1}, \quad s \geqslant 0,$$

and hence condition (4.4) in Lemma 4.3 is satisfied. Thus $\mathsf{E}[X] < \infty$. Moreover, $\mathsf{E}[pS_{N_0}] = \mathsf{E}[X] < \infty$, and hence $pS_{N_0} \stackrel{\mathrm{d}}{=} X$ by Lemma 4.1(ii). Finally, the conclusion follows from Theorem 3.2 with $\alpha = 1$. The proof is complete. ∎

PROOF OF THEOREM 4.3. (a) We first prove part (i). From Eq. (4.3) it follows that

$$\hat{F}_Z(s) \geqslant \hat{F}(s)\hat{F}_{TZ}(s), \quad s \geqslant 0, \tag{4.13}$$

where $\hat{F}_{TZ}(s) = \mathsf{E}[\exp(-sTZ)], \ \ s \geqslant 0$ (see Lemma 4.1(i)). Recall that

$$\hat{F}_Z(s) = \mathsf{E}[\mathrm{e}^{-sZ}] = \frac{-\hat{F}'(s)}{\mu}, \quad \int_0^s \hat{F}_Z(x)\,\mathrm{d}x = \frac{1-\hat{F}(s)}{\mu}, \ \ s \geqslant 0,$$

and define the function

$$\sigma(s) := \mu \int_0^s \hat{F}_{TZ}(x)\,\mathrm{d}x = \int_0^1 \frac{1-\hat{F}(ts)}{t}\,\mathrm{d}F_T(t), \ \ s \geqslant 0.$$

Then we have, by Eq. (4.13), that

$$\hat{F}(s) \leqslant \exp(-\sigma(s)) \leqslant [1+\sigma(s)]^{-1}, \ \ s \geqslant 0.$$

Lemma 4.4 applies and hence $\mathsf{E}[X^2] < \infty$, equivalently, $\mathsf{E}[Z] < \infty$. By Eq. (4.3) again, we have

$$\mathsf{E}[Z] \leqslant \mathsf{E}[X] + \mathsf{E}[TZ] = \mathsf{E}[X] + \mathsf{E}[T]\mathsf{E}[Z], \tag{4.14}$$

from which $\mathsf{CV}(X) = \sqrt{\mathsf{Var}(X)}/\mu \leqslant 1/\sqrt{5}$, because $\mathsf{E}[Z] = \mathsf{E}[X^2]/\mu$. This proves part (i).

(b) Suppose, in addition, $\mathsf{CV}(X) = 1/\sqrt{5}$. Then the equality in Eq. (4.14) holds true, namely, $\mathsf{E}[Z] = \mathsf{E}[X] + \mathsf{E}[TZ]$. This, together with Eq. (4.3), implies that $Z \stackrel{\mathrm{d}}{=} X + T\,Z$ (see Lemma 4.1(ii)). Therefore,

$$\hat{F}(s) = \left(\sqrt{3\mu s}/\sinh\sqrt{3\mu s}\right)^2, \ \ s \geqslant 0,$$

due to Pitman and Yor [30] (see Remark 4.1 above). Here, we apply the fact that for each $c > 0$, the random variable cZ obeys the length-biased distribution induced by $cX \sim F_{cX}(x) = F_X(x/c)$, where $x \geqslant 0$. Namely,

$$F_{cZ}(z) = F_Z(z/c) = \frac{1}{\mu}\int_0^{z/c} x\,\mathrm{d}F_X(x) = \frac{1}{c\mu}\int_0^{z} x\,\mathrm{d}F_{cX}(x), \ \ z \geqslant 0.$$

The proof is complete. ■

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