More on the product formula

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The fundamental theorem of arithmetic:

$$|n| = \Pi_p p^{v_p(n)}$$

relates the real norm $|x|_\infty$, with all p-adic norms.

$$v_{\mathbb{R}}(x) := -\log|x|$$

$$\Sigma_p \log(p) v_p(n) + (-\log|n|) = 0$$

$$\int_p v_p(n) dm(v) = 0$$

where m is a measure giving each v_p weight $\log(p)$, and v_∞ weight 1.

In this form, the formula is valid for any $n \in \mathbb{Q}^*$, in fact for any number field and global function field. In $L_{\omega_1,\omega}$, it axiomatizes these precisely (Artin-Whaples 1940).

We aim for a first-order, continuous-logic axiomatization.

The language

A field sort, $(F, +, \cdot, 0, 1)$, and a sort $(\mathbb{R}, +, <)$. Continuous logic is used to insist that the latter always has the standard interpretation.

On F, terms are polynomials over $\mathbb{Z}$; equality is a $\{0, 1\}$-valued relation as usual.

On $\mathbb{R}$, the "tropical terms" are terms in the signature $+, \min, 0, \alpha \cdot x (\alpha \in \mathbb{Q})$. Or allow the uniform closure of this, i.e. all continuous, positively homogeous functions $\mathbb{R}^m \to \mathbb{R}$.

Basic symbols I_t : A symbol I_t for each tropical term t; to be interpreted as a function $(F^*)^n \to \mathbb{R}$.

Local interpretation of I_t Let v be a valuation with value group $\mathbb{R}$, *or* a place $p \to \mathbb{C}$ with $v(x) = -\alpha \log |x|$. Interpret ${I_t}^v(x)$ as $t(vx_1, \ldots, vx_n)$.

Remark. The discreteness of F is unusual in continuous logic. It reflects a deep fact: discreteness of $\mathbb{Q}$ in the adeles, $=$ discreteness of $\mathbb{Z}$ in $\mathbb{R}$. An algebraic number can get close to 0 in the real topology, or in any p-adic topology, but not in all at once.

Universal axioms

Axioms GVF *for globally valued fields:*

1. $(F, +, \cdot)$ is a field.
2. The I_t are compatible with permutations of variables and dummy variables.
3. (Linearity:) $I_{t_1+t_2} = I_{t_1} + I_{t_2}$. $I_{\alpha t} = \alpha I_t$.
4. (Local-global positivity) If $I_t \geq 0$ in every local interpretation, then $I_t \geq 0$.
5. (Product formula) $I_x = 0$ for $x \neq 0$.

Proposition. *Let $F \models GVF$. Then there exists a measure m on a space Ω_F of absolute values of F, such that $v \mapsto v(a)$ is integrable ($a \in F^*$), and $I_t(a_1, \dots, a_n) = \int t(v(a_1), \dots, v(a_n)) dm(v) = 0$.*

We thus write $\int t(v(x_1), \dots, v(x_n))$ in place of $I_t(x_1, \dots, x_n)$.

Product formula:

$$\int v(x) dv = 0$$

(for $x \neq 0$.)

Plan: Let X be a smooth projective variety over a globally valued field F. A formula on X is a combination of:

1. Adelic formulas.
2. Néron-Weil character.
3. A positive affine map on a certain torsor of $NS(X')$, $X' \to X$ birational.

I will explain (1,2). This (along with finite dimensionality of $NS(X)$) will suffice for qf stability.

Adelic formulas

What formulas would you use to describe probability measures μ on $\mathbb{C}$, or $\mathbb{C}^n$, or $X(\mathbb{C})$?

– $\int \phi d\mu$, for various test functions ϕ on X.

The GVF language includes precisely this, with respect to the measure on $X(\mathbb{C})$

What measure?

An archimedean valuation v on $\mathbb{Q}(x_1, \ldots, x_n)$ = a point $(\alpha_1, \ldots, \alpha_n)$ of $\mathbb{C}^n$. (Namely, $v(x_i) = -\log|\alpha_i|$). So a GVF structure on the function field $\mathbb{Q}(X)$ includes the data: a conjugation-invariant measure on $X(\mathbb{C})$.

What test functions?

$t(1, x_1, \ldots, x_n)$, with $t(u, \ldots, x_n)$ a tropical term. Take t such that if $u = 0$, then $t = 0$; u is used to de-homegenize t.

Adelic formulas

Let Tr' be the set of terms $t(u, x_1, \ldots, x_m)$ such that if $v(u) = 0$ then $t = 0$.

Let $L[a]$ consist of formulas

$$\frac{1}{ht(a)} \int t(va^+, vx_1, \ldots, vx_m)$$

with $t \in Tr'$.

$\cup\{L(a) : a \in K\}$ is the K-adelic part of the language, over K.

Semantics: A GVF structure on K *includes* a probability measure on the space of valuations fo K with $v(a) = 1$. (These are a set of representatives for the valuations v with $v(a) > 0$.)

$L[a]$ = language of expectation operators for such measures.

Example: $F = \mathbb{Q}$; Ω_2=2-adic valuations. $\Omega_{1/2}$ == embeddings into $\mathbb{C}$; (conjugation-invariant) probability measures on $\mathbb{C}^n$. $\Omega_{6/7}$= above $\mathbb{Q}_2, \mathbb{Q}_3, \mathbb{R}$.

Recover randomized theory of valuations / absolute values. ($I_t/ht(a)$ is the *expected value* of t.)

Examples with $a \in \mathbb{Q}$, variable x ranging over X.

- Ω_2=2-adic valuations. Measures on Berkovich space, i.e. the n-type space of $VF_{0,2}$.
- $\Omega_{1/2}$ =$X(\mathbb{C})$-points. Probability measures on $\mathbb{C}^n$.
- $\Omega_{10/11}$= above $\mathbb{Q}_2, \mathbb{Q}_5, \mathbb{R}$.

m_{ad} = union over all $a \in F^*$.

Remark. This was originally considered in discrete logic, motivated roughly by adding an integral to the theory of the algebraic integers described by Van den Dries. We see actually a *relative* randomization, of ACVF over ACF; can be worked out for first order theories in general. (Setting out to axiomatize the fundamental theorem of arithmetic, one is formally led to randomizations!)

Example: height on $\mathbb{P}^n$

$$ht(x_0 : \cdots : x_n) := \int -\min_{i=0}^{n} v(x_i) dv$$

Well-defined in projective coordinates!

Example: for $a = (m_0 : \cdots : m_n) \in \mathbb{P}^n(\mathbb{Q})$, $ht(a) = \max \log |m_i|$ when the m_i are relatively prime integers.

If $g : V \to \mathbb{P}^n$ is a projective embedding, $ht_g(x) := ht(g(x))$.

For $x \in \mathbb{A}^1$, $ht(x) := ht(x : 1)$.

Example: multiplicative height 0

For $x \in \mathbb{Q}^{alg}$, $ht(x) = 0$ iff x is an *algebraic integer* and every Galois conjugate lies on unity circle. This is iff x is root of unity. (Kronecker.)

Let $\mu = \mu_{G_m}$ be the $(\bigwedge)$-definable subset of G_m defined by $ht(x) = 0$.

theorem. *The induced qf structure on μ is that of a pure group.*

(In the purely non-archimedean case, the induced qf structure on μ is that of a pure field.)

Corollary (Bilu). *A sequence of Galois orbits of algebraic integers, of heights approaching 0, is equidistributed on $\mathbb{C}$ along the circle $|z| = 1$.*

Proof. Take the ultraproduct of $(\mathbb{Q}^a, a_i)$, with a_i in the i'th orbit, to obtain (K, a) with $\mu(a) = 0$, a non-algebraic. On the other hand take the ultraproduct of $(\mathbb{Q}^a, \omega_i)$ with ω_i a primitive i'th root of 1, to obtain (K, a'). Then a, a' are non-algebraic elements of μ; so $a \equiv a'$. This includes in particular the complex measure. $\square$

Heights on Abelian varieties

Let $K \models GVF$, A an Abelian variety over K.

Let $[m] : A \to A$ denote multiplication by m.

Fix an embedding $D : A \to \mathbb{P}^N$, such that $[-1]$ is linear.

Theorem (Weil-Néron-Tate)**.** *The GVF formulas* $4^{-n} ht_D \circ [2^n]$ *converge uniformly to a limit denoted* $\widehat{h}_D$*. We have:*

- $ht_D - \hat{h}_D$ *is bounded, and*
- $\hat{h}_D$ *is a positive semi-definite quadratic form.*

bounded means: in any GVF extending K.

Interpretable Hilbert spaces

Proposition. *1. There exists a unique maximal ∞- definable subgroup μ of A (of bounded height).*

2. For any D as above, $\mu = \{x : \hat{h}_D(x) = 0\}$

3. A/μ carries a natural (hyper)definable $\mathbb{R}$-Hilbert space structure, $\widehat{A}_D$, determined by the class of D in $NS(A)$. Namely we have $|\hat{a}|^2 = \hat{h}_D(a)$, with $\hat{a}$ the image in A/μ of a.

By $\widehat{A}(F)$ we denote the *completion*, or the closure in some saturated extension, of the image of $A(F)$ in $\widehat{A}$.

$$0 \to \mu_A \to A \to \widehat{A} \to 0$$

$\widehat{A}$ is far from being a pure Hilbert space. Nevertheless we will see that the Hilbert space structure plays a critical role.

μ_A is conjecturally a pure module over $End(A)$, in GVFS containing $\mathbb{Q}[1]$. (And also in the purely non-archimedean case, if A has no isotrivial factors.) Close to theorems of Szpiro, Zhang, Ullmo, Gubler ... around equidistribution and the Bogomolov conjecture.

The Néron-Weil character: curves

Let X be a smooth, irreducible projective curve over a GVF F, $0 \in X(F)$. Let p be a qf GVF type on X, over F. We will define a homomorphism $NW_p : J(F) \to \mathbb{R}$ on the Jacobian J.

There exists a projective embedding g_0 and a hyperplane intersecting $g_0(X)$ in $6[0]$.

There exists a projective embedding g_b and a hyperplane intersecting $g_0(X)$ in $6[0] + [b]$.

$\phi_b = ht_{g_b} - ht_{g_0}$ is linear up to bounded: $\phi(2x) - 2\phi(x)$ is bounded.

The GVF formulas $2^{-n} ht_D \circ [2^n]$ converge uniformly to a limit denoted $\widehat{h}_b$.

$\phi_b(x) - \widehat{h}_b(x)$ is bounded *in* x, b.

For fixed a, $\phi_b(a)$ induces a linear map on J.

Define

$$NW_p(b) = \hat{h}_b(p)$$

The Néron-Weil character

Let X be a smooth, irreducible projective variety X over F, $X(F) \neq \emptyset$; let $\mathrm{alb} : X \to A$ be an Albanese map for X, and let $J = Pic^0(X)$ be the Picard variety. (When X is a curve, $A = J$ is the Jacobian.)

Let p be a GVF qf type on X over F.

Let $\mathcal{P} \leq A \times J$ be the Poincaré divisor.

Write $\mathcal{P} + D_1 = D_2$ for some D_1, D_2 that arise from projective embeddings; and set $\hat{h}_{\mathcal{P}} = \hat{h}_{D_2} - \hat{h}_{D_1}$. This formula defines a quadratic form.

Let $c \models p$, and let $a = \mathrm{alb}(c) \in A$.

Define the Néron character NW_p of p, a on $\widehat{J}(F)$, by:

$$NW_p(b) = \hat{h}_b(a) = \hat{h}_{\mathcal{P}}(a, b)$$

NW_p induces a continuous linear map on $\widehat{J}$

Hence, using self-duality of the Hilbert space $\widehat{J}_D$,
NW_p can be identified with an element $\mathfrak{h}(p)$ of $\widehat{J}_D(F)$.

theorem. *Any quantifier-free GVF type on a curve X over F is determined by:*

1. *The height (of the first nontrivial coordinate).*
2. *the F-adelic qf type. (values of F-adelic formulas.), and*
3. *the Néron character NW_p.*

An extension holds for any smooth projective variety X. It allows defining a *canonical extension* of a qf type over F to one over a GVF $K \geq F$. (1)-formulas over F. (2)- canonical extension of local types. (3) in Hilbert spaces. (4) mass zero to any exceptional divisor strictly over K.

Proof. We may assume F is algebraically closed, by taking a Galois-invariant extension of p, q and using $\widehat{J} = \widehat{J}^G \oplus \widehat{J}_G$ for the Galois group G.

Let p, q be two qf types of the same height, adelic type, and Néron character.

They determine measures on all valuations; can only differ on valuations of $F(X)/F$. These corresponds to points b of $X(F)$.

So we have masses $m_p(b), m_q(b)$. To show $m_p(b) = m_q(b)$.

Compute $m_p(b)$ from first height data, adelic data, and $\phi_b(p)$.

As $\phi_b(p) = \phi_b(q)$, conclude $m_p(b) - m_q(b)$ is bounded in b.

But $m_p - m_q$ is a homomorphism into $\mathbb{R}$ (product formula). So $m_p - m_q = 0$.

$\square$

Corollary. *Let $F = F \leq K$. Formulas on X over K are uniform limits of :*

- *algebraically bounded, finite formulas.*
- *formulas over F*
- *adelic formula $R_t(c, b, x)$ over K.*
- *formulas $(x, c)_{\mathcal{P}}$ giving values of the canonical bilinear map $A \times J$, with $c \in J(K^{alg})$, and*

To spell out the uniformity: for any such ϕ and $\epsilon > 0$ there exists a combination ψ of the three above forms, such that for any GVF structure L on $K(X)$ extending the given GVF structure on K, $|\phi - \psi|(K) < \epsilon$.

theorem. *The theory GVF is qf stable.*

I.e.: if (a_i, b_i) is a qf-indiscernible sequence and ϕ a formula, then $\phi(a_1, b_2) = \phi(b_1, a_2)$.

Ingredients:

1. $\mathbb{R}$ and $\mathbb{C}$

As treated in continuous logic, an indiscernible sequence is *constant.*

2. Valued fields

An indiscernible sequence (or invariant type) *orthogonal to the value group* is an indiscernible set (stably dominated). Here we *decree* that no type increase the value group, which is $\mathbb{R}$.

3. Randomizations of VF relative to F.

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T^{rand} - randomization. Each formula ϕ of VF is replaced by a real-valued $[\phi]$ understood to denote the expectation of ϕ. = Ben-Yaacov - Keisler randomization but *relative to* ACF, i.e. for any formula of the language of rings, impose $[\phi] = 0$ or $[\phi] = 1$.

A type $p(X)$ for T^{rand} = a probability measure on S_X in T, over a given field-theoretic type.

Consider an *indiscernible* probability measure on types in $(x_1, y_1), (x_2, y_2), \cdots$

If $[\phi(x_1, y_2)] < [\phi(x_2, y_1)]$,
$e_{ij} := \phi(x_1, y_2) < \phi(x_2, y_1)$ is an event of nonzero measure.

e_{ij} $\mathbb{N}^2$- indiscernible nonzero events in a probability space. NIP case: all are equal, (horizontally and vertically; otherwise, independence property.)

So find a single type p with $\phi(x_i, y_j) < \phi(x_j, y_i)$; contradicting stability of T.

4. Hilbert spaces.

(Von-Neumann; Krivine.)

First example truly belonging to continuous logic.

Independent = pairwise (!) orthogonal (over base). (Like pure sets.)

Indiscernible subspaces = orthogonal over their intersection. (1-based.)

theorem. *The theory GVF is qf stable.*

Proof. Let $(a_i : i \in \mathbb{Z})$ be a qf-indiscernible sequence over F_0. We wish to show that $tp(a_i, a_j)$ is symmetric. Let $F = F_0(a_i : i < 0)$, and prove that $(a_i : i \in \mathbb{N})$ is a qf Morley sequence over F, i.e. e.g. $c := a_0$ is independent from $K = F(a_1, a_2, \ldots)$ over F. We use the description above of $qftp(c/K)$.

F-adelic formulas

Stablity discussed above - these are $VF^{rand/F}$.

K-adelic formulas (given the F-adelic ones)

. Let $c_j \in F(a_j)$. Then

$$\mu\{v : v(c_j) > 0, v|F = 0\} = \mu\{v : v(c_j) > 0, v|F(a_0) = 0\}$$

by indiscernibility. For $v|F(a_0) = 0$, $t(a_0, a_j) = t'(a_j)$.

Néron character

Let $M = F(a_i : i \in \mathbb{Z})$. Then $\widehat{J}(F(a_i))$ are Hilbert subspaces of $\widehat{J}(M)$; the must be orthogonal over their intersection $\widehat{J}(F)$. So $\mathfrak{h}(p) \in \widehat{J}(F)$.

$\square$

Remark. Actually the proof decomposes $F_0(a)/F_0$ into a tower of one-dimensional extensions, and uses the notion of independence deduced from this and the canonical amalgam of curves. A posteriori, this is independent of any choices.

The description of qf types on curves does go through for higher dimensions, and gives a notion of canonical amlagamation. A direct proof using this is possible, but requires a separate argument on blowing up: If $K_i = f(V_i)$ are qf independent indiscernibles over F, an exceptional divisor above $V_1 \times V_2$ (above a correspondence $S \leq V_1 \times V_2$) cannot have positive mass.

The qf type as a limit of types on varieties

Let $F \models GVF$, X a smooth projective variety over F.

To any irreducible hypersurface $H \leq X$ corresponds a valuation v_H of $F(X)$; $v_H(f)$ is the order of vanishing of f along H. Such a valuation is called *divisorial*, or X-divisorial to emphasize that H is a hypersurface on X.

Definition. *A qf type on X is a GVF type on $F(X)$ with globalizing measure concentrating on valuations that are nontrivial on F, (the adelic part), and X-divisorial valuations.*

The part of the measure concentrating on divisorial valuations is equivalent to a positive linear map on the Cartier divisors on X. When F is trivially valued, but not in general, it factors through $Pic(X)$ and in fact $NS(X)$.

Let $S_{\mathbb{F}}(K)$ denote the space of GVF structures on K extending $\mathbb{F}$; let $S_{\mathbb{F}}(X)$ denote the space of qf types on X.

theorem. *There exists a canonical homeomorphism*

$$S_{\mathbb{F}}(K) \cong \underleftarrow{lim}_{X \in \mathfrak{X}} S_{\mathbb{F}}(X)$$

Proof. For v a valuation of K/F and D a Cartier divisor, define $v(D)$. In any Zariski open not disjoint from the center of v, D is defined by some (f), and $v(D) := v(f)$.

Let m be a globalizing measure on K, above the given one on F. Let m' be the *non-adelic* part of m, i.e. the part concentrating on valuations trivial on F.

Define m_X to give D mass equal to $\int v(D)dm'(v)$.

Show that along with m_{ad}, this gives a GVF structure. And that K is approximated by these. □

Now $NS(X) = Pic(X)/Pic^0(X)$ is generated by hyperplane pullbacks $\alpha_1, \ldots, \alpha_k$ from a finite number k of projective embeddings. Let H_i be the corresponding heights.

theorem. *Any quantifier-free GVF type p on a variety X over F is determined by the k values $H_1, \ldots, H_k$, the adelic part, and the Néron character NW_p.*

Remark. In case F is trivially valued, the type is determined by a homomorphism $NS(X) \to \mathbb{R}$, positive on the effective cone.

For proving $k(t)^{alg}$ *is existentially closed* this is the essential part.

See Orsay notes for some other connections.

Proposition. For indiscernible of transcendence degree 1, can also define and prove canonical amalgamation for for formulas with ACF-algebraically bounded quantifiers. Proof uses Hodge index theorem.