On Definable f-generics in Distal NIP Theories - Bedlewo

Ningyuan Yao

Fudan University

July 7, 2017

Tame unstable theories

In the 90s, Shelah developed stable theory into (tame) unstable environment.

Simple Theories + NIP Theories = Stable Theories.

Unstable Simple Theories

- Pseudo-Finite Fields
- Random Graphs.

Unstable NIP Theories

- o-mionimal structures
- p-adics fields
- algebraic closed valued fields

NIP Theories

Definition 1.1

A formula $\phi(x, y)$ has IP (Independent Property) if there is an indiscernible sequence $(a_i : i < \omega)$ and a tupe b such that

$$\models \phi(a_i, b) \iff i \text{ is even}$$

A theory T has NIP (Not Independent Property) if all formulas $\phi(x, y)$ do not have IP

Among NIP Theories, there are

- Non-Distal Theories: stable theories, ACVF
- Distal Theories: o-minimal theories, $Th(\mathbb{Q}_p)$

Distal NIP Theories

- 1-dim definable subsets in a o-minimal structure is defined by order and parameters.(intervals)
- 1-dim definable subsets in $\mathbb{Q}_p$ is defined by Boolean combinations of

$$v(x-a) > m \text{ and } C(x-a),$$

where v is the valuation map and C is a coset of n^{th} power.

Distal NIP Theories

- Distal NIP Theories are “Pure unstable parts” of NIP Theories.
- Definable subsets are controlled by orders.

Definable groups and its type space

- T is a complete theory of a first-order language $\mathcal{L}$.
- $\mathbb{M}$ is a monster model of T.
- $G = G(\mathbb{M})$ is a definable group, defined by the formula $G(x)$, in T.
- $S_G(\mathbb{M})$ is the space of global types containing the formula $G(x)$.
- Given $p \in S_G(\mathbb{M})$ and $g \in G$, $gp = \{\phi(g^{-1}x : \phi(x) \in p)\}$ is the (left) translate of p.
- $p \in S_G(\mathbb{M})$ is generic iff for any $\phi(x) \in p$, finitely many translate of $\phi(\mathbb{M})$ cover G.

Fundamental Theorem of Stable Groups

Theorem 1.2

Let T be a stable theory, G a group definable in a saturated model $\mathbb{M}$ of T, and $\mathcal{P}_G \subseteq S_G(\mathbb{M})$ the space of all global generic types of G. Then

- *$\mathcal{P}_G$ is nonempty;*
- *G^{00} exists;*
- *$\mathcal{P}_G$ is homeomorphic to G/G^{00}*

How to generalize the Fundamental Theorem to unstable NIP theories?

The Model theoretic invariants G^{00} and $\mathcal{P}_G$ may not exists in unstable context.

f-generics and weakly generics

- A type $p \in S_G(\mathbb{M})$ is f-generic if for any $g \in G$, and every L_{M_0}-formula $\phi \in p$, $g\phi$ does not **divide** over M_0. $(M_0 \prec \mathbb{M})$
- A formula $\phi(x)$ is weakly generic if there is a nongeneric formula $\psi(x)$ such that $\phi(x) \vee \psi(x)$ is generic.
- A type $p \in S_G(\mathbb{M})$ is weakly generic if every formula $\phi \in p$ is weakly generic.

The Connected Component G^{00}

- A a subgroup $H \leq G$ has **bounded index** if $|G/H| < |G|$.
- If G has a minimal bounded index type-definable subgroup, G^{00}, then we say that the type-definable **connected component** of G exist, which is G^{00}.
- For any NIP theories, the connected component exists (Shelah).

Invariants in NIP Theories

- Generics $\implies$ f-generics, weakly generics
- G^{00} exists
- But the space of f-generic (or weakly generic) types is NOT homeomorphic to G/G^{00}.

We need more invariants.

Invariants suggested by Topological Dynamics - Newelski

Consider the topological dynamics system $(G(\mathbb{M}), S_G(\mathbb{M}))$

- The minimal subflows $\mathcal{M} \subseteq \mathcal{S}_{\mathcal{G}}(\mathbb{M}))$
- almost periodic types $p \in \mathcal{M}$
- Space of weakly generic $\mathcal{WG} = cl$(almost periodic types)
- Enveloping semigroup $E(S_G(\mathbb{M}))$.
- Ellis subgroups $\mathcal{I}$ in $E(S_G(\mathbb{M}))$

In stable theories:

$$E(S_G(\mathbb{M})) = S_G(\mathbb{M})$$

$$\mathcal{P}_G = \mathcal{M} = \mathcal{WG} = \mathcal{I} \cong G/G^{00}$$

Newelski's Conjecture: Assuming NIP, $G/G^{00} \cong \mathcal{I}$.

Definably Amenable Groups

Theorem 1.3

(Chernikov-Simon) Assuming NIP, If G is definably amenable. Then

- Newelski's Conjecture holds
- weakly generic types $=$ f-generic types $=$ types with bounded orbit$=$ G^{00}-invariant types.

Definably amenable NIP groups are stable-like groups.

Definably amenable groups in o-minimal Structures

Recall that a structure $(M, <, ...)$ is o-minimal if $<$ is dense linear without endpoints, and every 1-dim definable subset of M is a finite union of intervals.

Theorem 2.1

(Conversano-Pillay) Assuming that T is an o-minimal expansion of RCF, a definable group G is definably amenable iff there exits a exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 0$$

where H is definable torsion-free and K is definably compact

Torsion free part H and compact part K are "orthogonal"

Torsion-free Part and Compact Part

Torsion-free Part H:

- H has a global f-generic type which is 0-definable;
- $H^{00} = H$, so every f-generic type is almost periodic.

Compact Part K:

- K has has a global f-generic type p s.t. every left translate p is finitely satisfiable in every small model;
- Generic types exist
- $\Rightarrow$ every f-generic type is almost periodic (Newelski).

Definable global types and finitely satisfiable global type are commute, so orthogonal (P. Simon).

dfg-groups and fsg-groups

Recall that: A definable group G has fsg if G admits a global type p such that every left translate of p is finitely satisfiable in every small model M_0.

- RCF: G has fsg iff G is definably compact (Hrushovski-Peterzil-Pillay).
- $Th(Q_p)$: G has fsg iff G is definably compact (Onshuus-Pillay).

Definition 2.2

A definable group G has dfg (definable f-generics) if G admits a global type p such that every left translate of p is 0-definable(p has a bounded orbit).

- RCF: G has dfg iff G is torsion-free.
- $Th(\mathbb{Q}_p)$: ?

dfg-groups

Question 1

Would dfg groups be suitable analogs of torsion free groups defined in o-minimal structures among other distal NIP theories, such as $Th(\mathbb{Q}_p)$ and Presburger Arithmetic?

$Th(Q_p)$ vs $Th(\mathbb{R})$

Theorem 3.1 (Pillay-Y)

Assuming NIP. If $p \in S_G(M)$ is dfg (every left translate of p is definable), then the orbit of p is closed, so p is almost periodic, and $G^{00} = G^0$.

Question 2

Assuming distality and NIP. Suppose that G has dfg. Is every f-generic type almost periodic?

The answer is positive in o-minimal expansion of RCF (Trivial).

Nontrivial Positive Examples I

We consider the Presburger Arithmetic: $T_{PA} = Th(\mathbb{Z}, +, <, 0)$.
Let G_a be the additive group and $G = G_a^n$.

Theorem 3.2 (Conant-Vojdani)

$p \in S_G(\mathbb{M})$ is f-generic iff p is 0-definable and every realization $(a_1, ..., a_n)$ of p is algebraic independent over $\mathbb{M}$.

- Every f-generic type of G is 0-definable;
- G has dfg;
- Every f-generic type is almost periodic.

"$(\mathbb{Z}^n, +)$" is an analogs of $(\mathbb{R}^n, +)$" (Informally)

Nontrivial Positive Examples II

We consider the p-adic field $\mathbb{Q}_p$. Let G_a be the additive group and G_m the multiplicative group of the field.

- Every f-generic type of ${G_a}^n$ is 0-definable;
- Every f-generic type of ${G_m}^n$ is 0-definable;
- Both ${G_a}^n$ and ${G_m}^n$ have dfg;
- Every f-generic type is almost periodic, in both ${G_a}^n$ and ${G_m}^n$.

"$({\mathbb{Q}_p}^n, +)$ and $({\mathbb{Q}_p}^{*n}, \times)$ are analogs of $(\mathbb{R}^n, +)$ and $(\mathbb{R}^{*n}, \times)$, respectively." (Informally)

Nontrivial Positive Examples III

- Let $\mathbb{M} \models Th(\mathbb{Q}_p)$;
- UT_n be subgroup of up triangle matrices in $GL(n, \mathbb{M})$;
- Let $\alpha = (\alpha_{ij})_{1 \leq i \leq j \leq n}$.

Theorem 3.3 (Pillay-Y)

Let $\Gamma_{\mathbb{M}}$ be the valuation group of $\mathbb{M}$. Then tp$(\alpha/\mathbb{M}) \in S_{UT_n}(M)$ is f-generic iff the following conditions hold:

- $v(\alpha_{ik}) < v(\alpha_{jk}) + \Gamma_{\mathbb{M}}$ *for all* $1 \leq k \leq n$ *and* $1 \leq i < j \leq k$.
- $v(\alpha) = (v(\alpha_{ij}))_{1 \leq i \leq j \leq n}$ *is algebraic independent over* $\Gamma_{\mathbb{M}}$

The above Theorem still holds if we replace UT_n by B_n, the standard Borel subgroup of $SL(n, \mathbb{M})$.

Nontrivial Positive Examples III

In $\mathbb{Q}_p$ context:

- Every f-generic type of UT_n is 0-definable;
- Every f-generic type of B_n is 0-definable;
- So both T_n and B_n have dfg;
- Every f-generic type is almost periodic, in both UT_n and B_n.

$UT_n(\mathbb{Q}_p)$ and $B_n(\mathbb{Q}_p)$ are analogs of $UT_n(\mathbb{R})$ and $B_n(\mathbb{R})$, respectively. (Informally)

The End

Thanks for your attention !