On continuous functions definable in expansions of the ordered real additive group

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References. Unless otherwise stated (or said) the results are from the following three papers:

- H.-Walsberg, *'Interpreting the monadic second order theory of one successor in expansions of the real line'*, Israel J. to appear,
- Fornasiero-H.-Walsberg, *'How to avoid a compact set'*, Preprint
- H.-Walsberg, *'On continuous functions definable in expansions of the ordered real additive group'*, Preprint soon

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'What about decidability of the theory? Just as biological taxonomy does not tell us whether a species is tasty, the classificaton here does not deal with decidability.' - Saharon Shelah

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Cor. If $\mathcal{R}$ expands the real field, then $\mathcal{R}$ can not be type B.

For $r \in \mathbb{N}_{\geq 2}$, consider a ternary predicate $V_r(x, u, k)$ that holds if and only if u is an integer power of r, $k \in \{0, \ldots, r-1\}$, and the digit of some base r representation of x in the position corresponding to u is k. Set $\mathcal{T}_r := (\mathbb{R}, <, +, V_r)$.

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Observation: $(D_r, \prec)$ is definable in $\mathcal{T}_r$ and has order type ω.

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It is every hard to expand type B structures without becoming type C:

- with α and C as above, $(\mathbb{R}, <, +, x \mapsto \alpha x, C)$ is type C.
- with α irrational and non-quadratic, $(\mathbb{R}, <, +, \mathbb{Z}, x \mapsto \alpha x)$ is type C.

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Use results of H. and Tychonievich.

An application to generic functions

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Theorem. The set of all $f \in C^k([0,1])$ such that $(\mathbb{R}, <, +, f)$ is type C, is co-meager in $C^k([0,1])$.

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Theorem. The set of all $f \in C^k([0,1])$ such that $(\mathbb{R}, <, +, f)$ is type C, is co-meager in $C^k([0,1])$.

Proof. It is well-known that the set of somewhere $(k+1)$-differentiable functions in $C^k([0,1])$ is meager. This rules out type A. It is left to show that the set of all $f \in C^k([0,1])$ such that $(\mathbb{R}, <, +, f)$ is type B is meager. When $k \geq 2$, these are just affine functions.

Definition. We say that $\mathcal{R}$ is of **field-type** if there is an interval $I \subseteq \mathbb{R}$ together with definable functions $\oplus, \otimes : I^2 \to I$ such that $(I, <, \oplus, \otimes)$ is an ordered field isomorphic to $(\mathbb{R}, <, +, \cdot)$.

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Marker-Pillay-Peterzil 92. Suppose $\mathcal{R}$ is o-minimal. If $\mathcal{R}$ defines a non-affine C^1 function $[0,1] \to \mathbb{R}$, then $\mathcal{R}$ is of field-type.

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Cor. Suppose that $\mathcal{R}$ is type B and does not interpret $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$. Then every definable C^1 function $f : [0,1] \to \mathbb{R}$ is affine.

Step 1. Let $a, b \subseteq \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be a definable non-affine C^1 function.

1. If f' is strictly increasing or strictly decreasing on some open subinterval of I, then $\mathcal{R}$ is of field-type.
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Proof:

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- D is dense in F.
- Let $\delta : D \to \mathbb{R}$ map d to the length of the connected component of U one of whose endpoints d is.
- Set $d \prec d'$ if either $\delta(d') < \delta(d)$ or $(\delta(d') = \delta(d)$ and $d < d')$.

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- Set $d \prec d'$ if either $\delta(d') < \delta(d)$ or $(\delta(d') = \delta(d)$ and $d < d')$. It is easy to see that $\prec$ is an ω-order on D.
- Conclude that $(F, <, \oplus, \otimes, D, \prec)$ defines $\iota^{-1}(\mathbb{Z})$.

An application to metric dimensions

H.-Miller 15. Let $\mathcal{R}$ be an expansion of $(\mathbb{R}, <, +, \cdot)$ that is not type C, then the Assouad dimension of any compact definable subset of $\mathbb{R}^n$ agrees with its topological dimension.

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Theorem. The Assouad dimension of any compact definable subset of $\mathbb{R}^n$ agrees with its topological dimension if one of the following holds:

1. $\mathcal{R}$ defines a non-affine C^1 function and does not interpret $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$.
2. $\mathcal{R}$ defines a non-affine C^2 function and is not type C.

Definition. A **decreasing sequence with decreasing gaps** is a strictly decreasing sequence $(s_n)_{n\in\mathbb{N}}$ with limit zero such that $(s_{n+1} - s_n)_{n\in\mathbb{N}}$ is also strictly decreasing. We say that $(s_n)_{n\in\mathbb{N}}$ has **exponential decay** if for some $\lambda < 0$ we have $s_n < \exp(\lambda n)$ for sufficiently large n. Otherwise, we say the sequence has **subexponential decay**.

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- S has positive Assouad dimension,
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Corollary. Suppose that $(s_n)_{n\in\mathbb{N}}$ has subexponential decay. Let $f : [0,1] \to \mathbb{R}$ be a non-affine function.

- If f is C^1, then $(\mathbb{R}, <, +, S, f)$ interprets $(\mathbb{R}, <, +, \cdot, \mathbb{Z})$.
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Optimality II. By van den Dries the expansion of $(\mathbb{R}, <, +, \cdot)$ by $\{2^{-n} : n \in \mathbb{N}\}$ is type A.