Geometric Representation in the Theories of Pseudo-finite Fields

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Pseudo-Finite Fields

- Infinite models of the theory of finite fields are called pseudo-finite fields.
- (Ax) If a field is F perfect, PAC, and $Gal(F) = \hat{\mathbb{Z}}$ then it is pseudo-finite.
- Non-trivial ultra-products of finite fields are pseudo-finite.
- If (A, σ) is a model of ACFA then $Fix(\sigma)$ is pseudo-finite.
- $Fix(\sigma)$ is pseudo-finite for almost all σ is in $Gal(\mathbb{Q})$.

Generalizations of Psf Fields

- We call a field F *bounded*, if it has finitely many extensions of degree n for each $n \in \mathbb{N}$. In this case the absolute Galois group $Gal(F)$ of F is called *small*.
- The theory of a perfect PAC field F is determined by its absolute Galois group $Gal(F)$, and the algebraic closure of the prime field in F.
- A field is called quasi-finite if it is perfect and its absolute Galois group is isomorphic to $\hat{\mathbb{Z}}$.

Geometric Representation

Definition

We say a finite group G **is geometrically represented** in a theory T (with elimination of imaginaries), if there are $M_0 \leq A \leq B \leq M$ such that,

- $M_0 \prec M \models T$
- $dcl(A) = A$,
- $B \subseteq acl(A)$,
- $Aut(B/A) \simeq G$.

Definition

A prime p **is geometrically represented** in a theory T if p divides the order of some finite group G which is geometrically represented in T.

Remarks

- $Aut(B/A)$ is the set of permutations of B over A preserving the truth value of the formulas computed in M.
- If M is saturated and of greater cardinality than A, $Aut(B/A)$ can also be described as the set of automorphisms of B fixing A that extends to an automorphism of M.
- If a finite group G is geometrically represented in a complete theory T over a model M_0 then it is represented over every elementary extension M of M_0.

Examples

Example

Let $T = ACF_0$ be the theory of algebraically closed fields of characteristic 0. Every finite group is represented in ACF_0. Since every finite group G is isomorphic to a subgroup of the symmetric group acting on G. We have
$\mathbb{C} \leq \mathbb{C}(x_1, \ldots, x_n)^G \leq \mathbb{C}(x_1, \ldots, x_n) \leq K$.

Roots of Unity

Let p be a prime, we will denote

- a primitive pth root of unity by ζ_p,
- the set of p^nth roots of unity by μ_{p^n},
- $\bigcup_{n\in\mathbb{N}} \mu_{p^n}$ by μ_{p^∞}.

We will also denote the maximal p extension of the prime field of characteristic p by Ω.

Theorem (B. , Hrushovski)

Let F be a quasifinite field, $char(F) \neq p$, if p is geometrically represented in $Th(F)$ then $\mu_{p^\infty} < F(\zeta_p)$.

More precisely:

Theorem (B. , Hrushovski)

Let F be a quasifinite field, p a prime. Assume p is geometrically represented in $Th(F)$. Then

- *if $char(F) \neq p$ then $F(\zeta_p)$ contains μ_{p^∞}*
- *if $char(F) = p$ then F contains Ω.*

Converse of the above theorem holds as well.

Theorem (B. , Hrushovski)

If a pseudo-finite field F contains μ_{p^∞} then $\mathbb{Z}/p^n\mathbb{Z}$ is geometrically represented in $Th(F)$ for every $n \in \mathbb{N}$.

Lemma

[B. , Hrushovski] Let T be a complete theory of pseudo-finite fields, if two finite groups G, H are geometrically represented in T then so is $G \times H$.

Observation

If F contains μ_{p^∞} for every prime p then Every finite abelian group is represented in $Th(F)$.

Question

Which are the finite groups that can be represented in theories of pseudo-finite fields?

Theorem (B., Chatzidakis)

Assume that F is a pseudo-finite field, A is a definably closed subfield of F, then we have:

- $G = Aut(acl(A)/A)$ *is abelian,*
- *for any prime p dividing $\sharp G$, $p \neq char(F)$,*
- $\mu_{p^\infty} \subset F$.

Let (K, v) be a valued field.

- We denote the valuation ring, its maximal ideal, the residue field and the value group by $\mathcal{O}_v, \mathcal{M}_v, K_v$.
- If v is Henselian, i.e. if v has a unique extension w to K^{sep} separable closure of K, then $Gal(K)$ is compatible with w, (i.e. $w(\sigma(x)) = w(x)$ for every $x \in K^{sep}$ and $\sigma \in Gal(K)$).
- This induces a canonical surjection π with

$$1 \to T \to Gal(K) \xrightarrow{\pi} Gal(K_v) \to 1$$

  where T, is the inertia subgroup of $Gal(K)$ with respect to w.
- If $char(K) = q > 0$, T has a characteristic subgroup V, the ramification subgroup with respect to the valuation w, which is the unique Sylow q subgroup of T, and T/V is abelian.

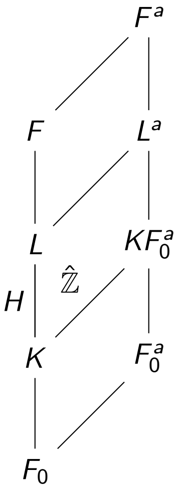
Theorem (Koenigsmann Product Theorem)

Let K be a field with $Gal(K) \simeq G_1 \times G_2$, where both G_1 and G_2 are non-trivial, let $\pi : Gal(K) \rightarrow Gal(K_v)$ be the canonical surjection, where v is the canonical surjection on K, then $Gal(K_v) = \pi(G_1) \times \pi(G_2)$ and $(\sharp\pi(G_1), \sharp\pi(G_2)) = 1$. If a prime p divides $(\sharp G_1, \sharp G_2)$, then, v is non-trivial, moreover:

- $char(K) \neq p$,
- $\mu_{p^\infty} \subset K(\zeta_p)$
- *there is a non-trivial Henselian valuation v on K.*

Theorem (Prestel)

Let K be a PAC field which is not separably closed, then K has no Henselian valuation on it.

- 
F^a
F
L^a
L
KF_0^a
$\hat{\mathbb{Z}}$
H
K
F_0^a
F_0

- Let K be an intermediate field of pseudo-finite fields $F_0 \prec F$, let L be the algebraic closure of K in F.
- Let $H = Gal(L/K)$ and we know that $Gal(KF_0^a/K) = \hat{\mathbb{Z}}$ Since L is linearly disjoint from KF_0^a over K and $LF_0^a = L^a = K^a$ we know that $Gal(K) = H \times \hat{\mathbb{Z}}$ so now we can use Koenigsmann Theorem.
- Since we assumed that p divides the order of H we know that $\mu_{p^\infty} \leq K(\zeta)$, and there is a Henselian valuation v on K such that v_F is Henselian, since F is PAC by Prestel, it has to be trivial.
- Therefore $\pi(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}$. Since every prime divides $\sharp\hat{\mathbb{Z}}$, $(\sharp\pi(H), \sharp\pi(\hat{\mathbb{Z}})) = (\sharp\pi(H), \sharp\hat{\mathbb{Z}}) = 1$. $\pi(H) = 1$, so H is in the inertia group, which is abelian, H is abelian.
- Moreover, we can show that $\mu_{p^\infty} \leq K$.