Cardinalities of definable sets in finite structures

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(joint work with Anscombe, Steinhorn, Wolf)

CDM Theorem

Theorem. [Chatzidakis, van den Dries and Macintyre 1992]
Let $\varphi(x_1, \ldots, x_n; y_1, \ldots, y_m)$ be a formula in the language of rings. Then there is a positive constant C and finitely many pairs (d_i, μ_i) $(1 \leq i \leq K)$, with $d_i \in \{0, 1, \ldots, n\}$ and $\mu_i \in \mathbb{Q}^{>0}$ a positive rational number such that for each finite field $\mathbb{F}_q$, where q is a prime power, and each $\bar{a} \in \mathbb{F}_q^m$, if the set $\varphi(\mathbb{F}_q^n, \bar{a})$ is nonempty, then

$$\left| \left| \varphi\left(\mathbb{F}_q^n, \bar{a}\right) \right| - \mu_i q^{d_i} \right| < C q^{d_i - (1/2)}$$

for some $i \leq K$. Moreover, for each pair (d_i, μ_i), there is a formula $\psi_i(y_1, \ldots, y_m)$ in the language of rings such that $\psi_i\left(\mathbb{F}_q^m\right)$ consists of those $\bar{a} \in \mathbb{F}_q^m$ for which the corresponding inequality with (μ_i, d_i) holds.

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Fact: Any ultraproduct of an asymptotic class is measurable.

We aim to broaden this framework, e.g. allow parts of the structure (sorts? coordinatising geometries?) to vary independently, not require that ultraproducts have finite rank, or even have simple theory, not be specific about the form of the functions giving approximate cardinalities (no longer just of form μq^d).

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Finite graphs of bounded degree

For a class $\mathcal{C}$ of finite $\mathcal{L}$-structures and a tuple $\bar{y}$ of variables, let $(\mathcal{C}, \bar{y})$ be the set $\{(M, \bar{a}) \mid M \in \mathcal{C}, \bar{a} \in M^{|\bar{y}|}\}$ of pairs consisting of a structure in $\mathcal{C}$ and a $\bar{y}$-tuple from that structure ('pointed structures in $\mathcal{C}$').

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A finite partition Φ of $(\mathcal{C}, \bar{y})$, is **$\emptyset$-definable** if for each $P \in \Phi$ there exists an $\mathcal{L}$-formula $\phi_P(\bar{y})$ without parameters such that

$$\phi_P(M) = \{\bar{b} \in M^{|\bar{y}|} \mid (M, \bar{b}) \in P\},$$

for each $M \in \mathcal{C}$.

Definition of R-m.a.c.

Let R be any set of functions $\mathcal{C} \longrightarrow \mathbb{R}_{\geq 0}$. A class $\mathcal{C}$ of finite $\mathcal{L}$-structures is an **R-multidimensional asymptotic class** (**R-m.a.c.**) if for every formula $\phi(\bar{x};\bar{y})$ there is a finite $\emptyset$-definable partition Φ of $(\mathcal{C},\bar{y})$ and a set $H_\Phi := \{h_P \in R \mid P \in \Phi\} \subset R$ such that for each $P \in \Phi$,

$$\left| |\phi(\bar{x};\bar{b})| - h_P(M) \right| = o(h_P(M)) \tag{1}$$

for $(M,\bar{b}) \in P$ as $|M| \longrightarrow \infty$.

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Weak R-m.a.c. (or R-m.e.c.) – drop the definability clause on the partition Φ.

Observations

1. To prove a class $\mathcal{C}$ is an R-m.a.c. or R-m.e.c it suffices to work with formulas $\phi(x, \bar{y})$ (with x a single variable), replacing R by the ring generated by R. (Fibering argument, using definability.)

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2. (Wolf) If $\mathcal{C}$ is a m.a.c. or m.e.c. then so is any class of finite structures uniformly bi-interpretable with $\mathcal{C}$. (Note: These conditions are not closed under uniform interpretability, as the definability clause may be lost.)

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3. Any class uniformly interpretable in a m.a.c. is a weak m.a.c..

Examples

1. (Garcia, M, Steinhorn) Class $\mathcal{C}$ of 2-sorted structures $(V, \mathbb{F}_q)$, with V finite dim. v.s. over $\mathbb{F}_q$. Given $\phi(\bar{x}, \bar{y})$ there is a finite set E_ϕ of polynomials $g(\mathbb{V}, \mathbb{F})$ over $\mathbb{Q}$ such that if $M = (V, F)$ then each $h_P(M)$ has form $g(|V|, |F|)$ for some $g \in E_\phi$. Ultraproducts of $\mathcal{C}$ are supersimple, but the V-sort may have rank ω.

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2. More generally, fix a quiver Q (digraph) of *finite representation type* (A_n, D_n, E_q, E_7, E_8). Over a field F, this has a finite-dimensional *path algebra* FQ, which has finitely many isomorphism types of indecomposable representations. Let

$$\mathcal{C}_Q := \{(V, FQ, F) : F \text{ finite field }, V \text{ finite module for } FQ\}$$

(3-sorted, with the natural language). Then $\mathcal{C}_Q$ is an R-m.a.c. with the functions h_P given by polynomials $g(F, W_1, \ldots, W_t)$, where the W_i variables correspond to the indecomposables.

Examples

3. (Bello Aguirre) In the language of rings, for fixed $d \in \mathbb{N}$, let $\mathcal{C}_d$ be the collection of all finite residue rings $\mathbb{Z}/n\mathbb{Z}$, where n is a product of powers of at most d primes, each with exponent at most d.

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Then $\mathcal{C}_d$ is a weak m.a.c., and a m.a.c. after appropriate expansion by unary predicates. If just one prime is involved, this is an asymptotic class. e.g. $\{\mathbb{Z}/p^2\mathbb{Z} : p \text{ prime}\}$ is a 2-dim asymptotic class. Ultraproducts are supersimple of finite SU-rank. (Idea: $\mathbb{Z}/p^d\mathbb{Z}$ is coordinatised uniformly by $\mathbb{Z}/p\mathbb{Z}$.)

Embedded m.a.c.s and m.e.c.s

(Work with Harrison-Shermoen) Notion (cf. 'embedded finite model theory') of an **embedded m.a.c.**: class $\mathcal{C}$ of structures of the form (M, N), where N is a finite substructure of the possibly infinite structure M. The formula $\phi(\bar{x}, \bar{y})$ is interpreted in (M, N) with $\bar{y}$ ranging over M, but we only consider cardinalities of definable sets in N or its powers. To show that $\mathcal{C}$ is an embedded m.a.c. it suffices to show that the corresponding class of finite structures N is a m.a.c. and that in all ultraproducts the N-part is fully (i.e. stably, canonically) embedded.

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4. For any ω-categorical ω-stable structure M, the class of structures (M,N) where N is a finite envelope.

Generalised measurable structures

Let $(S, +, \cdot, 0, 1, <)$ be a (commutative) ordered semiring (so $(S, +, 0)$, $(S, \cdot, 1)$ are commutative monoids, least element 0, etc.). Define $\sim$ on S with $a \sim b$ iff $a \leq b \leq na$ or $b \leq a \leq nb$ for some $n \in \mathbb{N}$. Put $D := S/\sim$, and $d : S \to D$ the natural 'dimension' map. Say S is a **measuring semiring** if

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Let S be a measuring semiring and let M be an L-structure. We say that M is **S-measurable** if there is a function $h : \mathrm{Def}(M) \longrightarrow S$ such that

1. *finite sets* $h(X) = |X|$ for finite X;
2. *finite additivity* h is finitely additive;
3. *mac condition* for each $\emptyset$-definable family $\mathcal{X}$ there is finite $F \subseteq S$ such that $h(\mathcal{X}) = F$ and for each $f \in F$, $h^{-1}(f)$ is a $\emptyset$-definable family;
4. *Fubini* Suppose $p : X \longrightarrow Y$ is a definable function for which there exists $f \in S$ such that for all $\bar{a} \in Y$, $h(p^{-1}(\bar{a})) = f$; then we have $h(X) = f \cdot h(Y)$.

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Proposition. Let M be weakly generalised measurable. Then
1. M does not have the strict order property
2. M is *functionally unimodular*, that is, if $f_i : A \to B$ (for $i = 1, 2$) are definable surjections with f_i k_i-to-1, then $k_1 = k_2$.

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Proposition Let M be S-measurable, and let
$S_0 := \{h(X) : X \subseteq M \text{ definable.}\}$.
If $d(S_0) = S_0/\sim$ is well-ordered then M is supersimple. (Idea: forking ensures drop in dimension.)

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Example (Anscombe). If M is a Fraïssé limit of a free amalgamation class then M is generalised measurable (note for example the generic triangle-free graph is such a Fraïssé limit and has TP1 and TP2 theory).

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Note. The above supersimplicity result applies to ultraproducts of examples like

$$\{(V, \mathbb{F}_q) : q \text{ prime power }, V \text{ finite dim. over } \mathbb{F}_q\}$$

and the quiver example, where the defining functions are given by polynomials in several variables, so the corresponding set of dimensions is well-ordered.

Examples of m.e.c.s

1. (Essentially by Pillay) Let M be any pseudofinite strongly minimal set. Then there is a m.e.c. whose infinite ultraproducts are all elementarily equivalent to M, with the functions determining cardinalities given as polynomials (over $\mathbb{Z}$) in the cardinalities of the finite structures.

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Note. Finite fields are not even a *weak* m.e.c..

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Proposition. If $\mathcal{C}$ is a m.e.c. of groups, then there is $d \in \mathbb{N}$ such that the groups in $\mathcal{C}$ have (uniformly definable) soluble radical $R(G)$ of index at most d, and $R(G)/F(G)$ has derived length at most d
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Problem. Find a m.e.c. with an ultraproduct with non-simple theory.

Homogeneous structures as limits of m.e.c.s

Conjecture. If M is a homogeneous structure over a finite relational language, then the following are equivalent.
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Remarks. 1. The direction (2) $\Rightarrow$ (1) follows from Lachlan + Wolf.

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2. The Paley graphs form a m.a.c (but not m.e.c) with limit the random graph, which is unstable.

Homogeneous structures as limits of m.e.c.s

Conjecture. If M is a homogeneous structure over a finite relational language, then the following are equivalent.
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3. The Conjecture holds for homogeneous graphs, by the Lachlan-Woodrow classification and...

Theorem. Let M be any of the following homogeneous structures. Then there is no m.e.c. with an ultraproduct elementarily equivalent to M.
(i) Any unstable homogeneous graph.
(ii) Any homogeneous tournament.
(iii) The digraph P_n for each $n \geq 3$ (universal subject to omitting an independent set I_n)
(iii) The generic bipartite graph.

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Questions. 1. Is the generic digraph a limit (in the above sense) of a m.e.c.?

2. Is the random graph a limit of a *weak* m.e.c.?

Proof of (ii) above, that the generic tournament is not a limit of a m.e.c..

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4. The sum of four odd numbers +2 is even!