Model theoretic dynamics in a Galois fashion

motivations, definitions, facts, no proofs...

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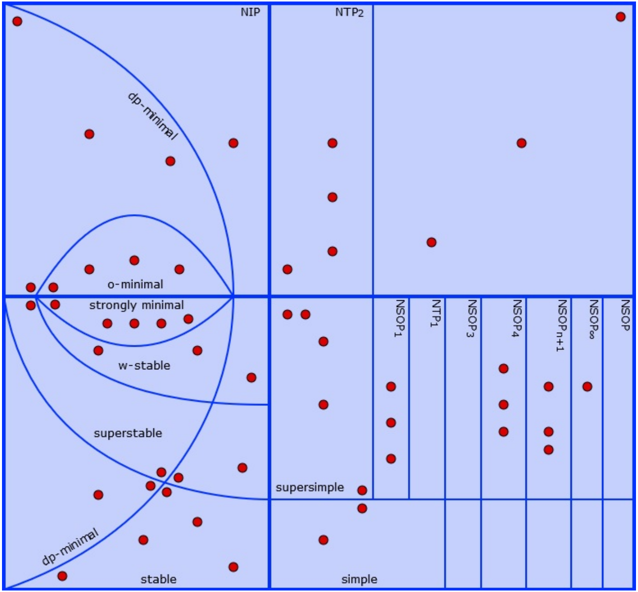
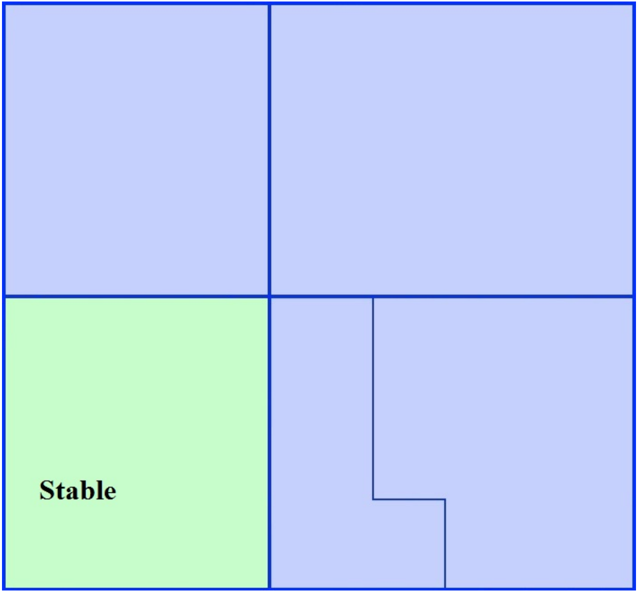
Będlewo, 7.07.2017

The Universe



The Universe (personal interpretation)



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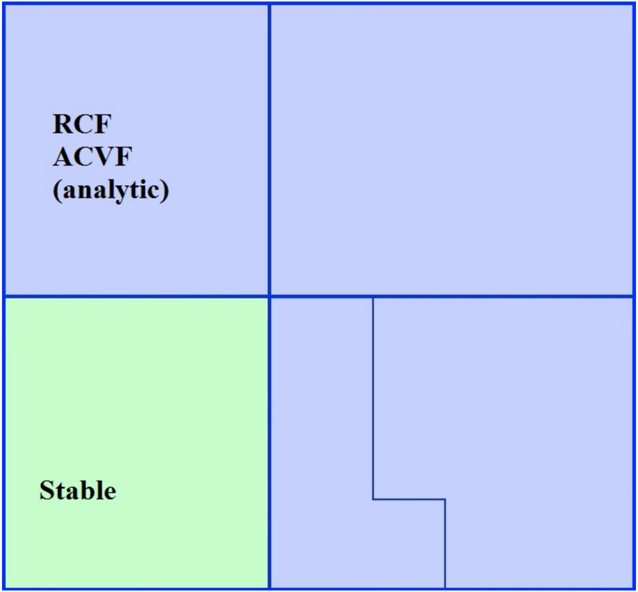

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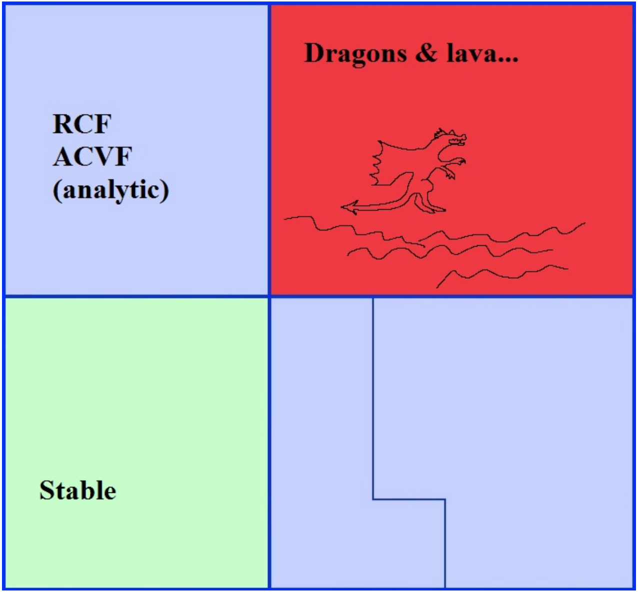

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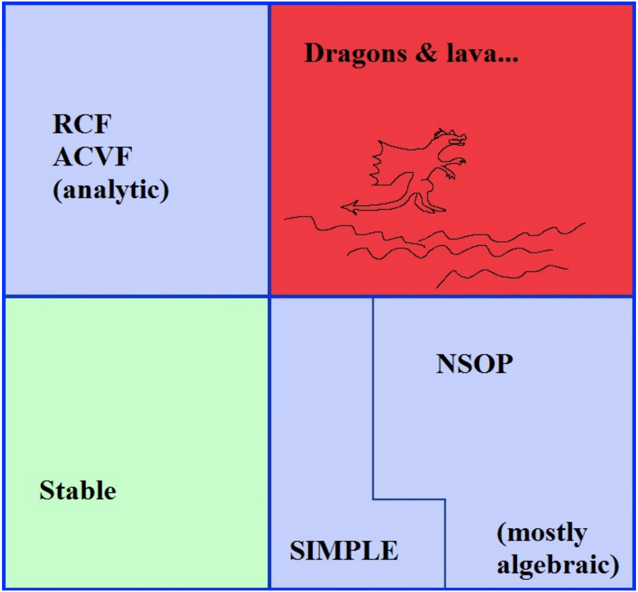

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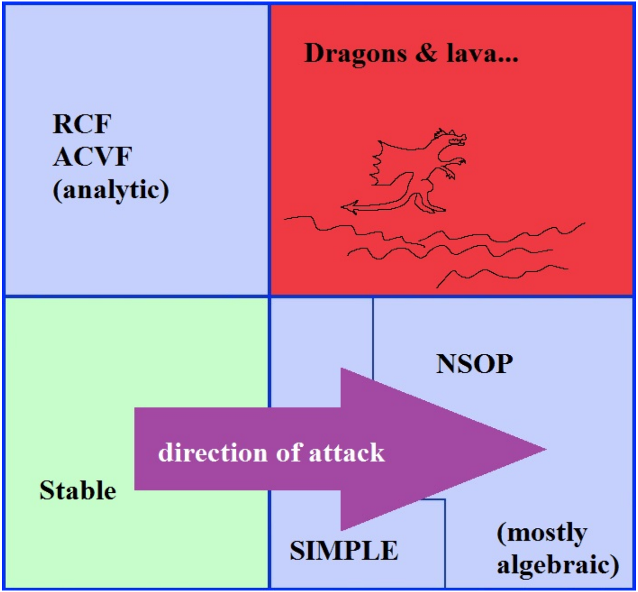

Main examples of simple theories

- ACFA, $\mathbb{Q}$ACFA,
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- random graph (other graph-like theories)

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- Are simple theories studied enough?
- Piotr's point of view...
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Long term plan

1. $T^{\mathtt{mc}}$ is stable $\implies T^{\mathtt{mc}}_G$ is simple (done, some results in this talk)
2. If T' is simple (probably more restrictions), then there exist $\mathscr{L}$, T and G such that $T^{\mathtt{mc}}_G$ exists and is bi-interpretable with T'.
3. Geometric description of simple theories in the style of ACFA.

Set-up

Assume that $T^{\mathtt{mc}}$ and $T^{\mathtt{mc}}_G$ exist and $T^{\mathtt{mc}}$ is stable and has QE and EI.

We fix a monster model $(\mathfrak{C},(\sigma_g)_{g\in G}) \models T^{\mathtt{mc}}_G$ and a $|\mathfrak{C}|^+$-saturated monster model $\mathfrak{D} \models T^{\mathtt{mc}}$ such that $\mathfrak{C} \subseteq \mathfrak{D}$.

After use of Galois theory, we can say more about how $\mathfrak{C}$ is contained in $\mathfrak{D}$, which is extremly important in proofs (not the case for $G = \mathbb{Z}$, e.g. ACFA).

E.g. $\operatorname{acl}^{\mathfrak{C}}_{\mathscr{L}^G}(A) = \operatorname{acl}^{\mathfrak{C}}_{\mathscr{L}}(G\cdot A) = \operatorname{acl}^{\mathfrak{D}}_{\mathscr{L}}(G\cdot A) \cap \mathfrak{C}$

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Why QE and EI?

Corollary

Assume that $E, A, B \subseteq \mathfrak{D}$*,* A *is regular over* E *- i.e.* $E \subseteq A$ *and*

$$\operatorname{dcl}^{\mathfrak{D}}_{\mathscr{L}}(A) \cap \operatorname{acl}^{\mathfrak{D}}_{\mathscr{L}}(E) = \operatorname{dcl}^{\mathfrak{D}}_{\mathscr{L}}(E)$$

- and $f_1, f_2 \in \operatorname{Aut}_{\mathscr{L}}(\mathfrak{D})$*,* $f_1|_E = f_2|_E$*.*
If $A \mathop{\smile\!\!\!\!\!\mid}^{\mathfrak{D}}_{E} B$ *and* $f_1(A) \mathop{\smile\!\!\!\!\!\mid}^{\mathfrak{D}}_{f_1(E)} f_2(B)$*, then there exists* $h \in \operatorname{Aut}_{\mathscr{L}}(\mathfrak{D})$ *such that* $h|_A = f_1|_A$ *and* $h|_B = f_2|_B$*.*

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Definition

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The above defines an independence relation.

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We say that algebraic closures split in $\mathfrak{D}$ over $\mathfrak{C}$ if for all small $(M, \bar{\sigma}) \preceq (\mathfrak{C}, \bar{\sigma})$ and all small $A \subseteq \mathfrak{C}$, such that $M \subseteq A$, it follows

$$\mathrm{acl}^{\mathfrak{D}}_{\mathscr{L}}(A) = \mathrm{dcl}^{\mathfrak{D}}_{\mathscr{L}}\left(\mathrm{acl}^{\mathfrak{D}}_{\mathscr{L}}(A) \cap \mathfrak{C},\ \mathrm{acl}^{\mathfrak{D}}_{\mathscr{L}}(M)\right)$$

Fact

- *Ok for $\mathfrak{C} = \mathfrak{D}$ (e.g. actions of $G = \mathbb{Z}$, like ACFA)*
- *Ok for finite G.*
- *Ok if $\mathfrak{C}$ is bounded (e.g. G is finitely generated)*

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Main result

Theorem

If algebraic closures split, then $\mathop{\smile\!\!\!\!\!\!\mid}^{\circ}$ satisfies the independence over a model theorem (T_G^{mc} is simple and $\mathop{\smile\!\!\!\!\!\!\mid}^{\circ}$ is forking independence).

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"NO hope" for:

- *stability of T_G^{mc} in general (ACFA)*
- *"superstable $\Longrightarrow$ supersimple" ($\mathbb{Q}$ACFA)*
- *removing "alg. clos. split" (alg. clos. do not split $\Longrightarrow$ not bounded, e.g D_∞-TCF exists and is not simple, because models are not bounded)*

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Other results

- $T_G^{\mathtt{mc}}$ has semi-QE (similar as in ACFA).
- $T_G^{\mathtt{mc}}$ codes finite tuples.

Theorem

If $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ is forking independence, then $T_G^{\mathtt{mc}}$ has geometric EI (each tuple is interalgebraic with a real tuple).

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