On Kim-independence in $NSOP_1$ theories

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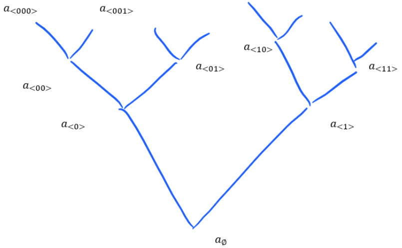
NSOP$_1$

Definition

The formula $\varphi(x; y)$ has SOP_1 if there is a collection of tuples $\langle a_\eta \,|\, \eta \in 2^{<\omega} \rangle$ so that

- For all $\eta \in 2^\omega$, $\left\{\varphi(x, a_{\eta|\alpha}) \,|\, \alpha < \omega \right\}$ is consistent.
- For all $\eta \in 2^{<\omega}$, if $\nu \trianglerighteq \eta \frown \langle 0 \rangle$, then $\{\varphi(x, a_\nu), \varphi(x, a_{\eta \frown \langle 1 \rangle})\}$ is inconsistent.

T is $NSOP_1$ if no formula in it has SOP$_1$.



Alternative definition

The following definition seems more accessible.

Definition

The formula $\varphi(x; y)$ has an SOP_1 *array* if there is a collection of pairs $\langle c_i, d_i \mid i < \omega \rangle$ and some $k < \omega$ so that

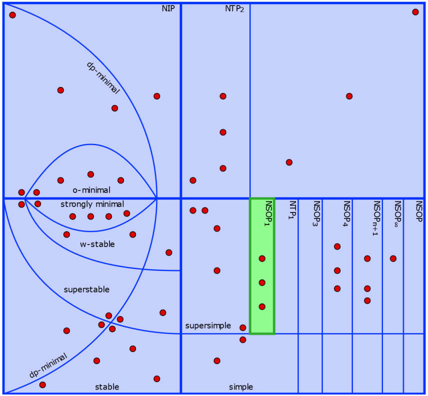
- $\langle \varphi(x, c_i) \mid i < \omega \rangle$ is consistent.
- $\langle \varphi(x, d_i) \mid i < \omega \rangle$ is k-inconsistent.
- $c_i \equiv_{c, d_{<i}} d_i$ for all $i < \omega$.

Fact

T is $NSOP_1$ iff no formula $\varphi(x, y)$ has an SOP_1-array.

The place of NSOP_1 in the universe

Every simple theory is NSOP_1, and every NSOP_1 theory is NTP_1, as illustrated in Gabe Conant's beautiful diagram



A map of the universe

SOP_1 was defined by Džamonja and Shelah (2004), and was later studied by Usvyatsov and Shelah where a first example of a non-simple $NSOP_1$ was introduced* (2008).
More recently, in their paper "on model-theoretic tree properties" (2016), Chernikov and Ramsey provided more information. They proved a version of the Kim-Pillay characterization for $NSOP_1$. Namely, if there is an independence relation satisfying certain properties, then the theory is $NSOP_1$.
This characterization can be used to provide many natural examples of $NSOP_1$-theories.

The Chernikov-Ramsey characterization

Theorem

(Chernkiov-Ramsey) Assume there is an $\mathrm{Aut}(\mathfrak{C})$*-invariant ternary relation* $\mathop{\smile\!\!\!\!\!\vert}$ *on small subsets of the monster which satisfies the following properties, for an arbitrary* $M \models T$ *and arbitrary tuples from* $\mathfrak{C}$.

- *Strong finite character: if* $a \not\mathop{\smile\!\!\!\!\!\vert}_M b$*, then there is a formula* $\varphi(x, b, m) \in tp(a/bM)$ *such that for any* $a' \models \varphi(x, b, m)$, $a' \not\mathop{\smile\!\!\!\!\!\vert}_M b$.
- *Existence over models:* $M \models T$ *implies* $a \mathop{\smile\!\!\!\!\!\vert}_M M$ *for any* a.
- *Monotonicity:* $aa' \mathop{\smile\!\!\!\!\!\vert}_M bb' \implies a \mathop{\smile\!\!\!\!\!\vert}_M b$.
- *Symmetry:* $a \mathop{\smile\!\!\!\!\!\vert}_M b \iff b \mathop{\smile\!\!\!\!\!\vert}_M a$.
- *The independence theorem:* $a \mathop{\smile\!\!\!\!\!\vert}_M b$, $a' \mathop{\smile\!\!\!\!\!\vert}_M c$, $b \mathop{\smile\!\!\!\!\!\vert}_M c$ *and* $a \equiv_M a'$ *implies there is* a'' *with* $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$ *and* $a'' \mathop{\smile\!\!\!\!\!\vert}_M bc$.

Then T *is* $NSOP_1$.

Examples of NSOP_1 theories

Here are some examples of non-simple NSOP_1 that were studied recently.

1. (Ramsey) The selector function: the model companion of the following theory. Two sorts, F and O. E is an equivalence relation on O, $eval : F \times O \to O$ is a function such that $eval\,(f, o)\ E\ o$ and if $o_1\ E\ o_2$ then $eval\,(f, o_1) = eval\,(f, o_2)$.
2. (Chernikov, Ramsey) Parametrized simple theory (T simple and is a Fraïssé limit of a universal class of finite relational language with no algebraicity, then add a new sort for generic copies of models of T).
3. (Chernikov, Ramsey) ω-free PAC fields (i.e., PAC fields with Galois group $\hat{F}_\omega$, the free profinite group with $\aleph_0$-generators. (Was extended to general Frobenius fields, essentially the same proof.)

Examples of NSOP_1 theories

4 (Kruckman, Ramsey) If T is a model complete NSOP_1 theory eliminating the quantifier $\exists^\infty$, then the generic expansion of T by arbitrary constant, function, and relation symbols is still NSOP_1. (Exists by a Theorem of Winkler, 1975.)

5. (Kruckman, Ramsey) With the same assumption, T can be extended to an NSOP_1-theory with Skolem functions.
6. (Chernikov, Ramsey) Vector spaces with a generic bilinear form (exist by Granger).
7. (d'Elbée, ...) Algebraically closed field of positive char. with a generic additive subgroup.

Kim independence

To complete the picture, it is natural to try and find an independence relation satisfying the criterion of Chernikov-Ramsey.

Definition

We say that $\varphi(x, b)$ *Kim-divides* over a model M if there is a global M-invariant type $q \supseteq \operatorname{tp}(b/M)$ such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent when $\langle a_i \mid i < \omega \rangle$ is a Morley sequence in q over M.

In other words, this is saying that $\varphi(x, b)$ divides but that moreover the sequence witnessing dividing is a Morley sequence generated by an invariant type ($a_0 \models q|_M$, $a_1 \models q|_{Ma_0}$, etc.). This notion was suggested by Kim in his Banff talk in 2009, and is also related to Hrushovski's *q-dividing* and Shelah and Malliaris' *higher formula*.

Definition

We say that $\varphi(x, b)$ *Kim-forks* over M if it implies a finite disjunction of Kim-dividing formulas.

Kim's Lemma for Kim independence

Lemma

($NSOP_1$) If $\varphi(x, b)$ Kim-divides and q is any global invariant type containing $\mathrm{tp}(b/M)$, then $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent where $\langle a_i \mid i < \omega \rangle$ is a Morley sequence generated by q.

Corollary

($NSOP_1$) If $\varphi(x, b)$ Kim-forks over M then it Kim-divides over M.

However, this is not true for forking.

Proof.

By the alternative definition of NSOP_1 using SOP_1 arrays. □

Definition

Write $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$ for $\mathrm{tp}(a/Mb)$ does not Kim-divide over M.

Properties of Kim independence

Theorem

($NSOP_1$) Kim-independence satisfies all the properties listed in the Ramsey-Chernikov criterion.

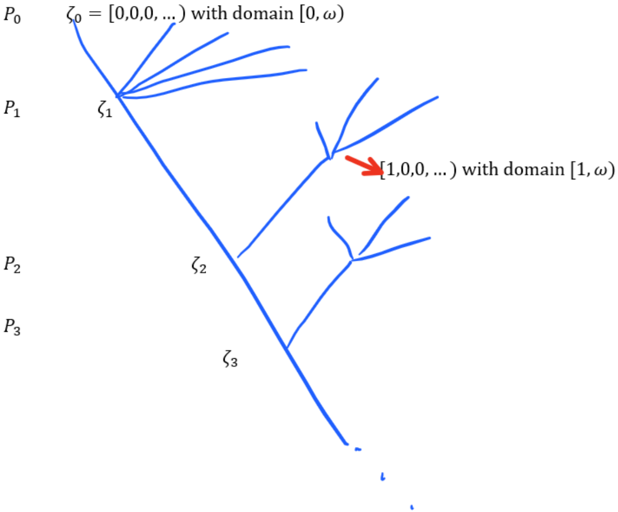
- *Strong finite character: if $a \mathop{\not\downarrow}^K_M b$, then there is a formula $\varphi(x, b, m) \in tp(a/bM)$ such that for any $a' \models \varphi(x, b, m)$, $a' \mathop{\not\downarrow}^K_M b$.*
- *Existence over models: $M \models T$ implies $a \mathop{\downarrow}^K_M M$ for any a.*
- *Monotonicity: $aa' \mathop{\downarrow}^K_M bb' \implies a \mathop{\downarrow}^K_M b$.*
- *Symmetry: $a \mathop{\downarrow}^K_M b \iff b \mathop{\downarrow}^K_M a$.*
- *The independence theorem: $a \mathop{\downarrow}^K_M b$, $a' \mathop{\downarrow}^K_M c$, $b \mathop{\downarrow}^K_M c$ and $a \equiv_M a'$ implies there is a'' with $a'' \equiv_{Mb} a$, $a'' \equiv_{Mc} a'$ and $a'' \mathop{\downarrow}^K_M bc$.*

Tree Morley sequences

The main tool in the proofs of all the nontrivial properties (symmetry and the independence theorem) was tree Morley sequences. These are sequences which are indexed by an infinite tree. They play a similar role to Morley sequences in simple theories. They can be defined to be of any height, but let me define the trees of height ω.

Definition

Let $\mathcal{T}_\omega$ be the set of functions $f : [n, \omega) \to \omega$ with *finite support*. We put a tree order $\mathcal{T}_\omega$ by $f \trianglelefteq g$ iff $f \subseteq g$. We let $f \wedge g = f \upharpoonright m$ where $m = \min \{n < \omega \mid f \upharpoonright [m, \omega) = g \upharpoonright [m, \omega)\}$. The n'th level of the tree is the set $P_n = \{f \mid \mathrm{dom}(f) = [n, \omega)\}$. We put a lexicographical order by $f <_{lex} g$ iff $f \trianglelefteq g$ or $f \wedge g \in P_{n+1}$ and $f(n) < g(n)$.
Let ζ_n be the zero function with domain $[n, \omega)$.



An illustration of $\mathcal{T}_\omega$

Tree Morley sequences

Definition

$\langle a_\eta \mid \eta \in \mathcal{T}_\omega \rangle$ is called a *Morley tree over M* if:

1. It is indiscernible with respect to the language $\{\trianglelefteq, <_{lex}, \wedge, \leq_{len}\}$ where $f \leq_{len} g$ iff f is lower than g in the tree.
2. For every $n < \omega$, there is some global invariant type q over M such that $\left\langle a_{\geq \zeta_{n+1} \frown \langle i \rangle} \mid i < \omega \right\rangle$ is a Morley sequence generated by q.

Definition

A sequence $\langle a_n \mid n < \omega \rangle$ is a Tree Morley sequence if there is a Morley tree as above such that $a_n = a_{\zeta_n}$ for all n.

Remark: to construct tree Morley sequences in practice, one usually constructs a very tall tree, and then extract using Erdös-Rado.

Tree Morley sequences

Theorem

($NSOP_1$)

1. *If $\langle a_i \mid i < \omega \rangle$ is a tree Morley sequence over M. Then $\varphi(x, a_0)$ Kim-divides over M iff $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent.*
2. *If $\langle a_i \mid i < \omega \rangle$ is a universal witness for Kim-dividing (if $\varphi(x; a_{<n})$ Kim-divides over M then the sequence of n-tuples from $\langle a_i \mid i < \omega \rangle$ witness this), then $\langle a_i \mid i < \omega \rangle$ is a Tree Morley sequence.*
3. *Tree Morley sequences exists: if $a \mathop{\smile\!\!\!\!\!\mid}^{K}_{M} b$ and $a \equiv_M b$ then there is a tree Morley sequence starting with a, b.*
4. *If $\langle a_i \mid i < \omega \rangle$ is a Morley sequence over M (i.e., an indiscernible sequence such that $a_i \mathop{\smile\!\!\!\!\!\mid}^{f}_{M} a_{<i}$), then $\langle a_i \mid i < \omega \rangle$ is a tree Morley sequence, and in particular witnesses Kim-dividing.*

Kim-independence and simple theories

Theorem

TFAE for an $NSOP_1$-theory T:

1. *T is simple.*
2. *$\mathop{\smile\hskip-0.9em ^{\vert}}^K = \mathop{\smile\hskip-0.9em ^{\vert}}^f$ over models.*
3. *$\mathop{\smile\hskip-0.9em ^{\vert}}^K$ satisfies base monotonicity over models.*

In particular we get a new proof of a (stronger result of Shelah that NTP_2+NTP_1=Simple).

Corollary

If T is $NSOP_1$ and NTP_2, then T is simple.

Proof.

It is known that in NTP_2, if $\varphi(x, b)$ divides over M then it Kim-divides over M. □

Note: Transitivity fails. It is possible that $ab \mathop{\smile\hskip-0.9em ^{\vert}}^K_M c$, $a \mathop{\smile\hskip-0.9em ^{\vert}}^K_M b$ but $a \not\mathop{\smile\hskip-0.9em ^{\vert}}^K_M bc$.

Local character

In my work with Ramsey, we proved a version of local character.

Theorem

($NSOP_1$) If $p \in S(M)$ then there is some $N \prec M$ of size $\leq 2^{|T|}$ over which p does not Kim-fork.

In joint work with Shelah we were able to considerably improve this theorem.

The club filter

Definition

For a set X, a family of C countable subsets of X is called a *club* if it is closed (a countable union of elements from C is in C) and unbounded (every countable subset of X is contained in some member of C).
The *club filter* is the filter of all clubs.
A family of countable subsets of X is called *stationary* if it intersects every club.

Fact

The Club filter on X is generated by sets of the form C_F where F is a collection of finitary functions on X and where C_F is the set of all countable $Y \subseteq X$ closed under members of F.
The definition of club and the fact generalize (in an appropriate way) to $[X]^\kappa$ instead of $[X]^\omega$.

The local character characterization

Theorem

TFAE for a theory T:

1. *T is $NSOP_1$.*
2. *If $p \in S(M)$ then for stationary many $N \prec M$ of size $|T|$, p does not Kim-divide over N.*
3. *If $p \in S(M)$ then for club many $N \prec M$ of size $|T|$, p does not Kim-divide over N.*
4. *If $p \in S(M)$ then there is a global extension q such that for are club many $N \prec M$ of size $|T|$, q does not Kim-fork over N.*

Note: the proof uses stationary logic where one is allowed to add quantifiers of the form $(aaS)\varphi(S)$ meaning that for club many sets S, $\varphi(S)$ holds.

Thank you

Thank you for your time!