Topological dynamics of stable groups

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Set-up

G is a stable group in countable language L
$T = Th(G)$
$G^* \succ G$ is a monster model of T
Can we do stable group theory without forking?

Definition

Let $p, q \in S(G)$
$p * q = tp(a \cdot b/M)$, where $a \models p$, $b \models q$ and $a \downarrow_G b$

- $(S(G), *)$ is a semi-group

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Definition

(1) X is a G-flow if

- X is a compact topological space
- G acts on X by homeomorphisms

(2) X is point-transitive if there is a dense G-orbit $\subseteq X$.
(3) $Y \subseteq X$ is a G-subflow of X if Y is closed and G-closed.

Example

Let X be a G-flow and $p \in X$. Then $cl(Gp)$ is a subflow of X generated by p.

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Topological dynamics

Let X be a point-transitive G-flow.

$G \ni g \rightsquigarrow \pi_g : X \overset{\approx}{\to} X$, $\pi_g(x) = g \cdot x$,

$$E(X) = cl(\{\pi_g : g \in G\}) \subseteq X^X$$

- cl is the topological closure w.r. to pointwise convergence topology in X^X
- $E(X)$ is the Ellis (enveloping) semigroup of X
- $E(X)$ is a point-transitive G-flow:
  1. for $f \in E(X)$ and $g \in G$, $g * f = \pi_g \circ f$
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Functional representation

Sometimes $X \cong E(X)$

Let $\mathcal{A} \subseteq \mathcal{P}(G)$ be a G-algebra of sets (i.e. closed under left translation in G).
Then $S(\mathcal{A})$ is a G-flow.
For $p \in S(\mathcal{A})$ we define $d_p : \mathcal{A} \to \mathcal{P}(G)$ by:

$$d_p(U) = \{g \in G : g^{-1}U \in p\}$$

Definition

$\mathcal{A}$ is *d-closed* if $\mathcal{A}$ is closed under d_p for every $p \in S(\mathcal{A})$.

Example

$\mathcal{A} = Def(G)$ is d-closed, because every $p \in S(G)$ is definable.

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Assume $\mathcal{A}$ is d-closed.

- For $p \in S(\mathcal{A})$, $d_p \in End(\mathcal{A}) := \{G\text{-endomorphisms of } \mathcal{A}\}$.
- Let $d : S(\mathcal{A}) \to End(\mathcal{A})$ map p to d_p. Then d is a bijection.
- d induces $*$ on $S(\mathcal{A})$ so that

$$d : (S(\mathcal{A}), *) \stackrel{\cong}{\to} (End(\mathcal{A}), \circ)$$

Theorem 1

$$(E(S(\mathcal{A})), \circ) \cong^1 (S(\mathcal{A}), *) \cong^2 (End(\mathcal{A}), \circ)$$

Proof

1. For $p \in S(\mathcal{A})$ let $l_p(q) = p * q$.
Then $l_p \in E(S(\mathcal{A}))$ and $p \mapsto l_p$ gives $\cong^1$.
2. This is d.

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$(S(G), *)$ in the definable realm

Definition

1. $\Delta \subseteq L$ is invariant if the family of relatively Δ-definable subsets of G is closed under left and right translation in G, and also under taking inverse.
2. Let $Inv = \{\Delta \subseteq_{fin} L : \Delta \text{ is invariant}\}$.

Fact

Inv is cofinal in $[L]^{<\omega}$.

Let $\Delta \in Inv$.

Notation

$$Def_\Delta(G) = \{\text{relatively } \Delta\text{-definable subsets of } G\}$$

$$S_\Delta(G) = S(Def_\Delta(G)),$$

the space of complete Δ-types over G.

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$$S_\Delta(G) = S(Def_\Delta(G)),$$

the space of complete Δ-types over G.

$(S(G), *)$ in the definable realm

Definition

1. $\Delta \subseteq L$ is invariant if the family of relatively Δ-definable subsets of G is closed under left and right translation in G, and also under taking inverse.
2. Let $Inv = \{\Delta \subseteq_{fin} L : \Delta \text{ is invariant}\}$.

Fact

Inv is cofinal in $[L]^{<\omega}$.

Let $\Delta \in Inv$.

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$(S(G), *)$ in the definable realm

1. $Def_{G,\Delta}(M)$ is a d-closed G-algebra of sets.
   (this relies on the full definability lemma in local stability theory)
2. $(S_{G,\Delta}(M), *) \cong (E(S_{G,\Delta}(M)), \circ) \cong (End(Def_{G,\Delta}(M)), \circ)$
   (this is by Theorem 1)
3. $$Def_G(M) = \bigcup_{\Delta \in Inv} Def_{G,\Delta}(M)$$
4. $\langle S_{G,\Delta}(M), \Delta \in Inv \rangle$ is an inverse system of G- flows and semi-groups
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$$S(G) \ni p \rightsquigarrow d_p : Def(G) \to Def(G)$$

$$S_\Delta(G) \ni p \rightsquigarrow d_p : Def_\Delta(G) \to Def_\Delta(G)$$

$$d_p \rightsquigarrow Ker(d_p),\ Im(d_p)$$

$$Ker(d_p) = \{U \in Def_\Delta(G) : [U] \cap cl(Gp) = \emptyset\}$$

Idea

- The larger the type $p \in S_G(M)$, $p \in S_{G,\Delta}(M)$
- The smaller the flow $cl(Gp)$.
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$Ker(d_p), Im(d_p)$: measures of the size of p.

Let $p \in S(G)$ (or $p \in S_\Delta(G)$...)

Let $p^{*n} = \underbrace{p * \cdots * p}_{n}$. So $d_{p^{*n}} = \underbrace{d_p \circ \cdots \circ d_p}_{n}$.

Let $R(p) = \langle RM_\Delta(p) : \Delta \in Inv \rangle$.

Lemma

1. $R(p^{*n})$ grow (coordinatewise), $Ker(d_{p^{*n}})$ grow and $Im(d_{p^{*n}})$ shrink with $n = 1, 2, 3, \ldots$.
2. The growth/shrinking of these three sequences is strictly correlated.

There are similar connections between RM_Δ, Ker and Im in $S_\Delta(G)$. In particular, if $p \in S_\Delta(G)$, $U \in Def_\Delta(G)$ and $RM_\Delta(U) < RM_\Delta(p)$, then $U \in Ker(d_p)$

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Assume $\mathcal{A} \subseteq Def_\Delta(G)$ is a G-subalgebra.

- $\mathcal{A}$ is scattered, has finite CB-rank, MR_Δ-rank.
- $\mathcal{A}$ is atomic.

For $g \in G$ let $U_g \in \mathcal{A}$ be the atom containing g.

- It exists:
  there is some atom $U \in \mathcal{A}$, then $U = U_h$ for any $h \in U$.
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- $U_1 < G$
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Atoms almost determine $\mathcal{A}$

Subalgebras of $Def_\Delta(G)$

Assume $\mathcal{A} \subseteq Def_\Delta(G)$ is a G-subalgebra.

- $\mathcal{A}$ is scattered, has finite CB-rank, MR_Δ-rank.
- $\mathcal{A}$ is atomic.

For $g \in G$ let $U_g \in \mathcal{A}$ be the atom containing g.

- It exists:
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Atoms almost determine $\mathcal{A}$

$\mathcal{A} := Im(d_p)$ explained

Let $p \in S_\Delta(G)$ and $\mathcal{A} = Im(d_p)$.
What is U_1?
For $V \in Def_\Delta(G)$:

$$1 \in d_p(V) \iff V \in p$$

Choose $V \in p$ with $CB(V) = CB(p)$ and $Mlt(V) = Mlt(p)$.
Here CB and Mlt is meant in $Def_\Delta(G)$.
Then

- $U_1 = d_p(V)$
- $U_g = gU_1$ for all $g \in G$
- In fact, $U_1 = Stab(p) = \{g \in G : gp = p\}$.

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Test case: the 2-step theorem

Assume $p \in S(G)$. Recall that $G^* \succ G$ is a monster model.

$p(G^*)$ generates a subgroup $\langle p(G^*) \rangle < G^*$, invariant under $Aut(G^*/G)$.

Let $\langle p \rangle$ be the minimal type-definable subgroup of G^* containing $\langle p(G^*) \rangle$.

Let $Cl_*(p) = cl(\{p^n : n \in \mathbb{N}^+\})$, where

$$p^n = \underbrace{p * \cdots * p}_{n}$$

So $p^n(G^*) \subseteq \langle p \rangle$.

Theorem 2

The generic types of $\langle p \rangle$ are precisely the types in $Cl_*(p)$ with maximal ranks $RM_\Delta, \Delta \subseteq L$ finite, invariant.

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The generic types of $\langle p \rangle$ are precisely the types in $Cl_*(p)$ with maximal ranks $RM_\Delta, \Delta \subseteq L$ finite, invariant.

The 2-step theorem

- $RM_\Delta(p_0 * p_1) \geq RM_\Delta(p_i)$
- $RM_\Delta(p^n), n < \omega$, is non-decreasing.
- If $q = \lim_i p^{n_i}$, then $RM_\Delta(q) \geq RM_\Delta(p^{n_i})$.

If $q \in Cl_*(p)$ is a generic type of $\langle p \rangle$, then q is an accumulation point of the set $\{p^n : n > 0\}$.

Is the converse true? Not necessarily.

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Assume q is an accumulation point of $\{p^n : n > 0\}$ and r is an accumulation point of $\{q^n : n > 0\}$. Then r is a generic type of $\langle p \rangle$.

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An example

Let $p \in S(G)$ or $S_\Delta(G)$ and $q = p^{-1} * p$. Then the group $\langle q \rangle$ is connected.

Proof. Wlog $p \in S_\Delta(G)$. So $q \in S_\Delta(G)$.

$$Im(d_{q^1}) \supseteq Im(d_{q^2}) \supseteq Im(d_{q^3}) \supseteq \dots$$

There is n s.t. $Im(d_{q^n}) = Im(d_{q^{n+k}})$ for all k.
Let $\mathcal{A} = Im(d_{q^n})$.
So $d_{q^{n+1}} = d_{p^{-1}} \circ d_p \circ d_{q^n}$.
We shall prove that

$$d_q|_{\mathcal{A}} = id_{\mathcal{A}}$$

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$d_p(U_1) \neq \emptyset$, so $U_s = sU_1 \in p$ for some $s \in G$.

$$U_1' = sU_1s^{-1} = U_1^s = d_p(U_s)$$

$d_p(U_h) = d_p(hs^{-1}U_s) = hs^{-1}d_p(U_s) = hs^{-1}U_1' = U'_{hs^{-1}}$
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Can take $t = s^{-1}$:
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The proof continued

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