A property of pseudofinite groups

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Pseudofinite groups: Definition and examples

Definition

A group G is **pseudofinite** if it is an infinite model of the first-order theory of finite groups.

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Similarly, one can define the concept of pseudofinite field.

- Examples are torsion-free divisible abelian groups, infinite extraspecial p-groups of odd exponent p, general linear groups over a pseudofinite field,...
- Non-examples are $(\mathbb{Z}, +)$, free groups, infinite dihedral subgroup, Higman group, ...

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Theorem (Burnside)

Any finite group admitting a fixed-point-free involutory automorphism is abelian.

Theorem (Brauer and Fowler, '55)

There are only a finite number of finite simple groups with a given centralizer of an involution.

Infinite abelian subgroup

Theorem (Hempel, P.)

A pseudofinite group has an infinite abelian subgroup.

Main ingredients are:

- The Feit-Thompson Theorem: A finite group without an involution is solvable.
- Basic notions and techniques from infinite group theory.

FC-center

The **FC-center** of a group G is defined as

$$\mathrm{FC}(G) = \{x \in G : [G : C_G(x)] \text{ is finite}\}.$$

Equivalently, for any x in G: $x \in \mathrm{FC}(G)$ iff x^G is finite.

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- It is a characteristic subgroup and its derived subgroup $\mathrm{FC}(G)'$ is a periodic subgroup.
- If there exists a natural number k such that

  $$\mathrm{FC}(G) = \{x \in G : [G : C_G(x)] \leq k\},$$

  then $\mathrm{FC}(G)'$ is finite (i.e. the group $\mathrm{FC}(G)$ is finite-by-abelian).

Example ($G = \mathrm{FC}(G)$)

1. Infinite direct sum of finite non-abelian groups.
2. An infinite extraspecial p-group of odd exponent p.

Main Lemma

Lemma

If G is a pseudofinite group containing an involution i with a finite centralizer, then either $C_G(i) \cap \mathrm{FC}(G)$ contains an involution or $G = C_G(i) \cdot \mathrm{FC}(G)$.

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Idea of the proof. Set $I(x)$ to denote the set of involutions in $C_G(x)$, and for any $g \in G$, let X_g be the finite set

$$I(i) \cup I(i^g) \cup g^{-1} C_G(i) \cup C_G(i) g^{C_G(i)}.$$

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- Using the fact that any two involutions are either conjugate or have an involution centralizing both of them, show that for any $h \in G$ the set $X_g \cap X_g^h \neq \emptyset$.
- Thus $G = \bigcup_{a,b \in X_g} \{u \in G : a^u = b\}$.

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- Thus $G = \bigcup_{a,b \in X_g} \{u \in G : a^u = b\}$.
- Then $X_g \cap \mathrm{FC}(G) \neq \emptyset$ by a lemma of B. H. Neumann.
- Note: either $I(i) \cap \mathrm{FC}(G) \neq \emptyset$ or $g^{-1} C_G(i) \cap \mathrm{FC}(G) \neq \emptyset$. □

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- As G is pseudofinite, there exists some non-trivial $h \in G$ such that $h^G \subseteq C_G(h)$, yielding that $G/C_G(h)$ is finite and so is G, a contradiction.

Proof of the Theorem (cont.)

Without loss of generality, assume G is periodic.

- Let x_0 be an element of G with infinite centralizer.
- Consider the pseudo-finite group $G_0 = C_G(x_0)/\langle x_0 \rangle$ and apply the first step to find an element x_1 in $C_G(x_0)$ whose class $\bar{x}_1$ in G_0 is non-trivial and has an infinite centralizer.
- Note: Since x_0 and x_1 commute, the group $\langle x_0, x_1 \rangle$ is finite.

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- Set $G_1 = C_G(x_0, x_1)/\langle x_0, x_1 \rangle$ and apply the first part to find an element x_2 in $C_G(x_0, x_1)$ whose class $\bar{x}_2$ in G_1 is non-trivial and has again an infinite centralizer.

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- Iterating this process ...
- We find infinitely many elements $x_0, x_1, x_2, \ldots$ that generate an infinite abelian subgroup. □

Comments and remarks

- It is reasonable to use the Feit-Thompson Theorem since it is also needed to show the existence of an infinite abelian subgroup in a locally finite group (due to Hall and Kulatilaka).
- The proof is surprisingly easy compared to the existence of an infinite abelian subgroup in any profinite group!

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At the level of finite groups one immediately obtains:

Corollary

For each n, there are only finitely many finite groups in which the centralizer of every non-trivial element has size at most n.

Definable subgroups

- Assuming a uniform chain condition on centralizers (up to bounded index) one easily obtain an infinite definable (finite-by-)abelian subgroup like in stable (simple) theories.
- Consequently, every pseudofinite group of thorn-rank one is finite-by-abelian-by-finite (Wagner, '15).

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Theorem (Wagner, '15)

There is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for any $n, m \in \mathbb{N}$, if G is a finite group in which one cannot find elements $a_0, \ldots, a_m$ satisfying

$$[C_G(a_0, \ldots, a_{i-1}) : C_G(a_0, \ldots, a_i)] \geq n$$

for every $i \leq m$, then there exists a subgroup H of G such that $|H'| \leq f(n, m)$ and $|G| \leq f(n, m)|H|^{f(n,m)}$.

Restricted centralizers I

Definition

A group has restricted centralizers if the centralizer of any element is finite or has finite index.

Examples:

- FC-groups.
- Infinite dihedral group.
- Tarski monsters.
- Any group of thorn-rank one.

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Theorem (Shalev, '94)

A profinite group having restricted centralizers is finite-by-abelian-by-finite.

In the proof Shalev uses

- The solution to the restricted Burnside Problem for compact groups.
- A result on finite groups due to Hartley-Meixner and Khukhro (CFSG).

Restricted centralizers II

Theorem (Hartley and Meixner '81; Khukhro '93)

There are two functions $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *and* $h : \mathbb{N} \to \mathbb{N}$ *such that given a finite group* G *admitting an automorphism of prime order* p *with centralizer of order* n*, there is a nilpotent subgroup of index bounded by* $f(n, p)$ *and whose nilpotency class is at most* $h(p)$*.*

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There is a function $f : \mathbb{N} \to \mathbb{N}$ such that if G is a finite group amitting an involutory automorphism α with $|C_G(\alpha)| \leq n$, then there is a normal subgroup H of G such that $[G : H] \leq f(n)$ and $H' \leq C_G(\alpha)$.

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- Using the Main Lemma we obtained a model-theoretic proof this.

Restricted centralizers III

Theorem (Hempel, P.)

The FC-center of any pseudo-finite group with restricted centralizers is definable and has finite index.

- We use the result of Khukhro, and Hartley and Meixner.
- If the pseudo-finite group is $\aleph_0$-saturated, then the FC-center is finite-by-abelian.

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Corollary (Hempel, P.)

There is a function $f : \mathbb{N} \to \mathbb{N}$ such that: if G is a finite group such that for any element x the size of $C_G(x)$ or $G/C_G(x)$ is at most n, then there is a characteristic subgroup H of G such that the size of G/H and H' are bounded by $f(n)$.