Model Theory and Combinatorial Geometry.

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Combinatorial geometry

Let X be a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ a family of subsets of X.

Let $\mathbb{I} \subseteq X \times \mathcal{F}$ be the incidence relation $\mathbb{I} = \{(x, F) \in X \times \mathcal{F} \colon x \in F\}$, and $\mathbb{G}_{\mathbb{I}}$ be the incidence structure $\mathbb{G}_{\mathbb{I}} = (X, \mathcal{F}, \mathbb{I})$.
We view $\mathbb{G}_{\mathbb{I}}$ as a bipartite graph.

In combinatorial geometry one is interested in combinatorial properties of the family $\mathcal{G}_{\mathbb{I}}$ of all finite (induced) subgraphs of $\mathbb{G}_{\mathbb{I}}$:

$$\mathcal{G}_{\mathbb{I}} = \{(X_0, \mathcal{F}_0, I) \colon X_0 \subseteq X, \mathcal{F}_0 \subseteq \mathcal{F} \text{ are finite, } I = \mathbb{I} \cap (X_0 \times \mathcal{F}_0)\},$$

Example

Let $X = \mathbb{R}^2$ and $\mathcal{F}$ be the set of all circles of radius one in $\mathbb{R}^2$.

Unit Distance Problem: What is the growth rate of

$$f(m, n) = \max\{|I| \colon (X_0, \mathcal{F}_0, I) \in \mathcal{G}, |X_0| = m, |\mathcal{F}_0| = n\},$$

as $m, n \to \infty$?

Setting

By a relation $\mathbb{I}$ we mean a subset of the Cartesian product of two sets $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$.

Often we view a relation $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ as the bipartite graph $\mathbb{G}_{\mathbb{I}} = (\mathbb{U}, \mathbb{V}, \mathbb{I})$.

For $a \in \mathbb{U}, b \in \mathbb{V}$ we often write $\mathbb{I}(a, b)$ instead of $(a, b) \in \mathbb{I}$;
Also for $b \in \mathbb{V}$ we denote by $\mathbb{I}(\mathbb{U}; b)$ the set

$$\mathbb{I}(\mathbb{U}; b) = \{u \in \mathbb{U} \colon (u, b) \in \mathbb{I}\}.$$

Let $\mathcal{G}_{\mathbb{I}}$ be the set of all finite subgraphs of $\mathbb{G}_{\mathbb{I}}$:

$$\mathcal{G}_{\mathbb{I}} = \{(U, V, I) \colon U \subseteq \mathbb{U}, V \subseteq \mathbb{V} \text{ are finite}, I = \mathbb{I} \cap (U \times V)\}.$$

Assume $\mathbb{I}$ is definable in a first order structure $\mathcal{M}$.

What are combinatorial properties of $\mathcal{G}_{\mathbb{I}}$ under some model-theoretic assumptions, e.g. stability, NIP?

These assumptions can be global, e.g assuming that $\mathrm{Th}(\mathcal{M})$ is NIP; or local, assuming only that $\mathbb{I}$ is NIP.

Example

The relation $\mathbb{I}$ from the unit circles problem is semialgebraic, namely

$$\mathbb{I} = \{(u, v) \in \mathbb{R}^2 \times \mathbb{R}^2 \colon (u_1 - v_1)^2 + (u_2 - v_2)^2 = 1\}.$$

In these talk we consider Strong Erdös–Hajnal Property under the assumption of local distality.

Strong Erdös–Hajnal Property

We say that a relation $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ has Strong Erdös–Hajnal Property if there is $\delta > 0$ such for any $(U, V, I) \in \mathcal{G}_{\mathbb{I}}$ there are $U_0 \subseteq U$, $V_0 \subseteq V$ with $|U_0| \geqslant \delta|U|$, $|V_0| \geqslant \delta|V|$ and either $(U_0 \times V_0) \cap I = \varnothing$ or $(U_0 \times V_0) \subseteq I$.

Theorem (Chernikov-S., 2015)

If a relation $\mathbb{I}$ is definable in a distal structure then $\mathcal{G}_{\mathbb{I}}$ has Strong Erdos-Hajnal Property.

Example

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$.
Let $\mathbb{I} \subseteq \mathbb{F}^2 \times \mathbb{F}^2$ be the set of all pairs (u, v) with $u_1 v_1 = u_2 + v_2$.
The family $\mathcal{G}_{\mathbb{I}}$ does not have Strong Erdos-Hajnal Property.

NIP and Distality

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ be a relation.

As usual for a subset $B \subseteq \mathbb{V}$ we will denote by $S_{\mathbb{I}}(B)$ the set of all complete $\mathbb{I}(u; v)$-types over B.

Definition

The relation $\mathbb{I}$ is NIP if there is $d \in \mathbb{N}$ such that for all finite $B \subseteq \mathbb{V}$ we have $|S_{\mathbb{I}}(B)| \in O(|B|^d)$, i.e. for some $C \in \mathbb{R}$ we have $|S_{\mathbb{I}}(B)| \leqslant C|B|^d$ for all finite $B \subseteq \mathbb{V}$.
A structure $\mathcal{M}$ is NIP if every definable in $\mathcal{M}$ relation is NIP.

To define distality we first introduce some terminology.

Definition

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ be a relation and $\Delta \subseteq \mathbb{U}$ a subset.

1. For $b \in \mathbb{V}$ we say that $\mathbb{I}(\mathbb{U}, b)$ crosses Δ if $\mathbb{I}(\mathbb{U}, b) \cap \Delta \neq \varnothing$ and $\neg\mathbb{I}(\mathbb{U}, b) \cap \Delta \neq \varnothing$.
2. For $B \subseteq \mathbb{V}$ we say that Δ is $\mathbb{I}$-complete over B if Δ is not crossed by any $\mathbb{I}(\mathbb{U}, b)$ with $b \in B$.

In other words, Δ is $\mathbb{I}$-complete over B if and only if any $a, a' \in \Delta$ realize the same $\mathbb{I}$-type over B.

Definition

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ be a relation.

1. Let $B \subseteq \mathbb{V}$ be a finite set. A family $\mathcal{F}$ of subsets of $\mathbb{U}$ is an (abstract) cell decomposition for $\mathbb{I}$ over B if $\mathbb{U} \subseteq \bigcup \mathcal{F}$ and every $\Delta \in \mathcal{F}$ is $\mathbb{I}$-complete over B.
2. An (abstract) cell decomposition for $\mathbb{I}$ is an assignment $\mathcal{T}$ that to each finite $B \subseteq \mathbb{V}$ assignes a cell decomposition $\mathcal{T}(B)$ for $\mathbb{I}$ over B.

Remark

Any relation $\mathbb{I} \subseteq \mathbb{U} \times V$ admits the smallest cell decomposition where $\mathcal{T}(B)$ is the partition of $\mathbb{U}$ to realizations of complete $\mathbb{I}$-types over B.

We can restate NIP:

Restatement of NIP

A relation $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ is NIP if and only if $\mathbb{I}$ admits a cell decomposition $\mathcal{T}$ with $\mathcal{T}(B) = O(|B|^d)$ for finite $B \subseteq \mathbb{V}$.

The idea of distality is to require that the sets in $\mathcal{T}(B)$ are uniformly definable.

Distality (Simon 2011; Chernikov–Simon 2012)

Definition

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ be a relation.

1. A cell decomposition $\mathcal{T}$ for $\mathbb{I}$ is called weakly distal if there is a relation $\mathbb{D} \subseteq \mathbb{U} \times \mathbb{V}^k$ such that for any finite $B \subseteq \mathbb{V}$ every $\Delta \in \mathcal{T}(B)$ is $\mathbb{D}$-definable over B^k, i.e. there are $b_1, \ldots, b_k \in B$ with $\Delta = \mathbb{D}(\mathbb{U}; b_1, \ldots, b_k)$.
2. We say that the relation $\mathbb{I}$ is distal if it admits a weak distal cell decomposition.

In addition if both $\mathbb{I}$ and $\mathbb{D}$ are definable in a structure $\mathcal{M}$ then we say that $\mathbb{I}$ is distal in $\mathcal{M}$.

Distality

Let $\mathcal{T}$ be a weak distal cell decomposition for a relation $\mathbb{I}$ witnessed by a relation $\mathbb{D} \subseteq \mathbb{U} \times \mathbb{V}^k$.

For a finite set $B \subseteq \mathbb{V}$ let $\mathcal{T}_{\mathbb{D}}(B)$ be the family of all $\mathbb{D}$-definable over B^k sets that are $\mathbb{I}$-complete over B.

Obviously $\mathcal{T}(B) \subseteq \mathcal{T}_{\mathbb{D}}(B)$, and $\mathcal{T}_{\mathbb{D}}$ is also a weak distal cell decomposition for $\mathbb{I}$.

We say that $\mathcal{T}$ is a distal cell decomposition for $\mathbb{I}$ if $\mathcal{T} = \mathcal{T}_{\mathbb{D}}$.

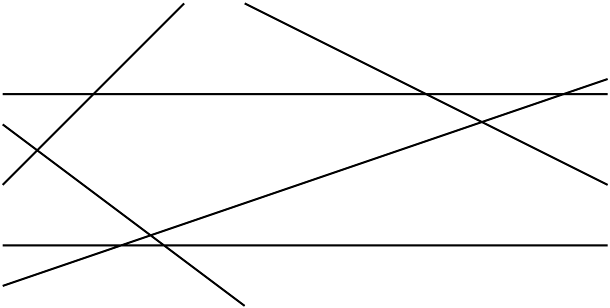
Remark

A distal cell decomposition can be viewed as uniformly definable:
Let $\mathcal{T}_{\mathbb{D}}$ be a distal cell decomposition for $\mathbb{I}$ given by $\mathbb{D} \subseteq \mathbb{U} \times \mathbb{V}^k$.
Let $\Theta \subseteq \mathbb{V} \times \mathbb{V}^k$ be the set of all pairs $(b, \beta) \in \mathbb{V} \times \mathbb{V}^k$ with $\mathbb{I}(\mathbb{U}, b)$ crossing $\mathbb{D}(\mathbb{U}, \beta)$.
Given a finite $B \subseteq \mathbb{V}$ we have

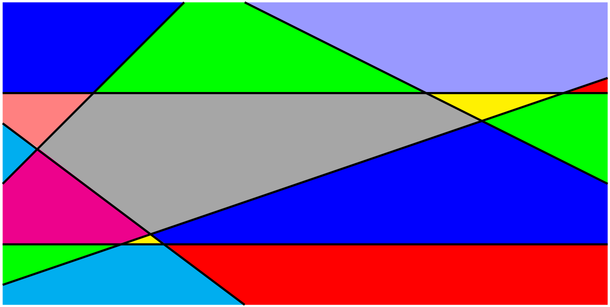
$$\mathcal{T}_{\mathbb{D}}(B) = \{\mathbb{D}(\mathbb{V}, \beta) \colon \beta \in B^k, (b, \beta) \notin \Theta \text{ for any } b \in B\}.$$

An example

Let $\mathbb{U} = \mathbb{R}^2$, $\mathbb{V}$ be the set of all affine lines and half-spaces, and $\mathbb{I}$ be the incidence relation.
We take B to be the set of the following 6 lines.



An example

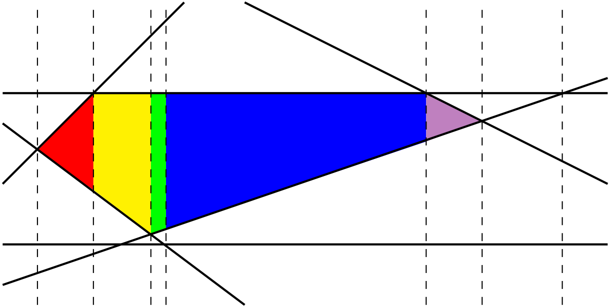


We get at least 15 two-dimensional convex regions that are $\mathbb{I}$-complete over B.

These convex regions can not be uniformly definable when B changes. So the smallest cell decomposition is not weakly distal.

An example

To get the o-minimal cell decomposition we add all vertical lines through the intersection points.



We get a weak distal cell decomposition, where $\mathbb{D}$-definable sets are vertical trapezoids.

Distality implies NIP

Remark

If a relation $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ is distal then $\mathbb{I}$ is NIP.

Indeed let $\mathcal{T} = \mathcal{T}_{\mathbb{D}}$ be a distal cell decomposition for $\mathbb{I}$ with $\mathbb{D} \subseteq \mathbb{U} \times \mathbb{V}^k$. For any finite $B \subseteq \mathbb{V}$ the size of $\mathcal{T}_{\mathbb{D}}(B)$, is bounded by the number of $\mathbb{D}$-definable over B^k sets, hence it is at most $|B|^k$.

NIP+Distality: Strong Erdös–Hajnal Property

Theorem (Chernikov-S., 2015)

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ be a relation. If $\mathbb{I}$ is distal in some NIP structure $\mathcal{M}$ then $\mathcal{G}_{\mathbb{I}}$ has Strong Erdos-Hajnal Property.

Main ingredient of the proof: Cutting Lemma.

ε-cutting

If $I \subseteq \mathbb{U} \times \mathbb{V}$ is a NIP relation then $S_{\mathbb{I}}(B) = O(|B|^d)$.

What is the number of approximate types?

Idea: for $\varepsilon \geqslant 0$ elements $a, a' \in \mathbb{U}$ have the same $(\mathbb{I}, \varepsilon)$-type over finite $B \subseteq \mathbb{V}$ if

$$I(a, b) \leftrightarrow I(a', b)$$

for all but $\varepsilon|B|$-many $b \in B$.

Definition

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ be a relation and $0 \leqslant \varepsilon \leqslant 1$.

1. Let $\Delta \subseteq \mathbb{U}$ be a subset and $B \subseteq \mathbb{V}$ be finite.
   For $0 \leqslant \varepsilon \leqslant 1$ we say that Δ is $(\mathbb{I}, \varepsilon)$-complete over B if
   $$|\{b \in B \colon \mathbb{I}(\mathbb{U}; b) \text{ crosses } \Delta\}| < \varepsilon |B|.$$
   In other words, there is $B_0 \subseteq B$ with $|B_0| \leqslant \varepsilon |B|$ such that Δ is $\mathbb{I}$-complete over $B \setminus B_0$.
2. The family $\Delta_1, \ldots, \Delta_t \subseteq \mathbb{U}$ is called an ε-cutting for $\mathbb{I}$ over B if $\mathbb{U} \subseteq \bigcup_{i=1}^{t} \Delta_i$ and every Δ_i is $(\mathbb{I}, \varepsilon)$-complete over B.

Cutting Lemma

Theorem (Cutting Lemma; Chernikov-S., 2015)

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ *be a relation. Assume* $\mathbb{I}$ *is distal in some NIP structure* $\mathcal{M}$.

For any $0 < \varepsilon \leqslant 1$ *there is* $T(\varepsilon)$ *such that for any finite* $B \subseteq \mathbb{V}$ *there is an* ε*-cutting for* $\mathbb{I}$ *over* B *of size at most* $T(\varepsilon)$.

Cutting Lemma implies Strong Erdös–Hajnal Property

Claim

Assume $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ satisfies the conclusion of the Cutting Lemma. For any $0 < \alpha < 1/2$ there is $0 < \beta < 1$ such that for any finite $A \subseteq \mathbb{U}$, $B \subseteq \mathbb{V}$ there are $A_0 \subseteq A$, $B_0 \subseteq B$ with $|A_0| \geqslant \beta|A|$, $|B_0| \geqslant \alpha|B|$ and either $(A_0 \times B_0) \cap \mathbb{I} = \varnothing$ or $(A_0 \times B_0) \subseteq \mathbb{I}$.

Proof.

Let $\varepsilon = 1 - 2\alpha$.
Let $A \subseteq \mathbb{U}$, $B \subseteq \mathbb{V}$ be finite.
By Cutting Lemma there are $\Delta_1, \ldots, \Delta_t \subseteq \mathbb{U}$ covering $\mathbb{U}$ with $t < T(\varepsilon)$ such that every Δ_i is $(\mathbb{I}, \varepsilon)$-complete over B.
Let $\beta = 1/T(\varepsilon)$.
For at least one i we have $|\Delta_i \cap A| \geqslant \beta|A|$. Let $A_0 = \Delta_i \cap A$.
Choose $B_1 \subseteq B$ with $|B_1| \geqslant (1 - \varepsilon)|B| = 2\alpha|B|$ such that A_0 is $\mathbb{I}$-complete over B_1.
For each $b \in B_1$ either $A_0 \cap \mathbb{I}(\mathbb{U}, b) = \varnothing$ or $A_0 \subseteq \mathbb{I}(\mathbb{U}, b)$. □

Proof of Cutting Lemma (based on Matoušek's idea)

Let $\mathbb{I} \times V \times \mathbb{U}$ be a relation with a distal cell decomposition $\mathcal{T}_{\mathbb{D}}$ given by some $\mathbb{D} \subseteq \mathbb{V} \times \mathbb{V}^k$ definable in a NIP structure $\mathcal{M}$.

Key Lemma

Let $0 < \varepsilon < 1$. There is $l(\varepsilon)$ such that for any finite $B \subseteq \mathbb{U}$ there is $S \subseteq B$ with $|S| < l(\varepsilon)$ such that $\mathcal{T}_{\mathbb{D}}(S)$ is ε-cutting for $\mathbb{I}$ over B. (Notice that $|\mathcal{T}_{\mathbb{D}}(S)| \leqslant |S|^k$).

Proof.

Let $\Theta \subseteq \mathbb{V} \times \mathbb{V}^k$ be the set of all pairs (b, β) such that $\mathbb{I}(\mathbb{U}; b)$ crosses $\mathbb{D}(\mathbb{U}; \beta)$. Clearly Θ is definable in $\mathcal{M}$, hence is *NIP*.
Fix $0 < \varepsilon \leqslant 1$.
Let $B \subseteq \mathbb{V}$ be finite.
By the ε-net theorem there is $S \subseteq B$ with $|S| < l(\varepsilon)$ such that for any $\beta \in B^k$ if $|\Theta(B, \beta) \geqslant \varepsilon |B|$ then $\Theta(\mathbb{U}, \beta) \cap S \neq \varnothing$.
In other words, if $\mathbb{D}(\mathbb{U}, \beta)$ is not $(\mathbb{I}, \varepsilon)$-complete over B then $\mathbb{D}(\mathbb{U}, \beta)$ is crossed by some $\mathbb{I}(\mathbb{U}, s)$ with $s \in S$, i.e. it is not $\mathbb{I}$-complete over S. □

Cutting Lemma for Distal Relations

Theorem (Cutting Lemma; Chernikov-Galvin-S.)

Let $\mathbb{I} \subseteq \mathbb{V} \times \mathbb{U}$ be a distal relation.
For any $0 < \varepsilon \leqslant 1$ there is $T(\varepsilon)$ such that for any finite $B \subseteq \mathbb{V}$ there is an ε-cutting for $\mathbb{I}$ over B of size at most $T(\varepsilon)$.

Corollary

If $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ is a distal relation then the family $\mathcal{G}_{\mathbb{I}}$ has Strong Erdös–Hajnal Property.

Corollary

Let $\mathbb{F}$ be an algebraically closed field of finite characteristic $p > 0$.
The relation $\mathbb{I} \subseteq \mathbb{F}^2 \times \mathbb{F}^2$ given by

$$\mathbb{I} = \{(u, v) \colon u_1 v_1 = u_2 + v_2\}$$

is not distal.

On the proof of Cutting Lemma

Key Lemma

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ be a distal relation with a distal cell decomposition $\mathbb{D}$. Let $0 < \varepsilon \leqslant 1$. For any finite $B \subseteq \mathbb{U}$ there is $S \subseteq B$ such that $\mathcal{T}_{\mathbb{D}}(S)$ is an ε-cutting for $\mathbb{I}$ over B and $|\mathcal{T}_{\mathbb{D}}(S)| < T(\varepsilon)$.

The main ingredient of the proof:
The notion of a distal cell decomposition provides an axiomatic setting for random sampling method of Clarkson and Shor (1989).

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ and $\mathbb{D} \subseteq \mathbb{U} \times \mathbb{V}^k$ be relations such that for any $b_1, \dots, b_k \in \mathbb{U}$ the set $\mathbb{D}(\mathbb{U}; b_1, \dots, b_k)$ is $\mathbb{I}$-complete over $\{b_1, \dots, b_k\}$. Let $B \subseteq \mathbb{U}$ be a finite set and μ be a binomial probability distribution on 2^B.

For $\varepsilon \geqslant 0$ and $S \subseteq B$ let $|\mathbb{D}(S)_{\geqslant\varepsilon}|$ be the number of $\mathbb{D}$-definable over S sets crossed by at lest $\varepsilon|B|$-many $b \in B$.

Clarkson and Shor provided a very useful estimate on $\mathbf{E}(|\mathbb{D}(S)_{\geqslant\varepsilon}|)$.

Comparing two cases

Key Lemma

Let $0 < \varepsilon < 1$. For any finite $B \subseteq \mathbb{U}$ there is $S \subseteq B$ such that $\mathcal{T}_{\mathbb{D}}(S)$ is an ε-cutting for $\mathbb{I}$ over B and

(a) $|S| \leqslant \ell(\varepsilon)$ *in distal+NIP case;*

(b) $|\mathcal{T}_{\mathbb{D}}(S)| \leqslant T(\varepsilon)$ *in distal case.*

The idea of proof in the case (a):

1. Predict $\ell(\varepsilon)$ and choose the uniform probability distribution on the space $\Omega = \binom{B}{\ell(\varepsilon)}$.
2. Show that $\Pr(\{S \in \Omega \colon \mathcal{T}_{\mathbb{D}}(S) \text{ is an } \varepsilon\text{-cutting for } \mathbb{I} \text{ over } B\}) > 0$.

In the case (b) after predicting $T(\varepsilon)$ we work in the space$\Omega = 2^B$ with a binomial distribution and use Clarkson–Shor method to show that the probability of the desired event is positive.

To some surprise, in both cases, we get the same so-called suboptimal bound: $|\mathcal{T}_{\mathbb{D}}(S)| = O((\frac{1}{\varepsilon})^d \log^d(1 + \frac{1}{\varepsilon}))$.

Optimal Cutting Lemma

Theorem (Chernikov-Galvin-S.)

Let $\mathbb{I} \subseteq \mathbb{U} \times \mathbb{V}$ *be a relation admitting a distal cell decomposition* $\mathcal{T}_{\mathbb{D}}$ *with* $\mathcal{T}_{\mathbb{D}}(B) = O(|B|^d)$.
For any $0 < \varepsilon < 1$ *there is a constant* C *such that for finite* $B \subseteq \mathbb{V}$ *there is an* ε*-cutting* $\Delta_1, \ldots, \Delta_t$ *for* $\mathbb{I}$ *over* B *with* $t \leqslant C(\frac{1}{\varepsilon})^d$.
Moreover each Δ_i *is an intersection two* $\mathbb{D}$*-definable over* B^k *sets.*

Remark

The exponent d plays an essential role in applications.

O-minimal case

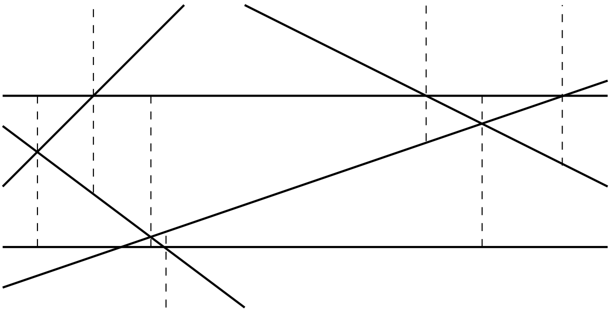
Example

Let $\mathbb{U} = \mathbb{R}^2$, $\mathbb{V}$ be the set of all affine half planes, and $\mathbb{I}$ the incidence relation.
For any finite $B \subseteq \mathbb{V}$ we have $|S_{\mathbb{I}}(B)| \approx |B|^2$.
Hence for any cell decomposition $\mathcal{T}$ we have $|\mathcal{T}(B)| \gtrapprox |B|^2$.
Let $\mathcal{T}$ be the standard o-minimal cell decomposition for $\mathbb{I}$. It is weakly distal with $\mathbb{D} \subseteq \mathbb{U} \times \mathbb{V}^6$ and $|\mathcal{T}(B)| = O(|B|^3)$.
There is a semi-cylindrical cell decomposition $\mathcal{T}^s$ that is distal with $\mathbb{D}^s \subseteq \mathbb{U} \times \mathbb{V}^4$ and $|\mathcal{T}^s_{\mathbb{D}}| = O(|B|^2)$, i.e. it is optimal.

Theorem (Chernikov-Galvin-S.)

Let $\mathbb{I} \subseteq M^2 \times M^n$ be a relation definable in an o-minimal structure $\mathcal{M}$. There is a distal cell decomposition $\mathcal{T}_{\mathbb{D}}$ for $\mathbb{I}$ definable in $\mathcal{M}$ with $|\mathcal{T}_{\mathbb{D}}(B)| = O(|B|^2)$.

An example of optimal distal cell decomposition



We add only vertical line segments where they are needed, i.e. from an intersection point to the first line above (or plus infinity) and the first line below (or minus infinity), as in the following picture.