PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 0, Fasc. 0 (0000), pp. 000–000
doi:10.19195/0208-4147.0.0.0

GENERALIZATIONS OF THE FOURTH MOMENT THEOREM

BY

NOBUAKI NAGANUMA* (KUMAMOTO)

Abstract. Azmoodeh *et al.* established a criterion regarding convergence of the *second* and *other* even moments of random variables in a Wiener chaos with fixed order guaranteeing the central convergence of the random variables. This was a major step in studies of the fourth moment theorem. In this paper, we settle even more generalizations of the fourth moment theorem by building on their ideas. More precisely, further criteria implying central convergence are provided: (i) the convergence of the *fourth* and *any other* even moment, (ii) the convergence of the *sixth* and *some other* even moments.

2010 AMS Mathematics Subject Classification: Primary: 60F05; Secondary: 33C45, 60H07.

Key words and phrases: the fourth moment theorem, Nualart-Peccati criterion, central convergence, Wiener chaos

1. INTRODUCTION

The fourth moment theorem (Nualart-Peccati criterion), discovered by Nualart and Peccati [9], provides a concise criterion for central convergence of random variables $\{Z_n\}_{n=1}^{\infty}$ belonging to a Wiener chaos of fixed order. More precisely, Nualart and Peccati showed that if $\boldsymbol{E}[Z_n^2] \to 1$ and $\boldsymbol{E}[Z_n^4] \to 3$ as $n \to \infty$,

* This work was supported by JSPS KAKENHI Grant Number 17K14202.

then $\{Z_n\}_{n=1}^{\infty}$ converges to a standard Gaussian random variable N in law. Subsequently, many researchers began studying generalizations and applications of the theorem. For example, Peccati and Tudor [11] extended it to the multidimensional case, and Nualart and Ortiz-Latorre [8] provided another proof for the theorem in terms of Malliavin calculus. Nourdin and Peccati [5] provided Berry-Esséen bounds in the Breuer–Major central limit theorem by combining Malliavin calculus and Stein's method.

An extension by Ledoux [3] was a major step in the ongoing study of the fourth moment theorem. He provided another proof for the fourth moment theorem in the framework of diffusive Markov generators inspired by a proof based on Malliavin calculus. More sophisticated and generalized results were provided by Azmoodeh, Campese, and Poly [1]. These papers were devoted to answering the following question stated in [2] by Azmoodeh, Malicet, Mijoule, and Poly.

What are the moment conditions that ensure central convergence?

This paper is also devoted to answering this question.

In order to go on discussion more precisely, we introduce some notation. Let $X = \{X(h)\}_{h\in\mathfrak{H}}$ be an isonormal Gaussian process over a real separable Hilbert space $\mathfrak{H}$. For every $p \in \mathbf{N} \cup \{0\}$, we write $\mathcal{H}_p$ to denote the pth Wiener chaos of X. For precise definitions, see [7], [6]. Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathcal{H}_p$ for some integer $p \geqslant 2$. We denote by $\mathcal{I}$ a finite subset of even numbers.

Then, the question above may be able to be reduced to equivalence of (CL) and (CM) for a finite subset $\mathcal{I}$ of even numbers:

(CL) $Z_n \to N$ in law as $n \to \infty$.

(CM) $\boldsymbol{E}[Z_n^{2i}] \to \boldsymbol{E}[N^{2i}]$ as $n \to \infty$ for all $2i \in \mathcal{I}$.

Of course, the fourth moment theorem involves equivalence of (CL) and (CM) for $\{2, 4\}$ and after shown the theorem some researchers wonder that the equivalence holds for *any* set of two distinct even numbers. The authors of [2] showed the equivalence of (CL) and (CM) for $\{2, 2k\}$ with $2k \geqslant 4$. One of their ingredients is a formulation of central convergence in terms of polynomials (this will be stated in Lemma 2.1). In this paper, we build on their formulation to suggest directions for generalization of the fourth moment theorem. Although we cannot provide a full answer of the question, we provide interesting examples of central convergence based on a lemma in [2]. Our main theorem is as follows.

THEOREM 1.1. *Let $\mathcal{I}$ be any of the following.*

1. $\mathcal{I} = \{2, 2k\}$, *where $2k \geqslant 4$ is an arbitrary even integer.*
2. $\mathcal{I} = \{4, 2k\}$, *where $2k \geqslant 6$ is an arbitrary even integer.*
3. $\mathcal{I} = \{6, 8\}, \{6, 10\}$.
4. $\mathcal{I} = \{6, 12, 14, 2k\}$, *where $2k \geqslant 16$ is an arbitrary even integer.*
5. $\mathcal{I} = \{6, 12, 18, 30, 32, 2k\}$, *where $2k \geqslant 34$ is an arbitrary even integer.*

Then, (CL) *and* (CM) *for $\mathcal{I}$ are equivalent.*

For readers' convenience, this theorem contains previous results; that is, Assertion 1, a part of Assertion 2 and Assertion 3, have already been demonstrated in [2, Theorem 1.2 and Section 5]. The cases of $\mathcal{I} = \{4, 6\}$, $\{4, 8\}$, $\{4, 10\}$ have already been treated, and we demonstrate that convergences of the *fourth* and *any* even moments imply central convergence in Assertion 2. Note that we can see that only in cases 1, 2, and 3, the method in [2] is effective in the proof of equivalence of (CL) and (CM) (this is one of contribution of this paper and stated in Proposition 3.1). Assertions 4 and 5 are entirely new. We make a remark on them.

· The case $\mathcal{I} = \{6, 12\}$ cannot be treated with the method in [2] due to Proposition 3.1, although a truely nontrivial case is $\mathcal{I} = \{6, 12\}$ after cases 1, 2, and 3. Hence the second smallest number in Assertions 4 and 5 should be greater than or equal to 12. If we replace 12 by 10, we see the equivalence due to Assertion 3.

· At this stage, we have no counterexample for the case $\mathcal{I} = \{6, 12\}$.

14 Jun 2022 04:10:23 CEST
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· Assertions 4 and 5 are not trivial and their proofs are interesting from the viewpoint of the properties of polynomials that appear in the proof.

For more discussion on our main theorem we should prepare more notation, and we postpone it until Section 4.

The remainder of this paper is organized as follows. Section 2 reviews the principal part of [2]. Section 3 is devoted to proving our main theorem. In Section 4, we discuss on our main theorem. Section 5 investigates asymptotic characteristics of the hypergeometric function.

Throughout this paper, we use the following notation. Let N be a standard Gaussian random variable and set $w(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, which is the density function of N. Set $\mu_i = \boldsymbol{E}[N^{2i}] = (2i-1)!!$ for $i \in \mathbf{N} \cup \{0\}$ with the convention $(-1)!! = 0$. We introduce the following functions.

· The Hermite polynomial: $H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$ for $n \in \mathbf{N} \cup \{0\}$.

· The Gamma function: $\Gamma(a) = \int_0^\infty u^{a-1} e^{-u}\, du$ for $a > 0$.

· The Beta function: $B(a,b) = \int_0^1 (1-u)^{a-1} u^{b-1}\, du = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ for $a, b > 0$.

· The Hypergeometric function:

$$F(a,b,c;z) = \frac{1}{B(a,c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-uz)^{-b}\, du$$

for $0 < a < c$ and $|z| < 1$.

We define $\{\kappa_i(m)\}_{m \geqslant i \geqslant 2}$ and $\{\xi_i(m)\}_{m,i \geqslant 2}$ as

$$\kappa_i(m) = B\left(i-1, \frac{1}{2}\right) F\left(i-1, -(m-i), i-\frac{1}{2}, \frac{1}{2}\right) = \int_0^1 u^{i-2}(1-u)^{-\frac{1}{2}} \left(1 - \frac{u}{2}\right)^{m-i} du, \tag{1.1}$$

and

$$\xi_i(m) = \begin{cases} \frac{(m-1)!}{(m-i)!} \kappa_i(m), & 2 \leqslant i \leqslant m, \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

2. REVIEW OF AZMOODEH *ET AL.*

In this section, we summarize most important part of [2] and extend it. For every $i \geqslant 2$, we define even polynomials W_i and ψ_i with degree $2i$ as

$$W_i(x) = (2i-1)\Phi[H_i H_{i-2}](x), \qquad \psi_i(x) = \boldsymbol{E}[W_i(xN)],$$

where Φ is defined as

$$\Phi[Q](x) = x\int_0^x Q(t)\,dt - Q(x).$$

Note that W_i is monic. Let T be a monic even polynomial with degree $2k \geqslant 4$ of the form

$$T(x) = \sum_{i=2}^{k} \alpha_i W_i(x) \tag{2.1}$$

for some $\alpha_2, \dots, \alpha_{k-1} \in \mathbf{R}$ and $\alpha_k = 1$. Then, the next lemma is a major component of [2].

LEMMA 2.1 ([2, Lemma 4.2]). *Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of elements in $\mathcal{H}_p$ for some integer $p \geqslant 2$, and let T be a monic even polynomial with degree $2k \geqslant 4$ of the form* (2.1) *with positive α_2, nonnegative $\alpha_3, \dots, \alpha_{k-1}$, and $\alpha_k = 1$. Then, $Z_n \to N$ in law as $n \to \infty$ if and only if $\boldsymbol{E}[T(Z_n)] \to 0$ as $n \to \infty$.*

Lemma 2.1 tells us that we can obtain central convergence of $\{Z_n\}_{n=1}^{\infty}$ by finding a suitable polynomial T. In general, a monic even polynomial T with degree $2k \geqslant 4$ is defined as

$$T(x) = \sum_{i=1}^{k} a_i x^{2i} + a_0 \tag{2.2}$$

for some $a_0, \dots, a_{k-1}$ and $a_k = 1$. To use Lemma 2.1, we seek to determine what conditions on $a_0, \dots, a_k$ imply T are of the form (2.1) with $\alpha_2, \dots, \alpha_k$, and we provide a formula for calculating $\alpha_2, \dots, \alpha_k$ from $a_0, \dots, a_k$. We know that

$$\boldsymbol{E}[T(N)] = \lim_{n\to\infty} \boldsymbol{E}[T(Z_n)] = 0$$

if $\{Z_n\}_{n=1}^{\infty} \subset \mathcal{H}_p$ satisfies $\boldsymbol{E}[Z_n^{2i}] \to \mu_i$ as $n \to \infty$. This is equivalent to $\phi(1) = 0$, where $\phi(x) = \boldsymbol{E}[T(xN)]$. The next proposition follows.

PROPOSITION 2.1. *Let T be an even polynomial with degree $2k \geqslant 4$ and set $\phi(x) = \boldsymbol{E}[T(xN)]$. The following are equivalent.*

1. $\phi(1) = \phi'(1) = 0$ *holds. In other words,*

$$\sum_{i=1}^{k} a_i \mu_i + a_0 = 0, \qquad \sum_{i=1}^{k} a_i 2i \mu_i = 0. \tag{2.3}$$

2. *There exist constants* $\alpha_2, \ldots, \alpha_k \in \mathbf{R}$ *such that* (2.1).

Proof. In this proof, we use $\psi_i(1) = \psi_i'(1) = 0$ for $i \geqslant 2$ (see [2, Lemma 4.1]).

We show that Assertion 1 implies Assertion 2. Since W_i is an even polynomial with degree $2i$, there exists a unique expansion of the form

$$T(x) = \sum_{i=2}^{k} \alpha_i W_i(x) + \beta x^2 + \gamma.$$

We see that $\beta = \gamma = 0$ as follows.

$$\phi(x) = \sum_{i=2}^{k} \alpha_i \boldsymbol{E}[W_i(xN)] + \beta \boldsymbol{E}[(xN)^2] + \gamma = \sum_{i=2}^{k} \alpha_i \psi_i(x) + \beta x^2 + \gamma.$$

Since $\phi(1) = \phi'(1) = 0$ and $\psi_i(1) = \psi_i'(1) = 0$ for $i \geqslant 2$, it follows that $\beta + \gamma = 0$ and $2\beta = 0$ so $\beta = \gamma = 0$. Hence, Assertion 2 holds.

Next, we show that Assertion 2 implies Assertion 1. The assumption implies that $\phi(x) = \sum_{i=2}^{k} \alpha_i \psi_i(x)$. This expression and the identity $\psi_i(1) = \psi_i'(1) = 0$ yield Assertion 1. ■

Hereafter, we assume $\phi(1) = \phi'(1) = 0$. Then, as a result of Proposition 2.1, $a_0, \ldots, a_k$ in (2.2) and $\alpha_2, \ldots, \alpha_k$ in (2.1) have good relations. We examine an explicit formula for $\alpha_2, \ldots, \alpha_k$ by $a_0, \ldots, a_k$. More precisely, setting $c_i = (2i-1)i!(i-2)!$ for $i \geqslant 2$, demonstrate the next proposition, an analogue of [2, Proposition 4.1] demonstrated in a similar manner.

PROPOSITION 2.2. *For every $2 \leqslant i \leqslant k$,*

$$\alpha_i c_i = \frac{1}{2^{i-1}} \sum_{m=i}^{k} \frac{m! \kappa_i(m)}{(m-i)!} a_m \mu_m.$$

Here, $\{\kappa_i(m)\}_{m \geqslant i \geqslant 2}$ are defined by (1.1).

The next corollary follows immediately from Proposition 2.2. It will be used in Section 3 and play an important role in the proof of main theorem.

COROLLARY 2.1. *Let* $1 \leqslant l < k$ *and assume that* $a_m = 0$ *for all* $1 \leqslant m \leqslant l-1$. *Then, for every* $2 \leqslant i \leqslant k$,

$$\alpha_i c_i = \frac{1}{2^{i-1}} \sum_{m=l+1}^{k} \{\xi_i(m) - \xi_i(l)\} m \mu_m a_m.$$

Here, $\{\xi_i(m)\}_{i,m \geqslant 2}$ *are defined by* (1.2).

Proof. From Proposition 2.2, for all $2 \leqslant i \leqslant k$, it follows that

$$\alpha_i c_i 2^{i-1} = \sum_{m=i}^{k} \xi_i(m) m \mu_m a_m = \sum_{m=2}^{k} \xi_i(m) m \mu_m a_m = \sum_{m=l}^{k} \xi_i(m) m \mu_m a_m.$$

In the above, we used $\xi_i(m) = 0$ for $2 \leqslant m \leqslant i-1$ and $a_m = 0$ for $1 \leqslant m \leqslant l-1$. Since $\phi'(1) = 0$ (see (2.3)) and $a_m = 0$ for $1 \leqslant m \leqslant l-1$ imply

$$0 = \sum_{m=1}^{k} m \mu_m a_m = \sum_{m=l}^{k} m \mu_m a_m,$$

we have

$$\xi_i(l) l \mu_l a_l = - \sum_{m=l+1}^{k} \xi_i(l) m \mu_m a_m.$$

Substituting this equality into $\alpha_i c_i 2^{i-1}$ yields the assertion. ∎

For readers' convenience, we provide a proof of Proposition 2.2. For details, see [2, Appendix A]. We introduce even polynomials Q and R with degree $2(k-1) \geqslant 2$ as

$$Q(x) = \sum_{i=2}^{k} \alpha_i (2i-1) H_i(x) H_{i-2}(x), \qquad R(x) = \sum_{i=1}^{k} a_i \mu_i \sum_{r=0}^{i-1} \frac{x^{2r}}{\mu_r}.$$

Then, $\Phi[Q] = T = \Phi[R]$ from direct computation, and $Q = R$ as a consequence of [2, Lemma A.2].

LEMMA 2.2. *For all* $1 \leqslant n \leqslant k-1$,

$$\int_{-\infty}^{\infty} Q(x)H_{2n}(x)w(x)\,dx = \frac{(2n)!}{(n-1)!(n+1)!}\sum_{m=n+1}^{k}\frac{\alpha_m c_m}{(m-(n+1))!},$$
$$\int_{-\infty}^{\infty} R(x)H_{2n}(x)w(x)\,dx = 2^n \sum_{m=n+1}^{k} a_m\mu_m \sum_{r=n}^{m-1}\frac{r!}{(r-n)!}.$$

Proof. We refer to [2, Lemma A.1]. The product formula and the orthogonality of the Hermite polynomials imply that

$$\int_{-\infty}^{\infty} H_i(x)H_{i-2}(x)H_{2n}(x)w(x)\,dx = \frac{(2n)!}{(n+1)!(n-1)!}\frac{i!(i-2)!}{(i-(n+1))!}1_{n+1\leqslant i}.$$

Hence, the first equality holds. The second assertion follows from

$$\frac{1}{\mu_r}\int_{-\infty}^{\infty} x^{2r}H_{2n}(x)w(x)\,dx = \frac{1}{\mu_r}\frac{(2r)!}{2^{r-n}(i-n)!}1_{n\leqslant r} = \frac{2^n r!}{(r-n)!}1_{n\leqslant r}.$$

This completes the proof. ∎

Proof of Proposition 2.2.. Set

$$f(x) = \sum_{i=2}^{k}\frac{\alpha_i c_i}{(i-1)!}x^{i-1}, \qquad g(x) = \sum_{i=1}^{k} a_i\mu_i \sum_{r=0}^{i-1} x^r.$$

Since $f^{(n)}(0) = \alpha_{n+1}c_{n+1}$ holds for every $1 \leqslant n \leqslant k-1$, we look for other expressions of $f^{(n)}(0)$. First, we show that

$$f(1-2x) - f(1) = \int_0^1 (1-u)^{-\frac{1}{2}}u^{-1}\frac{d}{du}\{ug(1-ux)\}\,du \tag{2.4}$$

and next we consider nth derivatives of both sides at $x = 1/2$. We obtain the assertion as a consequence.

For every $n \in \mathbf{N}$,

$$f^{(n)}(x) = \sum_{i=n+1}^{k}\frac{\alpha_i c_i}{(i-(n+1))!}x^{i-(n+1)},$$
$$g^{(n)}(x) = \sum_{i=n+1}^{k} a_i\mu_i \sum_{r=n}^{i-1}\frac{r!}{(r-n)!}x^{r-n}.$$

Combining Lemma 2.2 with the above yields

$$\frac{(2n)!}{(n-1)!(n+1)!}f^{(n)}(1) = 2^n g^{(n)}(1).$$

Since $\frac{(2n)!}{(n-1)!(n+1)!} = \frac{2^{2n}}{n+1}\frac{1}{B\left(\frac{1}{2},n\right)}$ as a consequence of [10, (5.4.6), (5.5.5) and (5.12.1)],

$$f^{(n)}(1) = \frac{n+1}{2^n} B\left(\frac{1}{2}, n\right) g^{(n)}(1).$$

By the above,

$$\begin{aligned} f(1-2x) - f(1) &= \sum_{n=1}^{k-1} \frac{f^{(n)}(1)}{n!}(-2x)^n \\ &= \sum_{n=1}^{k-1} \frac{1}{n!}\left(\frac{n+1}{2^n}\int_0^1 (1-u)^{-\frac{1}{2}} u^{n-1}\, du\right) g^{(n)}(1)(-2)^n x^n \\ &= \int_0^1 (1-u)^{-\frac{1}{2}}\left(\sum_{n=1}^{k-1} \frac{g^{(n)}(1)}{n!}(n+1)u^{n-1}\,(-1)^n x^n\right) du. \end{aligned}$$

Here, noting that $g(1) = \sum_{i=1}^{k} a_i \mu_i i = 0$ and applying the Taylor formula to $g(1-ux)$ yield

$$\begin{aligned} \frac{d}{du}\{ug(1-ux)\} &= \frac{d}{du}\left\{u\sum_{n=1}^{k-1}\frac{g^{(n)}(1)}{n!}(-ux)^n\right\} \\ &= \sum_{n=1}^{k-1}\frac{g^{(n)}(1)}{n!}(n+1)(-ux)^n. \end{aligned}$$

The two equalities imply (2.4).

Next, we consider the nth derivative of (2.4) at $x = 1/2$. Substituting

$$\begin{aligned} \frac{d}{du}\{ug(1-ux)\} &= \frac{d}{du}\sum_{m=1}^{k}\alpha_m\mu_m u\frac{1-(1-ux)^m}{1-(1-ux)} \\ &= \sum_{m=1}^{k} a_m\mu_m \cdot m(1-ux)^{m-1}, \end{aligned}$$

into (2.4) yields

$$f(1-2x) - f(1) = \sum_{m=1}^{k} a_m\mu_m \cdot m\int_0^1 (1-u)^{-\frac{1}{2}}u^{-1}(1-ux)^{m-1}\, du.$$

Furthermore, for every $n \leqslant m-1$,

$$\begin{aligned} &\frac{d^n}{dx^n}(1-u)^{-\frac{1}{2}}u^{-1}(1-ux)^{m-1} \\ &\qquad = \frac{(m-1)!}{(m-1-n)!}(1-u)^{-\frac{1}{2}}u^{-1}(-u)^n(1-ux)^{m-1-n} \end{aligned}$$

and

$$\sup_{x\in(1/4,3/4)} |(\text{RHS of the above})| \leqslant \frac{(m-1)!}{(m-1-n)!}(1-u)^{-\frac{1}{2}}u^{n-1}\left(1-\frac{u}{4}\right)^{m-1-n}.$$

Hence, by Lebesgue's derivative theorem,

$$(-2)^n f^{(n)}(0) = (-1)^n \sum_{m=n+1}^{k} a_m\mu_m \cdot \frac{m(m-1)!}{(m-1-n)!}\kappa_{n+1}(m),$$

where $\kappa_{n+1}(m)$ is a constant defined by (1.1). This is the conclusion of the proposition. ■

3. EXPRESSION OF T AND PROOF OF MAIN THEOREMS

In this section, we consider the positivity of $\{\alpha_i\}_{2\leqslant i\leqslant k}$ for several cases and present our main theorem. Set $k \geqslant 2$, and write $\tilde{\alpha}_i = \tilde{\alpha}_i(k) = \frac{\alpha_i c_i 2^{i-1}}{k\mu_k}$. From Corollary 2.1, we have

$$\tilde{\alpha}_i = \tilde{\alpha}_i(k) = \sum_{m=l+1}^{k} \{\xi_i(m) - \xi_i(l)\}\frac{m\mu_m}{k\mu_k}a_m. \tag{3.1}$$

3.1. $T(x) = x^{2k} + a_l x^{2l} + a_0$**.** Consider an even polynomial $T(x) = x^{2k} + a_l x^{2l} + a_0$ for $2k > 2l \geqslant 2$. The function $\phi(x) = \boldsymbol{E}[T(xN)]$ satisfies $\phi(1) = \phi'(1) = 0$ if and only if $a_l = -\frac{k\mu_k}{l\mu_l}$ and $a_0 = \left(\frac{k}{l} - 1\right)\mu_k$. In this subsection, we show the next proposition.

PROPOSITION 3.1. *The polynomial $T(x) = x^{2k} + a_l x^{2l} + a_0$ is expressed as* (2.1) *with positive coefficients $\alpha_2, \ldots, \alpha_k$ if and only if $k > l = 1$ or $k > l = 2$ or* $(k,l) = (4,3), (5,3)$.

P r o o f. Substituting $a_{l+1} = \cdots = a_{k-1} = 0$ and $a_k = 1$ into (3.1) we have $\tilde{\alpha}_i(k) = \xi_i(k) - \xi_i(l)$ for every $2 \leqslant i \leqslant k$. We should recall $\xi_i(l) \neq 0$ only for $2 \leqslant i \leqslant l$ due to (1.2).

We can obtain the "only if" part of the assertion by focusing on α_2. If $l = 3$, then

$$\tilde{\alpha}_2(k) = \xi_2(k) - \xi_2(3) \leqslant \xi_2(6) - \xi_2(3) = \frac{166}{63} - \frac{8}{3} < 0$$

for all $k \geqslant 6$ (see Proposition 5.2 Assertion 3 and Proposition 5.1). If $l \geqslant 4$, then $\tilde{\alpha}_2(k) = \xi_2(k) - \xi_2(l) < 0$ for all $k > l$ (see Proposition 5.2 Assertion 3). This yields the "only if" part.

Now, we show the "if" part. If $l = 1$, then $\tilde{\alpha}_i(k) = \xi_i(k) > 0$ for all $2 \leqslant i \leqslant k$. If $l = 2$, then

$$\tilde{\alpha}_i(k) = \begin{cases} \xi_i(k), & 3 \leqslant i \leqslant k, \\ \xi_2(k) - \xi_2(2), & i = 2. \end{cases}$$

Hence we have $\tilde{\alpha}_i(k) > 0$ for $i = 2$ (resp. $3 \leqslant i \leqslant k$) due to Proposition 5.2 Assertion 1 (resp. $\xi_i(k) > 0$). If $l = 3$, then $\tilde{\alpha}_i(k) = \xi_i(k) - \xi_i(3) > 0$ for $k = 4, 5$ due to the same reason with the case $l = 2$. This completes the proof. ■

3.2. $T(x) = x^{2k} + a_7 x^{14} + a_6 x^{12} + a_3 x^6 + a_0$**.**

Select a natural number $k \geqslant 8$, and set $T(x) = x^{2k} + ax^{14} + bx^{12} + a_3 x^6 + a_0$. Here, a_3 and a_0 is chosen to ensure that $\phi(1) = \phi'(1) = 0$. Then, from Corollary 2.1,

$$\tilde{\alpha}_i(k) = \{\xi_i(k) - \xi_i(3)\} + \{\xi_i(7) - \xi_i(3)\} \frac{7\mu_7}{k\mu_k} a + \{\xi_i(6) - \xi_i(3)\} \frac{6\mu_6}{k\mu_k} b.$$

In what follows, we consider the case $\alpha_7 = 0$ and $\alpha_6 = 0$ and show that $\alpha_k, \dots, \alpha_8$, α_5, α_4, α_3, α_2 are positive. In this case, it is necessary that

$$\frac{7\mu_7}{k\mu_k} a = -\frac{\xi_7(k)}{\xi_7(7)}, \qquad \frac{6\mu_6}{k\mu_k} b = \frac{\xi_7(k)}{\xi_6(7)} \frac{\xi_6(7)}{\xi_6(6)} - \frac{\xi_6(k)}{\xi_6(6)}.$$

Hence,

$$\begin{aligned} \tilde{\alpha}_i(k) &= \{\xi_i(k) - \xi_i(3)\} + \{\xi_i(7) - \xi_i(3)\} \left(-\frac{\xi_7(k)}{\xi_7(7)} \right) \\ &\quad + \{\xi_i(6) - \xi_i(3)\} \left(\frac{\xi_7(k)}{\xi_7(7)} \frac{\xi_6(7)}{\xi_6(6)} - \frac{\xi_6(k)}{\xi_6(6)} \right) \\ &= \xi_i(k) + \left[-\frac{\xi_i(7) - \xi_i(3)}{\xi_7(7)} + \frac{\xi_6(7)(\xi_i(6) - \xi_i(3))}{\xi_7(7)\xi_6(6)} \right] \xi_7(k) \\ &\quad + \left[-\frac{\xi_i(6) - \xi_i(3)}{\xi_6(6)} \right] \xi_6(k) - \xi_i(3). \end{aligned}$$

Since $\tilde{\alpha}_i(k) = \xi_i(k) > 0$ for $8 \leqslant i \leqslant k$, we consider $\tilde{\alpha}_i(k)$ for $i = 2, 3, 4, 5$.

14 Jun 2022 04:10:23 CEST
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LEMMA 3.1. *Let $k \geqslant 8$. Then, $\tilde{\alpha}_i(k) > 0$ for any $i = 2, 3, 4, 5$.*

P r o o f. For $8 \leqslant k < 3000$, the assertion follows by direct computation. We use Mathematica for this calculation. For the source code used, see Listing 1. Next we show the assertion for $k \geqslant 3000$. As a consequence of Proposition 5.3, $\{\xi_i(k)\}_{k=2}^{\infty}$ converges to $2^{i-1}(i-2)!$ as $k \to \infty$, and we estimate the error of this convergence. Setting $r_i(k) = \xi_i(k) - 2^{i-1}(i-2)!$ yields

$$\begin{aligned}\tilde{\alpha}_i(k) &= \begin{cases} \xi_2(k) + \frac{1}{3072}\xi_6(k) + \frac{1}{15360}\xi_7(k) - \frac{8}{3}, & i = 2, \\ \xi_3(k) - \frac{29}{768}\xi_6(k) + \frac{121}{7680}\xi_7(k) - \frac{8}{3}, & i = 3, \\ \xi_4(k) - \frac{7}{32}\xi_6(k) + \frac{1}{12}\xi_7(k), & i = 4, \\ \xi_5(k) - \frac{5}{8}\xi_6(k) + \frac{29}{160}\xi_7(k), & i = 5, \end{cases} \\ &= \begin{cases} r_2(k) + \frac{1}{3072}r_6(k) + \frac{1}{15360}r_7(k) + \frac{1}{12}, & i = 2, \\ r_3(k) - \frac{29}{768}r_6(k) + \frac{121}{7680}r_7(k) + \frac{280}{3}, & i = 3, \\ r_4(k) - \frac{7}{32}r_6(k) + \frac{1}{12}r_7(k) + 488, & i = 4, \\ r_5(k) - \frac{5}{8}r_6(k) + \frac{29}{160}r_7(k) + 1008, & i = 5, \end{cases}\end{aligned}$$

which implies

$$\begin{aligned}\tilde{\alpha}_i(k) &\geqslant \begin{cases} -(|r_2(k)| + \frac{1}{3072}|r_6(k)| + \frac{1}{15360}|r_7(k)|) + \frac{1}{12}, & i = 2, \\ -(|r_3(k)| + \frac{29}{768}|r_6(k)| + \frac{121}{7680}|r_7(k)|) + \frac{280}{3}, & i = 3, \\ -(|r_4(k)| + \frac{7}{32}|r_6(k)| + \frac{1}{12}|r_7(k)|) + 488, & i = 4, \\ -(|r_5(k)| + \frac{5}{8}|r_6(k)| + \frac{29}{160}|r_7(k)|) + 1008, & i = 5, \end{cases} \\ &> 0.\end{aligned}$$

The last inequality follows from Proposition 5.3. This completes the proof. ■

Listing 1. Proof of Lemma 3.1

```
kappa[i_,m_]:=Beta[i-1,1/2]*Hypergeometric2F1[i-1,-(m-i)
    ,i-1/2,1/2];
xi[i_,m_]:=(m-1)!/(m-i)!*kappa[i,m]/;m>=i;
xi[i_,m_]:=0/;m<i;
tildeA[i_,k_]:=xi[i,k]+(-(xi[i,7]-xi[i,3])/xi[7,7]+xi
    [6,7]*(xi[i,6]-xi[i,3])/(xi[7,7]*xi[6,6]))*xi[7,k
    ]+(-(xi[i,6]-xi[i,3])/xi[6,6])*xi[6,k]-xi[i,3]

(*Are tildeA[i,k]>0 for i=2,3,4,5 and k<3001?*)
Table[Map[tildeA[#,k]&,{2,3,4,5}],{k,8,3000}];
AllTrue[Flatten[%],Positive]

(*Are tildeA[i,k]>0 for i=2,3,4,5 and k>3000?*)
Map[tildeA[#,k]&,{2,3,4,5}]/.Array[xi[#,k]->2^(#-1)
    *(#-2)!+r[#,k]&,7,2]//Expand;
```

```
CoefficientArrays[%,Map[r[#,k]&,Range[2,16]]]//Normal;
Map[Abs,%];
%[[1]]+%[[2]].Map[-r[#,k]&,Range[2,16]];
%/.MapThread[#1->2^#2&,{Map[r[#,k]&,Range
    [2,16]],{-18,-9,-5,-2,2,6,10,14,18,23,28,32,37,42,47}}];

AllTrue[Flatten[%],Positive]
```

3.3. $T(x) = x^{2k} + a_{16}x^{32} + a_{15}x^{30} + a_9x^{18} + a_6x^{12} + a_3x^6 + a_0$**.**

Select a natural number $k \geqslant 17$, and set $T(x) = x^{2k} + ax^{32} + bx^{30} + cx^{18} + dx^{12} + a_3x^6 + a_0$. Here, a_3 and a_0 are chosen to ensure that $\phi(1) = \phi'(1) = 0$. Then, from Corollary 2.1,

$$\begin{aligned}\tilde{\alpha}_i(k) = \{\xi_i(k) - \xi_i(3)\} + \{\xi_i(16) - \xi_i(3)\}\frac{16\mu_{16}}{k\mu_k}a + \{\xi_i(15) - \xi_i(3)\}\frac{15\mu_{15}}{k\mu_k}b \\ + \{\xi_i(9) - \xi_i(3)\}\frac{9\mu_9}{k\mu_k}c + \{\xi_i(6) - \xi_i(3)\}\frac{6\mu_6}{k\mu_k}d.\end{aligned}$$

Here, we choose a, b, c, d to ensure that $\alpha_i = 0$ for all $i \in \{6, 7, 12, 13\}$. It follows from this expression that $\tilde{\alpha}_i(k) = \xi_i(k) > 0$ for any $17 \leqslant i \leqslant k$, and we can demonstrate the next lemma in the same manner as Lemma 3.1.

LEMMA 3.2. *Let* $k \geqslant 17$*. For every* $i \in \{2, \ldots, 16\} \setminus \{6, 7, 12, 13\}$*, we have* $\tilde{\alpha}_i(k) > 0$.

3.4. Proof of main theorem.

P r o o f o f T h e o r e m 1.1.. Proposition 3.1 implies that $T(x) = x^{2k} - \frac{k\mu_k}{l\mu_l}x^{2l} + \left(\frac{k}{l} - 1\right)\mu_k$ can be written as (2.1) with positive $\alpha_2, \ldots, \alpha_k$ for $k > l = 1$ or $k > l = 2$ or $(k, l) = (4, 3), (5, 3)$. Combining this fact with Lemma 2.1 yields Assertions 1, 2 and 3.

In the same manner as the above, combining Lemmas 2.1, 3.1 and 3.2 yields Assertions 4 and 5. ■

4. DISCUSSION ON MAIN THEOREM

After [9], [2] and the present paper, the next conjecture is still open:

CONJECTURE 4.1. *Let* $\mathcal{I} = \{2l, 2k\}$ *for* $6 \leqslant 2l < 2k$. *Then,* (CL) *and* (CM) *for* $\mathcal{I}$ *are equivalent.*

As stated in Section 1, we cannot give a proof to Conjecture 4.1 by the method in [2] (see Proposition 3.1). Here we reconsider this fact intensely. Write $\mathcal{I} = \{2l_1, \dots, 2l_M, 2k\}$ with $2 \leqslant 2l_1 < \cdots < 2l_M < 2k$.

Since our proof of "(CM) $\Rightarrow$ (CL)" relies on Lemma 2.1, T should be expressed as (2.1) and $\phi(1) = \phi'(1) = 0$ should be satisfied (see Proposition 2.1). Note that the conditions $\phi(1) = \phi'(1) = 0$ give a system of two linear equations (2.3) with k unknowns $a_0, a_1, \dots, a_{k-1}$ ($a_k = 1$ because T is monic). Since we should obtain $\boldsymbol{E}[T(Z_n)] \to 0$ from convergence of moments in $\mathcal{I}$, we should set $a_i = 0$ for $i \in \{0, \dots, k-1\} \setminus \{l_1, \dots, l_M\}$. Hence we have two linear equations with $(M+1)$ unknowns $a_0, a_{l_1}, \dots, a_{l_M}$. Of course, we may be able to obtain a unique solution only for $M = 1$ and we have choices in $a_0, a_1, \dots, a_{k-1}$ for $M \geqslant 2$. After finding $a_0, a_1, \dots, a_{k-1}$, we can calculate $\alpha_2, \dots, \alpha_{k-1}$ ($\alpha_k = 1$ since T and W_k are monic) from $a_0, a_1, \dots, a_{k-1}$ due to Proposition 2.2

For the case $M = 1$ (that is, $\mathcal{I} = \{2l, 2k\}$ with $2 \leqslant 2l < 2k$), $a_0, a_1, \dots, a_{k-1}$ are uniquely determine and do so $\alpha_2, \dots, \alpha_{k-1}$. Furthermore, for some cases (*e.g.* $\mathcal{I} = \{6, 12\}$), we have $\alpha_2 < 0$ and cannot show the equivalence of (CM) and (CL). If $M \geqslant 2$, $a_0, a_1, \dots, a_{k-1}$ are not uniquely determine and does not so $\alpha_2, \dots, \alpha_{k-1}$. Hence we may be able to choose $a_0, a_1, \dots, a_{k-1}$ so that $\alpha_2, \dots, \alpha_{k-1}$ are nonnegative.

From the observation above, we found $\mathcal{I}$ in Assertions 4 and 5 of Theorem 1.1 so that $\alpha_2, \dots, \alpha_{k-1}$ are nonnegative. This procedure needs numerical calculation (Listing 2 is the source code in Mathematica, which is used to find $\mathcal{I}$). Other than $\mathcal{I}$ in Assertions 4 and 5, we observe the next examples:

· The largest number of $\mathcal{I}$ in Assertions 4 ($\mathcal{I} = \{6, 12, 14, 2k\}$) and 5 ($\mathcal{I} = \{6, 12, 18, 30, 32, 2k\}$) of Theorem 1.1 is arbitrary, however it does not holds in general. For example, all of $\alpha_2, \dots, \alpha_k$ are (resp. are not) nonnegative for $\mathcal{I} = \{6,$ $12,$ $16,$ $2k\}$ with $18 \leqslant 2k \leqslant 40$ (resp. $42 \leqslant 2k \leqslant 100$)

· The smallest number of $\mathcal{I}$ may be arbitrary. For example, $\alpha_2, \dots, \alpha_k$ are nonnegative for $\mathcal{I} = \{8,$ $12,$ $14,$ $18,$ $26,$ $32,$ $34,$ $36,$ $38,$ $1000\}$, $\mathcal{I} = \{8,$ $12,$ $14,$ $18,$ $28,$ $30,$ $34,$ $36,$ $38,$ $1000\}$ and $\mathcal{I} = \{10,$ $14,$ $16,$ $18,$ $24,$ $28,$ $30,$ $32,$ $34,$ $36,$ $38,$ $1000\}$.

However we can not describe a rule which the nonnegativity of $\alpha_2, \dots, \alpha_{k-1}$ obeys clearly. These examples suggest the following conjecture, which is a relaxed version of Conjecture 4.1.

CONJECTURE 4.2. *Let* $2l_1 \geqslant 8$ *be an arbitrary even integer, and choose* $(M-1)$ *suitable integers* $l_2, \dots, l_M$ *with* $l_1 < l_2 < \cdots < l_M$*, where* $M \geqslant 1$*. Let* $2k \geqslant 2(l_M+1)$ *be an arbitrary even integer. Set* $\mathcal{I} = \{2l_1, \dots, 2l_M, 2k\}$*. Then,* (CL) *and* (CM) *for* $\mathcal{I}$ *are equivalent.*

Of course the cases $2l_1 = 2, 4, 6$ are obtained in Theorem 1.1 and this conjecture might be shown by the method in [2].

Listing 2. How to find examples

```
he[k_,x_]:=he[k,x]=2^(-k/2)HermiteH[k,x/Sqrt[2]];
(* Define w *)
w[l_,x_]:=w[l,x]=Module[{coeffList},coeffList=
   CoefficientList[he[l/2,t]*he[l/2-2,t],t];
(2*l/2-1)*({0,0}~Join~(coeffList*Map[1/#&,Range[Length[
   coeffList]]])-(coeffList~Join~{0,0})).Map[x^#&,Range
   [0,l]]]//Expand;
(* Set list={l,...,k}. Consider an identity with respect
    to x so that a_0+a_lx^l+...+a_kx^k = b_4 w[4,x]+...+
    b_k w[k,x] *)
equalities[list_]:=equalities[list]=Map[#==0&,
   CoefficientList[Plus@@Map[Subscript[a, #]*x^#&,{0}~
   Join~list]-Plus@@Map[Subscript[b, #]*w[#,x]&,Range[4,
   Last[list],2]],x^2]];
(* Find example a_k,...,a_1,b_1,...,b_k so that a_k=1,
   b_k=1, b_k,...,b_4 are nonnegative *)
example[list_]:=FindInstance[Join[{Subscript[a, Last[
   list]]==1,Subscript[b, Last[list]]==1},Map[Subscript[
   b, #]>=0&,Range[4,Last[list],2]],equalities[list]],
   Map[Subscript[a, #]&,{0}~Join~list]~Join~Map[
   Subscript[b, #]&,Range[4,Last[list],2]]]
list = {6, 12, 16, 100};
```

```
equalities[list]
example[list]
```

5. APPENDIX

In this section, we study properties of $\xi_i(k)$ defined by (1.2). First, we obtain the next proposition by direct calculation.

PROPOSITION 5.1. *The first few exact values of* $\{\xi_i(m)\}_{m \geqslant i \geqslant 2}$ *are*

$$\xi_2(2) = 2, \quad \xi_2(3) = \frac{8}{3} = 2.66\ldots, \quad \xi_2(4) = \frac{14}{5} = 2.8,$$
$$\xi_2(5) = \frac{96}{35} = 2.74\ldots, \quad \xi_2(6) = \frac{166}{63} = 2.63\ldots, \quad \xi_2(7) = \frac{584}{231} = 2.52\ldots,$$
$$\xi_3(3) = \frac{8}{3} = 2.66\ldots, \quad \xi_3(4) = \frac{24}{5} = 4.8, \quad \xi_3(5) = \frac{208}{35} = 5.94\ldots.$$

In addition, we can obtain more information regarding $\xi_i(k)$ by studying the hypergeometric function, as follows.

PROPOSITION 5.2. *Let* $i = 2$. *Then,*

1. $\xi_2(2) = 2$ *and* $2 < \xi_2(k)$ *for* $k \geqslant 3$.
2. $\xi_2(2) < \xi_2(3) < \xi_2(4)$.
3. $\xi_2(4) > \xi_2(5) > \xi_2(6) > \cdots$.

PROPOSITION 5.3. *For every* $i \geqslant 2$, $\{\xi_i(k)\}_{k=2}^{\infty}$ *converges to* $2^{i-1}(i-1)!$ *as* $k \to \infty$. *In addition, for all* $2 \leqslant i \leqslant 16$ *and* $k \geqslant 3000$, $r_i(k) = \xi_i(k) - 2^{i-1}(i-2)!$ *satisfies*

$$|r_i(k)| \leqslant 2^{p_i}$$

The values of p_i *are as listed in Table* 1.

TABLE 1. Definition of p_i

i	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
p_i	-18	-9	-5	-2	2	6	10	14	18	23	28	32	37	42	47

5.1. Proof of Proposition 5.2.

In this subsection, we demonstrate Proposition 5.2.

P r o o f o f P r o p s o t i o n 5.2.. First,

$$(5.1)\qquad (1-u)^{-\frac{1}{2}} \leqslant \left(1-\frac{u}{2}\right)^{-2} \qquad \text{for} \quad 0 \leqslant u \leqslant 1/2,$$

$$(5.2)\qquad (1-u)^{-\frac{1}{2}} \geqslant \left(1-\frac{u}{2}\right)^{-1} \qquad \text{for} \quad 0 \leqslant u \leqslant 1.$$

Then, Assertions (5.2) and (5.2) for $k = 2, 3$ follow from Proposition 5.1. For $k \geqslant 4$, it follows from (5.2) that

$$\xi_2(k) \geqslant (k-1)\int_0^1 \left(1-\frac{u}{2}\right)^{-1}\left(1-\frac{u}{2}\right)^{k-2} du = 2 + \frac{2}{k-2}\left(1-\frac{k-1}{2^{k-2}}\right).$$

Since the last term is positive for $k \geqslant 4$, $\xi_2(k) > 2$ for $k \geqslant 4$.

Now, we demonstrate (5.2). Since $\xi_2(4) > \xi_2(5) > \xi_2(6) > \xi_2(7)$ from Proposition 5.1, we show that $\xi_2(k) > \xi_2(k+1)$ for $k \geqslant 7$. If we set $\delta_k = \xi_2(k+1) - \xi_2(k)$, then

$$\delta_k = \int_0^1 (1-u)^{-\frac{1}{2}}\left(1-\frac{u}{2}\right)^{k-2}\left(1-\frac{k}{2}u\right) du.$$

Noting that $1-\frac{k}{2}u \gtreqless 0$ for $u \lesseqgtr \frac{2}{k}$ and using estimates (5.1) and (5.2) yield

$$\begin{aligned}\delta_k &\leqslant \int_0^{2/k}\left(1-\frac{u}{2}\right)^{k-4}\left(1-\frac{k}{2}u\right) du + \int_{2/k}^1 \left(1-\frac{u}{2}\right)^{k-3}\left(1-\frac{k}{2}u\right) du \\ &= 2\Bigg[-\frac{2}{(k-3)(k-2)} \\ &\qquad + \frac{3k^2}{(k-3)(k-2)(k-1)^2}\left(1-\frac{1}{k}\right)^k + \frac{2(k^2-2k+2)}{(k-2)(k-1)}\left(\frac{1}{2}\right)^k\Bigg].\end{aligned}$$

Here, the fact that (i) $k \mapsto \left(1-\frac{1}{k}\right)^k$ is increasing and converges to $1/e(< 7/19)$ as $k \to \infty$, that (ii) $k \mapsto \frac{k^2}{(k-1)^2}$ is decreasing, and that (iii) $k \mapsto \frac{2(k^2-2k+2)}{(k-2)(k-1)}$ is decreasing, yields that for $k \geqslant 7$,

$$
\begin{aligned}
\delta_k \leqslant 2 \Bigg[&-\frac{2}{(k-3)(k-2)} \\
&\qquad + \frac{3\cdot 7^2}{(k-3)(k-2)(7-1)^2}\frac{7}{19} + \frac{2(7^2-2\cdot 7+2)}{(7-2)(7-1)}\left(\frac{1}{2}\right)^k \Bigg] \\
= 2 \Bigg[&-\frac{113}{228}\frac{1}{(k-3)(k-2)} + \frac{37}{15}\left(\frac{1}{2}\right)^k \Bigg] \\
= -&\frac{2\cdot 37}{15(k-3)(k-2)2^k}\left[\frac{15}{37}\frac{113}{228}2^k - (k-3)(k-2)\right].
\end{aligned}
$$

Since the last term is negative if $k \geqslant 7$, the assertion is demonstrated. This completes the proof. ∎

5.2. Proof of Proposition 5.3.

Now, we examine the hypergeometric function $F(a,b,c;z)$.

LEMMA 5.1 (Watson's lemma, [4, Proposition 2.1]). *Let $\phi\colon (0,1) \to \mathbf{R}$ be an integrable function on $(0,1)$. Assume that there exist constants $\sigma > 0$ and $0 < \rho < 1$ and a smooth function ψ on $[0,\rho]$ such that $\phi(s) = \psi(s)s^{\sigma-1}$. Then,*

$$
\begin{aligned}
&\left| \int_0^1 \phi(s)e^{-\lambda s}\,ds - \frac{\psi(0)\Gamma(\sigma)}{\lambda^\sigma} \right| \\
&\qquad \leqslant \frac{2|\psi(0)|}{\rho\lambda e^{\rho\lambda}} + \frac{(\max_{0\leqslant s\leqslant\rho}|\psi'(s)|)\Gamma(\sigma+1)}{\lambda^{\sigma+1}} + \frac{1}{e^{\rho\lambda}}\int_\rho^1 |\phi(s)|\,ds
\end{aligned}
$$

for any $\lambda \geqslant 2\sigma/\rho$.

Proof. Following a proof in [4] and using the monotonicity of the function $t \mapsto e^{-\frac{t}{2}}t^{\sigma-1}$ on $[2\sigma,\infty)$ in estimating an incomplete Gamma function yield the estimate. ∎

LEMMA 5.2. *Let $a \geqslant 1$, $0 < c-a < 1$, and $0 < z < 1$. Then,*

$$
\left| B(a,c-a)F(a,-b,c;z) - \frac{\Gamma(a)}{z^a(b+1)^a} \right| \leqslant M_{a,c;z}(-(b+1)\log(1-z))
$$

for any $-(b+1)\log(1-z) > 2a/\rho$. *Here,* $0<\rho<1$ *is an arbitrary constant and* $M_{a,c;z}$ *is defined as*

$$M_{a,c;z}(\lambda) = \frac{1}{(1-z)^{a+1}}\left(\frac{2}{\rho\lambda e^{\rho\lambda}} + \frac{(1-\rho)^{c-a-2}\Gamma(a+2)}{\lambda^{a+1}} + \frac{B(a,c-a)}{e^{\rho\lambda}}\right).$$

Proof. We expand $B(a,c-a)F(a,-b,c;z) = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-zu)^b\,du$ with respect to $b+1$ making use of Lemma 5.1.

Set $v = \frac{\log(1-zu)}{\log(1-z)}$, $\xi = \frac{-\log(1-z)}{z}$, $\eta = \frac{-(1-z)\log(1-z)}{z}$, and $h(w) = \frac{e^w-1}{w}$. Then, the fact that

$$\frac{u}{v} = \xi h(v\log(1-z)), \quad \frac{1-u}{1-v} = \eta h((v-1)\log(1-z)), \quad 1-zu = e^{v\log(1-z)},$$

yields

$$\begin{aligned} & u^{a-1}(1-u)^{c-a-1}(1-zu)^b \\ &\qquad = \left(\frac{u}{v}\right)^{a-1}\left(\frac{1-u}{1-v}\right)^{c-a-1} v^{a-1}(1-v)^{c-a-1}(1-zu)^b \\ &\qquad = \xi^{-1}\phi(v)e^{vb\log(1-z)} \\ &\qquad = \xi^{-1}\psi(v)v^{a-1}e^{vb\log(1-z)}, \end{aligned}$$

where

$$\phi(v) = \psi(v)v^{a-1}, \quad \psi(v) = Kg(v)(1-v)^{c-a-1},$$
$$K = \xi^a\eta^{c-a-1}, \quad g(v) = h(v\log(1-z))^{a-1}h((v-1)\log(1-z))^{c-a-1}.$$

Combining this with $\frac{du}{dv} = \xi e^{v\log(1-z)}$ and writing $\lambda = -(b+1)\log(1-z)$ yield

$$B(a,c-a)F(a,-b,c;z) = \int_0^1 \phi(v)e^{-\lambda v}\,dv.$$

In what follows, we expand the integral above with respect to λ making use of Lemma 5.1. Here, we list properties of h, as follows.

- h is strictly increasing and positive;
- $h(\log(1-z)) = \xi^{-1}$, $h(0)=1$ and $h(-\log(1-z)) = \eta^{-1}$;
- h'/h is strictly increasing and $0 < (h'/h)(w) < 1$ for $w \in \mathbf{R}$;
- $(h'/h)(0) = 1/2$ and $(h'/h)(-\log(1-z)) = 1/z + 1/\log(1-z)$;
- $0 < (h'/h)'(w) \leqslant |(h'/h)'(0)| = 1/12$ for $w\in\mathbf{R}$.

First, $\psi(0) = Kg(0) = Kh(0)^{a-1}h(-\log(1-z))^{c-a-1} = \xi^a$. From this and $0 < \xi \leqslant (1-z)^{-1}$, it follows that

$$\frac{\psi(0)\Gamma(a)}{\lambda^a} = \frac{\Gamma(a)}{z^a(b+1)^a}, \qquad |\psi(0)| \leqslant (1-z)^{-a}.$$

Next, we estimate $\max_{0\leqslant v\leqslant\rho} |\psi'(v)|$. Note that $g'(v) = g(v)f(v)$, where

$$f(v) = \left\{(a-1)\frac{h'(v\log(1-z))}{h(v\log(1-z))} \right.\\ \left. +(c-a-1)\frac{h'((v-1)\log(1-z))}{h((v-1)\log(1-z))}\right\}\log(1-z),$$

implying that

$$\psi'(v) = Kg(v)(1-v)^{c-a-2}\{f(v)(1-v) - (c-a-1)\}.$$

It follows from $a-1 \geqslant 0$, $-1 < c-a-1 < 0$, and the properties of h that

$$\max_{0\leqslant v\leqslant 1} |g(v)| \leqslant h(0)^{a-1}h(0)^{c-a-1} = 1,$$
$$\max_{0\leqslant v\leqslant 1} |f(v)| \leqslant \{|a-1| + |c-a-1|\}|\log(1-z)| \leqslant a|\log(1-z)|.$$

Hence, using $K = \xi^{c-1}(1-z)^{c-a-1} \leqslant (1-z)^{-a}$ $(c \geqslant 1)$ and $|\log(1-z)| \vee 1 \leqslant (1-z)^{-1}$ for $0 < z < 1$ yields

$$\max_{0\leqslant v\leqslant\rho} |\psi'(v)| \leqslant K(1-\rho)^{c-a-2}\{a|\log(1-z)| + 1\}\\ \leqslant (1-\rho)^{c-a-2}(a+1)(1-z)^{-(a+1)}.$$

and finally,

$$\int_\rho^1 |\phi(v)|\,dv \leqslant K\max_{0\leqslant s\leqslant 1}|g(s)|\int_0^1 v^{a-1}(1-v)^{c-a-1}\,dv \leqslant (1-z)^{-a}B(a,c-a).$$

Hence, the remainder is bounded by

$$\frac{1}{(1-z)^a}\left(\frac{2}{\rho\lambda e^{\rho\lambda}} + \frac{(1-\rho)^{c-a-2}(a+1)}{1-z}\frac{\Gamma(a+1)}{\lambda^{a+1}} + \frac{1}{e^{\rho\lambda}}B(a,c-a)\right).$$

This bound and $1 \leqslant (1-z)^{-1}$ complete the proof. ■

Proof of Proposition 5.3.. Recalling (1.1) and applying Lemma 5.2 with $a = i-1$, $b = k-i$ and $c = i - \frac{1}{2}$ yield $r_i(k) = \xi_i(k) - 2^{i-1}(i-2)! = r_{1,i}(k) + r_{2,i}(k)$, where

$$r_{1,i}(k) = \frac{(k-1)!}{(k-i)!}\left\{B\left(i-1, \frac{1}{2}\right) F\left(i-1, -(k-i), i-\frac{1}{2}; \frac{1}{2}\right) - \frac{2^{i-1}(i-2)!}{(k-i+1)^{i-1}}\right\},$$

$$r_{2,i}(k) = \frac{(k-1)!}{(k-i)!}\frac{2^{i-1}(i-2)!}{(k-i+1)^{i-1}} - 2^{i-1}(i-2)!.$$

Write $c_i(k) = \frac{(k-1)!}{(k-i)!(k-i+1)^{i-1}}$. Then, $c_i(k) = \prod_{\alpha=1}^{i-1}\left(1 + \frac{\alpha-1}{k-i+1}\right)$ is monotonically convergent to 1 as $k \to \infty$.

Setting $\lambda = (k-i+1)\log 2$ yields

$$|r_{1,i}(k)| \leqslant \frac{(k-1)!}{(k-i)!} M_{i-1,i-\frac{1}{2};\frac{1}{2}}(\lambda) = \frac{c_i(k)}{(\log 2)^{i-1}} \cdot \lambda^{i-1} M_{i-1,i-\frac{1}{2};\frac{1}{2}}(\lambda)$$

for all $\lambda \geqslant 2(i-1)/\rho$. Since, for every $n \geqslant 0$ the functions $\lambda \mapsto \lambda^n e^{-\rho\lambda}$ and $\lambda \mapsto \lambda^{-n}$ are decreasing on $[n/\rho, \infty)$, the function $[0, \infty) \ni \lambda \mapsto \lambda^{i-1} M_{i-1,i-\frac{1}{2};\frac{1}{2}}(\lambda)$ is decreasing on $[(i-1)/\rho, \infty)$ and converges to 0 as $\lambda \to \infty$. In addition,

$$0 \leqslant r_{2,i}(k) = 2^{i-1}(i-2)!\{c_i(k) - 1\}.$$

From the above, it follows that $\xi_i(k) \to 2^{i-1}(i-2)!$ as $k \to \infty$.

Choose $k_0 \in \mathbf{N}$ and $i \in \mathbf{N}$ to ensure that $(k_0 - i + 1)\log 2 \geqslant 2(i-1)/\rho$, in other words, $\frac{(k_0+1)\rho\log 2 + 2}{\rho \log 2 + 2} \geqslant i$. Then, for all $k \geqslant k_0$,

$$|r_i(k)| \leqslant \frac{c_i(k_0)}{(\log 2)^{i-1}} \cdot \lambda^{i-1} M_{i-1,i-\frac{1}{2};\frac{1}{2}}(\lambda)\Big|_{\lambda=(k_0-i+1)\log 2} + 2^{i-1}(i-2)!\{c_i(k_0) - 1\}.$$

Since $\frac{(k_0+1)\rho\log 2+2}{2+\rho\log 2} \geqslant 64$ for $k_0 = 3000$ and $\rho = 2^{-4}$, we can choose $i = 2, \ldots, 16$ and obtain the estimate of $|r_i(k)|$ for $i = 2, \ldots, 16$. ■

Acknowledgments. The author thanks anonymous referees for careful reading a manuscript of the paper.

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Kumamoto University
2-39-1 Kurokami Chuo-ku Kumamoto 860-8555
JAPAN
E-mail: naganuma@kumamoto-u.ac.jp

Received on ;
revised version on xx.xx.xxxx