Definable groups in PRC fields

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Pseudo algebraically closed fields (PAC fields)

Definition (Ax)

A PAC field is a field M such that M is existentially closed (in $\mathcal{L}_{ring} := \{+, -, \cdot, 0, 1\}$) into each regular field extension of M.

Regular extensions: Let M and N be fields of characteristic 0 such that $M \subseteq N$. We say that N is a *regular extension* of M if $N \cap M^{alg} = M$.

Examples: The algebraically closed fields and the pseudo finite fields.

Pseudo real closed fields (PRC fields)

Definition

A field M of characteristic 0 is PRC if M is existentially closed (respect to $\mathcal{L}_{ring}$) into each regular field extension N to which all orderings of M extend.

[Prestel] The class of PRC fields is axiomatizable in $\mathcal{L}_{ring}$.

Examples of PRC fields:

1. PAC fields of characteristic 0.
2. Real closed fields.
3. Let $\mathbb{Q}_{tr}$ be the maximal totally real extension of $\mathbb{Q}$, that is the fixed field in $\mathbb{Q}^{alg}$ of all involutions of $G(\mathbb{Q})$. Pop showed that $\mathbb{Q}_{tr}$ is PRC.
4. For each field K and each system $(<_i)_{i \in I}$ of orderings on K, there is a regular extension M of K such that M is PRC and every ordering $<_i$ extends to M.

Pseudo real closed fields (PRC fields)

Bounded fields: A field M is called bounded if for any integer n, M has only finitely many extensions of degree n.

Notation: We fix a bounded PRC field K, and $K_0 \prec K$. Let $\mathcal{L} := \mathcal{L}_{ring} \cup \{c_m : m \in K_0\}$.

Bounded: There is n such that K has only n orders

Let $\{<_1, \ldots, <_n\}$ be the orders on K.

Remark: Each $<_i$ is definable by an existential $\mathcal{L}$-formula.

Let $T_{prc} := Th_{\mathcal{L}}(K)$.

Theorem

Let $M \models T_{prc}$. Let $A \subseteq M$. Then $A^{alg} \cap M = acl(A) = dcl(A)$.

Geometric structure with a good notion of dimension.

Forking

Denote by $M^{(i)}$ a fixed real closure of M with respect to $<_i$.

Denote by $a \downarrow_A B$ if $tp^M(a/AB)$ does not fork over A. and by $a \downarrow_A^i B$ if a is $tp^{M^{(i)}}(a/AB)$ does not fork over A

Theorem (M)

Let $M \models T_{prc}$.

1. *forking equals dividing.*
2. *$a \downarrow_A B$ if and only if $a \downarrow_A^i B$, for all $1 \leq i \leq n$*

Theorem

Let M be a PRC field. Then $Th_{\mathcal{L}_{\mathcal{R}}}(M)$ is NTP_2 if and only if M is bounded.

Multi-cells

Let $(M, <_1, \ldots, <_n)$ be a model of T_{prc}.
Denote by τ_i the topolgy induced in M by the order $<_i$.

[Prestel] If $i \neq j$, then $\tau_i \neq \tau_j$.

Definition

Let $(M, <_1, \ldots, <_n)$ be a model of T_{prc}.

A multi-cell in M^k is a non-empty set of the form $C = \bigcap_{i=1}^{n} C^i \cap M$,

where C^i is a $<_i$-cell in $(M^{(i)})^k$.

Multi-topolgy

Multi-cells in M:

1. Points
2. **Multi-intervals:** $I := \bigcap_{i=1}^{n} I^i \cap M$, where $I^i \subseteq M^{(i)}$ is a non-empty open $<_i$-interval in $M^{(i)}$

Definition

We define the *multi-topolgy* τ *as the topology in M generated by the multi-intervals and τ^m its product topology in M^m*

Definition

Multi-semialgebraic set = Union of multi-cells.

Density Theorem

Theorem

Let $(M, <_1, \ldots, <_n)$ be a model of T_{prc}. Let $A \subseteq M$ and let $S \subset M^k$ be an $\mathcal{L}(A)$-definable set. Then there are $m \in \mathbb{N}$ and $C_1, \ldots, C_m$ multi-cells in M^k such that:

1. $S \subseteq \bigcup_{j=1}^{m} C_j$,
2. *S is τ^k-dense in C_j for all $1 \leq j \leq m$,*
3. *C_j is $\mathcal{L}(A)$-definable in M, for all $1 \leq j \leq m$.*

Definable groups in bounded PAC fields

Virtually Isogeny: A virtual isogeny between two groups, say G and H is an isogeny (morphism of groups that is surjective and has a finite kernel) between subgroups G_1 of finite index in G and H_1 of finite index in H.

Theorem (Hrushovski-Pillay)

Let G be a definable group in a bounded PAC field M. Then there is an algebraic group H defined over M and a definable (in M) virtual isogeny between G and $H(M)$

Definition

Let G be a M-definable group

1. $X \subseteq G$ *is* *generic* *if finitely many translates cover* G.
2. *Let* $H < G$ *(type-definable).* H *has* *bounded index* *in* G *if:* $|G(M^*)/H(M^*)| < |M^*|$ *for some* $M^* \succeq M$ *saturated.*
3. G_M^{00} *:= smallest type* M*-definable subgroup of bounded index*

Definition

Let T *be a theory* T, $\mathbb{M}$ *a monster model of* T *and* G *a definable group. A global type* $p \in S_G(\mathbb{M})$ *is* *strongly (left)* f*-generic over* A *if for all* $g \in G(\mathbb{M})$, $g \cdot p$ *does not fork over* A ($g \cdot p = tp(g \cdot a/\mathbb{M})$, $a \models p$).

Assume T is NTP2. Let M be a model of T. Denote by μ_M the ideal of formulas which do not extend to a global type, strongly f-generic over M.

Theorem (M, Onshuus, Simon)

Let G be a definable group on M. Assume that G has strong f-generics. Let $p \in S_G(M)$ be f-generic.
Then $G^{00}_M = St_{\mu_M}(p)^2 = (pp^{-1})^2$ and $G^{00}_M \setminus St_{\mu_M}(p)$ is contained in a union of non-wide M-definable sets.

$St(p) = \{g : gp \cap p \text{ is wide}\}$.
$p(x)$ in G is μ-*wide* if it is not contained in a set $D \in \mu$

Theorem (M, Onshuus, Simon)

Let M be a model of T_{prc}, ω-saturated. Let G be an M-definable group with strong f-generics. Then there are G_1, K, H and H_1 such that:

1. *H is a multi-semialgebraic group, M-definable.*
2. *$G_1 \leq G$, $H_1 \leq H$ of finite index and M-definable,*
3. *$K \leq G_1$ is finite and central in G_1,*
4. *G_1/K is definable isomorphic to H_1.*

Group configuration Theorem

Let M be a model of T_{prc} and let $M_0 \prec M$, M is $|M_0|^+$-saturated.

Let G be a definable group in M and let p be a global type in G strongly f-generic over M_0.

Let $a \models p|M$, $b \models p|Ma$, and $c = ab$.

Then $tp(c/Ma)$ is strongly f-generic over M_0 and there is an M-definable algebraic group H and dimension-generic elements $a', b', c' \in H(\mathcal{U})$ such that:

1. $a' \cdot b' = c'$
2. $acl(Ma) = acl(Ma')$
3. $acl(Mb) = acl(Mb')$
4. $acl(Mc) = acl(Mc')$

Theorem (M, Onshuus, Simon)

Let μ and λ be M-invariant ideals on G as above, stable under left and right multiplication, and such that μ is S1 in any $X \in \lambda$. Assume we are given a wide and medium type p in G and the following conditions are satisfied:

1. *for any types q, r, if for some $(c, d) \models q \times_{nf} r$, $tp(cd/M)$ or $tp(dc/M)$ is medium, then q is medium;*
2. *for any $(a, b) \in p \times_{nf} p$, $tp(a^{-1}b/M)$ is medium;*
3. *there are $(a, b) \models p \times_{nf} p$ such that $tp(a/Mb)$ does not fork over M.*

Then $Stab(p) = St(p)^2 = (pp^{-1})^2$ is a connected type-definable, wide and medium group. Also $Stab(p) \setminus St(p)$ is contained in a union of non-wide M-definable sets.

[NTP_2 + f-generics + Group configuration theorem + Stabilizer Theorem]

There are H and K such that:

1. H is algebraic and definable.
2. $K < G \times H$ type-definable, connected, medium, wide.
3. $\pi_1(K) = G^{00}_M$.

We can suppose that π_1 and π_2 are injective

$$K^* = \pi_1(K)(\subseteq G) \cong K := \pi_2(K)(\subseteq H)$$

.

In H:

$K \subseteq \overline{K}^{\tau} \subseteq H$ (K has bounded index in $\overline{K}^{\tau}$).

$\overline{K}^{\tau}/K$ is profinite.

There is $X \subseteq H$ definable, and a decreasing sequence $\{U_k : k \in \omega\}$ definable such that:

1. $U_0 = \overline{X}^{\tau}$.
2. $\overline{K}^{\tau} = \bigcap_{k \in \omega} U_k$
3. U_k τ-open and multi semialgebraic.
4. X τ-dense in U_k.
5. $\langle X^* \rangle = G$.

Select points $\{\alpha_k : k < p\} \subseteq G$ such that

$$G = \bigcup_{k<p} \alpha_k \cdot_G (U_4 \cap X)^*$$

$W := (U_4 \cap X) \times \{0, \ldots, p-1\}$. $\quad G \cong W/E$

$W^- := U_4 \times \{0, \ldots, p-1\}$. $\quad G_0 = W^-/E^-$.

$$G \hookrightarrow G_0$$