Model theory of measure-preserving group actions

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Model theory, Będlewo

The basic setup of ergodic theory

- Γ – countable group;
- $(X, \mathcal{X}, \mu)$ – a probability space;
- $\Gamma \curvearrowright X$ by measure-preserving transformations:

$$\mu(\gamma A) = \mu(A) \quad \text{for all } \gamma \in \Gamma, A \in \mathcal{X}.$$

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We identify $A_1, A_2 \in \mathcal{X}$ if $\mu(A_1 \triangle A_2) = 0$. Thus $\mathcal{X}$ becomes a (complete) metric space with metric

$$d(A, B) = \mu(A \triangle B).$$

$\mathcal{X}$ also carries the the structure of a Boolean algebra and Γ acts on $\mathcal{X}$ by isometric isomorphisms.

Examples

- The Bernoulli shift $\Gamma \curvearrowright 2^\Gamma$:

$$(\gamma \cdot x)(\gamma') = x(\gamma^{-1}\gamma').$$

  The measure on 2^Γ is $(p\delta_0 + (1-p)\delta_1)^\Gamma$ for some $p \in (0,1)$.

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- Compact actions: if $\rho\colon \Gamma \to K$ is a homomorphism to a compact group and $L \leq K$ is a closed subgroup, then $\Gamma \curvearrowright K/L$ by

$$\gamma \cdot kL = \rho(\gamma)kL.$$

  The measure on K/L is the quotient of the Haar measure on K.

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- Substructures: A substructure $\mathcal{Y} \subseteq \mathcal{X}$ is a closed, Γ-invariant subalgebra. In ergodic theory, this is known as a factor (and is usually viewed dually, as a Γ-equivariant, measure-preserving map $X \to Y$.

Continuous logic

- Structures are complete metric spaces;
- The equality predicate is replaced by the metric $d(\cdot,\cdot)$;
- Predicates are real-valued, bounded, uniformly continuous functions;
- Connectives are continuous functions $f\colon \mathbf{R}^k \to \mathbf{R}$. We can take as a complete set of connectives the constants, addition, and multiplication;
- Quantifiers are of the form $\inf_y \phi(\bar{x}, y)$ and $\sup_y(\bar{x}, y)$;
- Uniform limits of formulas are again formulas;
- If $\phi(\bar{x})$ is an n-ary formula and M is a model, the interpretation of ϕ in M is a uniformly continuous, bounded function $M^n \to \mathbf{R}$, where the modulus of continuity and the bound can determined syntactically from ϕ.

Measure theory in continuous logic

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Axioms:

- it is a boolean algebra, for example,

$$\sup_{A,B,C} d\big(A \cap (B \cup C), (A \cap B) \cup (A \cup C)\big) = 0.$$

- μ is a probability measure on $\mathcal{X}$:

$$\sup_{A,B} \mu(A \cap B) + \mu(A \cup B) - \mu(A) - \mu(B) = 0.$$

- μ is non-atomic:

$$\sup_{A} \inf_{B} |\mu(A \cap B) - \mu(A)/2| = 0.$$

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These axioms define a complete, ω-categorical, ω-stable theory.

Ergodic theory in continuous logic

To code the group action, one adds a function symbol for every element $\gamma \in \Gamma$ and the axioms:

- each γ is an automorphism of $\mathcal{X}$;
- $\sup_A d\big((\gamma_1 \cdots \gamma_n)A, A\big) = 0$ for every $\gamma_1, \dots, \gamma_n$ such that $\gamma_1 \cdots \gamma_n = 1_\Gamma$.

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Axioms: for every $\gamma \neq 1_\Gamma$,

$$\sup_A \inf_B \mu(B \setminus A) + \mu(B \cap \gamma B) + |\mu(B) - \mu(A)/3| = 0.$$

We call $\mathrm{FR}(\Gamma)$ the resulting theory of free, measure-preserving actions of Γ on a non-atomic probability space.

The case of amenable Γ

A group Γ is called amenable if there exists a left-invariant, finitely additive, probability measure on Γ. Examples are finite groups, abelian (and more generally, solvable) groups. The free group $\mathbf{F}_2$ (and any group containing it) is not amenable.

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Theorem (ess. Ben Yaacov–Berenstein–Henson–Usvyatsov)

Let Γ be an amenable group. Then $\mathrm{FR}(\Gamma)$ is a complete, stable theory that eliminates quantifiers. It is ω-categorical iff Γ is finite.

(They considered the case of $\Gamma = \mathbf{Z}$ but the proof extends, using the machinery of Ornstein–Weiss, to amenable group actions.)

Ergodicity

The action $\Gamma \curvearrowright X$ is called ergodic if there are no non-trivial invariant sets:

$$\forall A \in \mathcal{X} \quad (\forall \gamma \in \Gamma \; \gamma A = A) \implies \mu(A) \in \{0, 1\}.$$

The ergodic decomposition theorem states that every measure-preserving action decomposes as an integral of ergodic components.

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However, ergodicity is not, in general, an elementary property. This can be seen, for example, from the previous theorem.

Strong ergodicity

A sequence $(A_n)_{n \in \mathbf{N}}$ of elements of $\mathcal{X}$ is called asymptotically invariant if

- $\mu(A_n)(1 - \mu(A_n)) \not\to 0$; and
- for every $\gamma \in \Gamma$, $\mu(\gamma A_n \triangle A_n) \to 0$.

An action $\Gamma \curvearrowright X$ is called strongly ergodic if it admits no asymptotically invariant sequences. Equivalently, if its ultrapower is ergodic.

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Amenable groups do not admit strongly ergodic actions: Γ is amenable iff $\Gamma \curvearrowright 2^\Gamma$ is not strongly ergodic iff every action of Γ is not strongly ergodic.

Existential theories and weak containment

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where ϕ is quantifier free.

A system $\mathcal{Y}$ is weakly contained in $\mathcal{X}$ (notation $\mathcal{Y} <_w \mathcal{X}$) if for every existential sentence ϕ, $\phi^{\mathcal{X}} \leq \phi^{\mathcal{Y}}$ (equivalently, if $\mathcal{Y}$ is a factor of an ultrapower of $\mathcal{X}$). Two systems are weakly equivalent if they have the same existential theory.

The notion of weak containment was introduced by Kechris with a different, but equivalent, definition.

Weak containment (cont.)

Some basic facts:

- If Γ is amenable, there is only one weak equivalence class of free Γ-actions.
- An action is strongly ergodic iff the trivial action is not weakly contained in it.
- (Abért–Weiss) The Bernoulli shift $\Gamma \curvearrowright 2^\Gamma$ is weakly contained in any free action of Γ.
- (Bowen–Tucker-Drob) If Γ contains $\mathbf{F}_2$, then there exist uncountably many weak equivalence classes of free, strongly ergodic actions of Γ.

  It is an open question whether this holds for every non-amenable Γ.

Rigidity for compact actions

Theorem (Ioana–Tucker-Drob)

If $\mathcal{X}$ is a compact action, $\mathcal{Y}$ is strongly ergodic, and $\mathcal{X} <_w \mathcal{Y}$, then $\mathcal{X}$ is a factor of $\mathcal{Y}$.

An analogous result had been previously shown for a finite $\mathcal{X}$ by Abért and Elek.

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Corollary

If $\mathcal{X}$ and $\mathcal{Y}$ are compact, strongly ergodic, and weakly equivalent, then they are isomorphic.

This holds because compact actions are coalescent, i.e., every self-embedding is an automorphism.

Distal actions

Let $(X, \mathcal{X}, \mu)$ be a measure-preserving system and K be a compact group. A cocycle is a measurable map $\alpha \colon \Gamma \times X \to K$ that satisfies

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 \cdot x)\alpha(\gamma_2, x).$$

A compact extension of $\mathcal{X}$ is a system $(Y, \mathcal{Y}, \nu)$, where $Y = X \times K$, $\nu = \mu \times \mathrm{Haar}(K)$ and

$$\gamma \cdot (x, k) = (\gamma \cdot x, \alpha(\gamma, x)k).$$

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Beleznay and Foreman proved that for $\Gamma = \mathbf{Z}$, for any ordinal $\beta < \omega_1$, distal actions of distal rank β exist.

An example of Parry and Walters

Let $\mathbf{T}$ be the circle, $\alpha, \beta \in \mathbf{T}$, and $\theta \colon \mathbf{T} \to \mathbf{T}$.

Define $S \colon \mathbf{T} \times \mathbf{T}^{\mathbf{N}} \to \mathbf{T} \times \mathbf{T}^{\mathbf{N}}$ by

$$S(z, w_1, w_2, w_3, \ldots) = (z+\alpha, w_1+\theta(z), w_2+\theta(z+\beta), w_3+\theta(z+2\beta), \ldots).$$

Then S is a distal transformation of rank 2 (compact extension of the irrational rotation $z \mapsto z + \alpha$) and α, β, and θ can be chosen so that it is ergodic.

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Define $\sigma\colon \mathbf{T} \times \mathbf{T}^{\mathbf{N}} \to \mathbf{T} \times \mathbf{T}^{\mathbf{N}}$ by

$$\sigma(z, w_1, w_2, \ldots) = (z + \beta, w_2, w_3, \ldots).$$

Then σ commutes with S and defines a factor map which is not an isomorphism.

Thus there exist ergodic, distal systems that are not coalescent.

Rigidity for distal actions

Theorem (Ioana–Tucker-Drob)

If $\mathcal{X}$ is a distal action, $\mathcal{Y}$ is strongly ergodic, and $\mathcal{X} <_w \mathcal{Y}$, then $\mathcal{X}$ is a factor of $\mathcal{Y}$.

However, the corollary from before does not immediately extend to distal actions because of the example of Parry and Walters.

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Theorem

Every strongly ergodic, distal system is coalescent. In particular, if two strongly ergodic, distal systems are weakly equivalent, then they are isomorphic.

Algebraic closure

Let M be a model and $A \subseteq M$. A subset $B \subseteq M$ is definable over A if the distance predicate $d(\cdot, B)$ is definable over A.

An element $a \in M$ is algebraic over A if it belongs to a compact set definable over A.

The algebraic closure of A, $\mathrm{acl}(A)$, is the set of all elements algebraic over A.

The existential algebraic closure of A, $\mathrm{acl}^{\exists}(A)$, is the union of compact subsets of M that are definable by (a uniform limit of) existential formulas over A.

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Let $\mathcal{X}$ be a strongly ergodic system and $\mathcal{Y}$ be a distal factor of $\mathcal{X}$. Then $\mathcal{Y} \subseteq \operatorname{acl}^{\exists}(\varnothing)$.

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We have two different proofs of the theorem. The first is by induction on the distal rank of $\mathcal{Y}$; the successor step is proved by showing that if $\mathcal{Z}$ is a compact, strongly ergodic extension of $\mathcal{Y}$, then $\mathcal{Z} \subseteq \operatorname{acl}(\mathcal{Y})$. Then one passes to $\operatorname{acl}^{\exists}$ by a diagram argument.

The second argument uses a different characterization of distality (the Furstenberg–Zimmer structure theorem) and a bit of stability theory.

Coalescence

Corollary

Every strongly ergodic, distal system is coalescent.

Proof: Let $\mathcal{X}$ be strongly ergodic and distal and let $\sigma: \mathcal{X} \to \mathcal{X}$ be an embedding. Then σ maps 0-sets of existential formulas to themselves and an isometric injection of a compact set into itself is automatically surjective.

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Question

Do strongly ergodic, distal, non-compact actions exist?

Property (T)

Definition

A group Γ has Kazhdan's property (T) if all of its ergodic actions are strongly ergodic, or equivalently, if ergodicity of measure-preserving actions of Γ is an elementary property.

The equivalence of this definition with the original one of Kazhdan is due to Connes–Weiss and Schmidt.

Examples: $\mathrm{SL}(3, \mathbf{Z})$ or, more generally, lattices in high rank, real, simple Lie groups. Also, *random groups* (in the sense of Gromov), etc.

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Theorem

Let Γ have property (T). Then every distal Γ-action is compact.

This theorem was also proved by Chifan and Peterson using different methods (from operator algebras).