Some model theory of the free group

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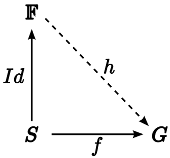
1. Historical Remarks
2. Model Theoretic Study of the Free Group
3. Further research

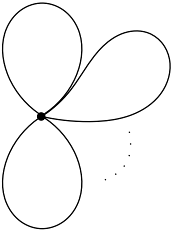
- A group $\mathbb{F}$ is free, if it has the **universal property** (over a subset $S \subset \mathbb{F}$) for the class of groups.



- **Universal property:** for every group G and every function $f : S \to G$, there exists a unique homomorphism $h : \mathbb{F} \to G$ such that the above diagram commutes;
- the subset S is called the **basis** of $\mathbb{F}$; and
- the cardinality of S is called the **rank** of $\mathbb{F}$.

Free Groups - Topology

- A group $\mathbb{F}$ is free, if it is isomorphic to the fundamental group of a bouquet of circles:

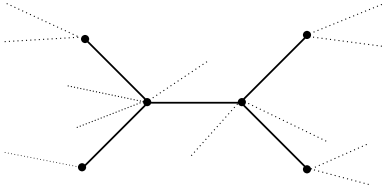
$\mathbb{F} \cong \pi_1($  $)$

- The **fundamental group** of a pointed topological space $(X, \bullet)$ is the group of homotopy classes of loops of X that start and end at $\bullet$ (where the group law is induced by the composition of loops).

Free Groups - Geometry

- A group $\mathbb{F}$ is free, if it admits a **free action** without inversion on a **tree** (a nonoriented connected graph without cycles):

$\mathbb{F} \curvearrowright$



- An action (by graph automorphisms) of a group G on a graph $\mathcal{G}$ is **free**, if $g.x \neq x$ for each $g \in G \setminus \{1\}$ and every vertex $x \in \mathcal{G}$.

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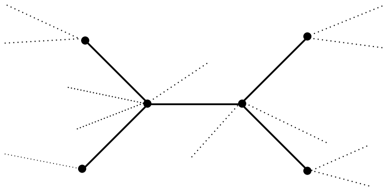

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Theorem (Nielsen-Schreier):

A subgroup of a free group is a free group.

Question (Tarski):

Do nonabelian free groups share the same common first-order theory ?

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- Free abelian groups, $\mathbb{Z}^n$, of different ranks have different first-order theories;
  - since $[\mathbb{Z}^n : 2\mathbb{Z}^n] \neq [\mathbb{Z}^m : 2\mathbb{Z}^m]$ for $m \neq n$.

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Question (Malcev):

Suppose $\mathbb{F}_n$ is a free group of rank n. Is the derived subgroup $[\mathbb{F}_n, \mathbb{F}_n]$ definable in $\mathbb{F}_n$?

Remark: the quotient group $\mathbb{F}_n/[\mathbb{F}_n, \mathbb{F}_n]$ is isomorphic to $\mathbb{Z}^n$.

Tarski's Problem

Theorem (Sela 2001 / Kharlampovich-Miasnikov):

Nonabelian free groups share the same common first-order theory.

As a matter of fact the following chain is elementary:

$$\mathbb{F}_2 \leq \mathbb{F}_3 \leq \ldots \leq \mathbb{F}_n \leq \ldots$$

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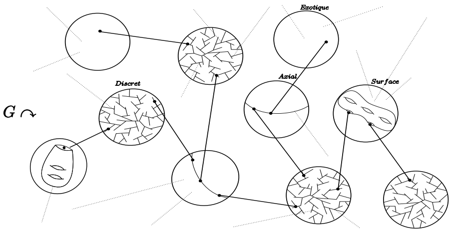

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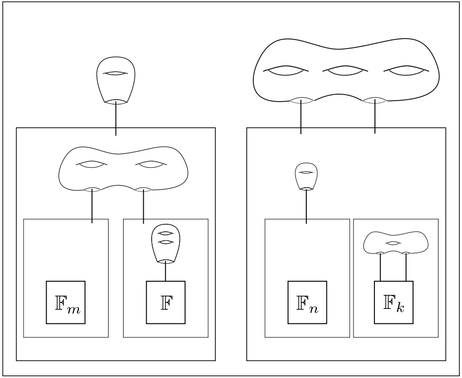

First model theoretic results by Sela

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The theory of the free group (weakly) eliminates imaginaries up to adding some "reasonable" sorts.

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$\mathbb{F}_\omega$ is not superstable.

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Theorem (Pillay):

- An element of a nonabelian free group is generic if and only if it is primitive, i.e. it is part of some basis.
- Any maximal independent set of realizations of the generic type in $\mathbb{F}_n$ is a basis of $\mathbb{F}_n$.

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Theorem (Pillay / S.):

The generic type has infinite weight.

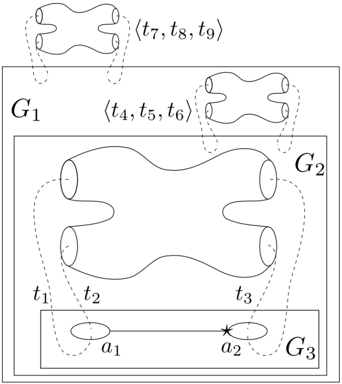
Theorem (Louder-Perin-S.):

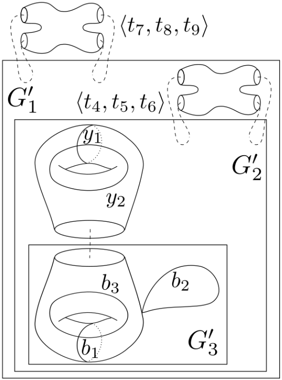
There exists a finitely generated group $G \models T_{fg}$ and two (finite) maximal independent sequences of realizations of the generic type in G of different length.

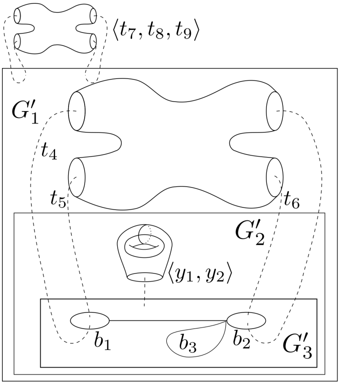
Theorem (Brück):

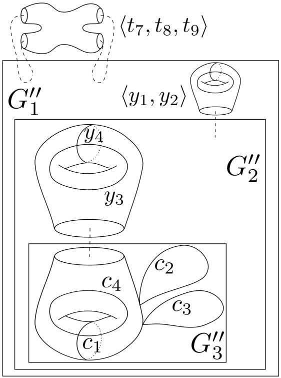
For every $n < \omega$, there exists a finitely generated group $G_n \models T_{fg}$ and two (finite) maximal independent sequences of realizations of the generic type in G_n for which the ratio of their lengths is greater than n.

Arbitrarily Large Weight






$\langle t_7, t_8, t_9 \rangle$
G'_1
t_4
t_5
t_6
G'_2
$\langle y_1, y_2 \rangle$
b_1
b_3
b_2
G'_3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Theorem (Perin-S. / Ould Houcine):

$\mathbb{F}_n$ is homogeneous.

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Homogeneity

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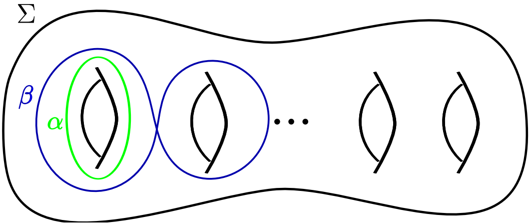
Theorem (S.):

Each uncountable free group is not $\aleph_1$-homogeneous.

Most of the surface groups are not homogeneous.



Theorem (Dehn-Nielsen-Baer):

$$Aut(\pi_1(\Sigma)) \cong Homeo(\Sigma)$$

Forking Independence

Theorem (Perin-S.):

Let $\mathbb{F}$ be a nonabelian free group and $\bar{b}, \bar{c} \subset \mathbb{F}$. Then $\bar{b}$ is independent from $\bar{c}$ over $\emptyset$ if and only if $\mathbb{F}$ admits a free splitting as $B * C$ with $\bar{b} \subset B$ and $\bar{c} \subset C$.

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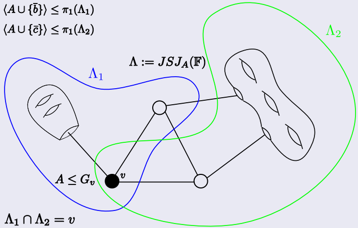
Theorem (Perin-S.):

Let $\mathbb{F}$ be a nonabelian free group and $\bar{b}, \bar{c}, A \subset \mathbb{F}$. Then $\bar{b}$ is independent from $\bar{c}$ over A if and only if



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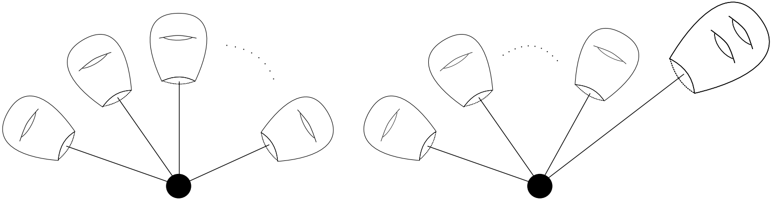

Remark: the main tool for confirming the algebraic conditions of ampleness is Thurston's pseudo-Anosov homeomorphisms.

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Theorem:

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Theorem (Perin / Byron-S.):

Centralizers of elements in nonabelian free groups are pure groups, i.e. the induced structure on a centralizer can be defined by multiplication alone.

Remark: this is the first example of a stable group which is ample but no infinite field is interpretable in it.

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Theorem (Sela):

Any free product of stable groups is stable.

Definable Groups

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Theorem (Perin-Pillay-S.-Tent / Kharlampovich-Miasnikov / Bestvina-Feighn):

Any proper definable subgroup of a nonabelian free group is cyclic.

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Conjecture:

The only definable groups in the free group are the “obvious” ones.

Finite Index Subgroups

Theorem (Sela):

Let G be a finitely generated model of the free group. Then G is a hyperbolic tower.

Examples: nonabelian free groups, surface groups, $\pi_1(\Sigma)$ with $\chi(\Sigma) < -1$, and free products of these groups.

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Let G be a free product of nonabelian groups and surface groups $\pi_1(\Sigma)$ (with $\chi(\Sigma) < -1$). Then any finite index subgroup of G is elementarily equivalent to G.

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Theorem (Guirardel-Levitt-S.):

Let G be a finitely generated model of the free group. Then either it is the free product of free groups and surface groups, or it has infinitely many subgroups of finite index pairwise non-elementarily equivalent.

Superstable part

Theorem (Perin-S.):

Let $\phi(x)$ be a formula over $\mathbb{F}_n$. Suppose $\phi(\mathbb{F}_n) \neq \phi(\mathbb{F}_\omega)$. Then $\phi(x)$ is not superstable.

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Conjecture:

Let $\phi(x)$ be a formula over $\mathbb{F}_n$. Then $\phi(x)$ is superstable if and only if $\phi(\mathbb{F}_n) = \phi(\mathbb{F}_\omega)$.

Theorem (Sela / Kharlampovich-Miasnikov):

The free group admits quantifier elimination up to boolean combinations of $\forall\exists$ formulas.

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Question:
Does the free group admit a model companion?

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