Hardy Fields and Transseries

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Overview

- Part I: Orders of Infinity and Hausdorff Fields
- Part II: Hardy Fields
- Part III: Transseries

Parts I and II: mostly historical, and background. Part III: includes report on ongoing work with Matthias Aschenbrenner and Joris van der Hoeven

I. Orders of Infinity and Hausdorff Fields

Orders of infinity originate with Du Bois-Reymond, a rival of Cantor. Around 1910 his work was put on a firm basis and further developed by Hardy and Hausdorff. (Hardy: DBR is at times exceedingly obscure.)

Time Line

Du Bois-Reymond:	*Orders of Infinity*	(1870s)
Hausdorff:	*Hausdorff fields*	(1910)
Hardy:	*Logarithmic-exponential functions*	(1910)
Bourbaki:	*Hardy fields*	(1950s)
Conway:	*Surreal numbers*	(1970s)
Écalle:	*Transseries*	(1980s)

Rosenlicht and Boshernitzan should also be mentioned as having revived interest in Hardy fields in the seventies and eighties

Du Bois-Reymond

$\mathcal{C}$: ring of germs at $+\infty$ of real-valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$ on which the function is continuous; *germs*: identify two such functions iff they agree on such an interval. For $f, g \in \mathcal{C}$,

$$f \preccurlyeq g \;:\Longleftrightarrow\; |f(t)| \leqslant c|g(t)|, \text{ eventually, for some positive constant } c.$$

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$$f \preccurlyeq g \ :\Longleftrightarrow \ |f(t)| \leqslant c|g(t)|, \text{ eventually, for some positive constant } c.$$

The transitive and reflexive relation $\preccurlyeq$ on $\mathcal{C}$ yields an equivalence relation

$$f \asymp g \ :\Longleftrightarrow \ f \preccurlyeq g \text{ and } g \preccurlyeq f.$$

and $\preccurlyeq$ induces a partial ordering on the set of equivalence classes; these equivalence classes are essentially DBR's "orders of infinity".

Hausdorff fields

A *Hausdorff field* is a subfield of $\mathcal{C}$. Examples:

$$\mathbb{Q}, \qquad \mathbb{R}, \qquad \mathbb{R}(x), \qquad \mathbb{R}(\sqrt{x}), \qquad \mathbb{R}(x, e^x, \log x)$$

For any expansion $\tilde{\mathbb{R}}$ of the field of reals the following are equivalent:

- $\tilde{\mathbb{R}}$ is o-minimal;
- the germs at $+\infty$ of its definable functions $\mathbb{R} \to \mathbb{R}$ form a Hausdorff field.

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Let H be a Hausdorff field. For every nonzero $g \in H$ there is an $h \in H$ with $gh = 1$, so $g(t) \neq 0$, eventually, and thus $g(t) > 0$, eventually, or $g(t) < 0$, eventually.

Thus H is naturally a (totally) ordered field. Moreover, the set of orders of infinity in H is totally ordered: $f \preccurlyeq g$ or $g \preccurlyeq f$ for all $f, g \in H$.

Extending Hausdorff fields

- *H has a unique algebraic Hausdorff field extension that is real closed*

Hausdorff was particularly interested in "maximal" objects, and their order type.
Every Hausdorff field is contained in a maximal Hausdorff field (by Zorn).
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Maximal Hausdorff fields have uncountable cofinality, in fact:

- *The underlying ordered set of a maximal Hausdorff field H is an η_1-set*: if $A, B \subseteq H$ are countable and $A < B$, then $A < h < B$ for some $h \in H$.

Corollary

(CH) *All maximal Hausdorff fields are isomorphic.*

II. Hardy Fields

A Hardy field is basically a Hausdorff field whose germs can be differentiated, with the resulting derivation as an extra primitive.

This leads to a much richer theory and an intriguing open problem:

What can we say about maximal Hardy fields?

Hardy fields

$$\mathcal{C}^1 \ := \ \{f \in \mathcal{C} : \ f \text{ is eventually continuously differentiable}\}.$$

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A **Hardy field** is a subfield of $\mathcal{C}^1$ that is closed under $f \mapsto f'$. Hardy fields are thus not only ordered fields but also differential fields. For f in a Hardy field we have either $f' > 0$, or $f' = 0$, or $f' < 0$, so f is either eventually strictly increasing, or eventually constant, or eventually strictly decreasing (not necessarily true for f in a Hausdorff field).

Hardy fields continued

Let H be a Hardy field. Its ordering and derivation interact: if $f \in H$ and $f > n$ for all n, then $f' > 0$. Asymptotic relations in H can be "differentiated and integrated":

$$f \preccurlyeq g \Longleftrightarrow f' \preccurlyeq g' \qquad \text{(for nonzero } f, g \not\asymp 1\text{)}$$

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Hardy's Theorem: consider the functions $f(x)$ that can be expressed in terms of x using real constants, $+$, $\times$, $1/-$, exp, log, and which are defined for all sufficiently large real arguments; their germs form a Hardy field. (Main thing to show: such a function is eventually positive, constant, or negative.)

Extending Hardy fields

We now see this theorem as a byproduct of some basic extension results on Hardy fields H:

- H has a unique algebraic Hardy field extension that is real closed
- if $h \in H$, then e^h generates a Hardy field $H(e^h)$
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Special cases of the last item: $H(\mathbb{R})$ and $H(x)$ are Hardy fields, and if $h \in H^>$, then $H(\log h)$ is a Hardy field. Thus maximal Hardy fields contain $\mathbb{R}$, are real closed, and closed under exponentiation and integration.

Aims for Hardy fields

Work in progress (ADH) has among its goals to prove for Hardy fields H:

1. **for any differential polynomial** $P(Y) \in H[Y, Y', Y'', \ldots]$ **and** $f < g$ **in** H **with** $P(f) < 0 < P(g)$ **there exists a** y **in a Hardy field extension of** H **such that** $f < y < g$ **and** $P(y) = 0$
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Conjecture 1 $\Longrightarrow$ *all maximal Hardy fields are elementarily equivalent*

Conjectures 1,2 $\Longrightarrow$ *all maximal Hardy fields are isomorphic* (CH)

III. Transseries

Hardy: the logarithmic-exponential functions seem to cover all orders of infinity that occur naturally in mathematics. But he also suspected that the order of infinity of the compositional inverse of $(\log x)(\log\log x)$ differs from that of any logarithmic-exponential function; in fact, his suspicion is correct.

For a more revealing view of orders of infinity and a more comprehensive theory we need transseries. For example, transseries lead to an easy proof of Hardy's suspicion.

What are transseries?

Also called **logarithmic-exponential series**, they are formal series in a variable x involving typically exp and log. One can get a sense by considering an example like:

$$\mathrm{e}^{\mathrm{e}^{x}+\mathrm{e}^{x/2}+\mathrm{e}^{x/4}+\cdots} - 3\,\mathrm{e}^{x^2} + 5x^{\sqrt{2}} - (\log x)^{\pi} + 1 + x^{-1} + x^{-2} + \cdots + \mathrm{e}^{-x}\,.$$

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The field $\mathbb{T}$ of transseries has a somewhat lengthy inductive definition. For each transseries there is a finite bound on the "nesting" of exp and log in its transmonomials: series like

$$\frac{1}{x} + \frac{1}{e^x} + \frac{1}{e^{e^x}} + \frac{1}{e^{e^{e^x}}} + \cdots, \qquad \frac{1}{x} + \frac{1}{x\log x} + \frac{1}{x\log x\log\log x} + \cdots$$

are excluded. ("$\mathbb{T}$ is not spherically complete.")

$\mathbb{T}$ as an exponential field

The construction of $\mathbb{T}$ involves building an isomorphism $f \mapsto e^f$ of the ordered additive group of $\mathbb{T}$ onto its multiplicative group $\mathbb{T}^>$ of positive element, with inverse $g \mapsto \log g$

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As an ordered exponential field, $\mathbb{T}$ is an elementary extension of the ordered exponential field of real numbers.

$\mathbb{T}$ as a differential field

- Every $f \in \mathbb{T}$ can be *differentiated* term by term:

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- We obtain a *derivation* $f \mapsto f' : \mathbb{T} \to \mathbb{T}$ on the field $\mathbb{T}$:

$$(f+g) = f' + g', \quad (f \cdot g)' = f' \cdot g + f \cdot g'.$$

Its *constant field* is $\{f \in \mathbb{T} : \ f' = 0\} = \mathbb{R}$.

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- Every $f \in \mathbb{T}$ has an *antiderivative* in $\mathbb{T}$:

$$\int \frac{\mathrm{e}^{x}}{x}\,dx \;=\; \text{const} + \sum_{n=0}^{\infty} n!x^{-1-n}\,\mathrm{e}^{x} \quad \text{(diverges)}.$$

The dominance relation $\preccurlyeq$ on $\mathbb{T}$

For $f, g \in \mathbb{T}$,

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad |f| \leqslant c|g| \text{ for some } c \in \mathbb{R}^{>} \\ &\Longleftrightarrow \quad (\text{leading transmonomial of } f) \ \leqslant\ (\text{leading transmonomial of } g) \\ f \asymp g \quad &:\Longleftrightarrow \ f \preccurlyeq g \text{ and } g \preccurlyeq f \\ f \prec g \quad &:\Longleftrightarrow \ f \preccurlyeq g \text{ and } f \not\asymp g \end{aligned}$$

For example $\qquad 0 \prec \mathrm{e}^{-x} \prec x^{-10} \prec 1 \prec \log x \prec x^{1/10} \prec \mathrm{e}^{x} \prec \mathrm{e}^{\mathrm{e}^{x}}$

As in Hardy fields, $f > \mathbb{R} \Rightarrow f' > 0$, and we can differentiate and integrate dominance:

$$f \preccurlyeq g \iff f' \preccurlyeq g' \qquad \text{for nonzero } f, g \not\asymp 1.$$

$\mathbb{T}$ as a model-theoretic structure

We shall consider $\mathbb{T}$ as a *valued ordered differential field*, and model-theoretically as an $\mathcal{L}$-structure where the language $\mathcal{L}$ has primitives

$$0, \quad 1, \quad +, \quad -, \quad \cdot\,, \quad \partial \text{ (derivation)}, \quad \leqslant \text{ (ordering)}, \quad \preccurlyeq \text{ (dominance)}.$$

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More generally, let K be any ordered differential field with constant field $C = \{f \in K : \ f' = 0\}$. This yields a dominance relation $\preccurlyeq$ on K by

$$f \preccurlyeq g \ :\Longleftrightarrow\ |f| \leqslant c|g| \text{ for some positive } c \in C$$

and we view K accordingly as an $\mathcal{L}$-structure. We also introduce the valuation ring $\mathcal{O}$ of K,

$$\mathcal{O} \ := \ \{f \in K : f \preccurlyeq 1\} \ = \ \text{convex hull of } C \text{ in } K$$

with its maximal ideal $\scriptstyle\mathcal{O}$ $:= \ \{f \in K : \ f \prec 1\}$ of infinitesimals.

H-fields

An H-**field** is an ordered differential field K such that:

1. $f > C \Rightarrow f' > 0$;
2. $\mathcal{O} = C + \scriptstyle{\mathcal{O}}$.

Examples: Hardy fields that contain $\mathbb{R}$; differential subfields of $\mathbb{T}$ that contain $\mathbb{R}$.

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Examples: Hardy fields that contain $\mathbb{R}$; differential subfields of $\mathbb{T}$ that contain $\mathbb{R}$.

In particular, $\mathbb{T}$ is an H-field, but $\mathbb{T}$ has further basic elementary properties that do not follow from this: its derivation is *small*, and it is *Liouville closed*.

Here an H-field K is said to have **small derivation** if it satisfies $f \prec 1 \Rightarrow f' \prec 1$, and is said to be **Liouville closed** if it is real closed and for every $f \in K$ there are $g, h \in K$ such that $g' = f$ and $h \neq 0$ and $h'/h = f$.

IVP

We say that an H-field K has **IVP** (the Intermediate Value Property) if for every differential polynomial $P(Y) \in K[Y, Y', Y'', \ldots]$ and all $f < g$ in K with $P(f) < 0 < P(g)$ there is a $y \in K$ such that $f < y < h$ and $P(y) = 0$.

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Theorem

$\mathbb{T}$ *is completely axiomatized by the requirements:*

- *being an H-field with small derivation;*
- *being Liouville closed;*
- *having IVP.*

Actually, IVP is a bit of an afterthought. We mention it here for expository reasons and because it explains why Conjecture 1 on maximal Hardy fields from Part II implies that all maximal Hardy fields are elementarily equivalent.

Model Completeness

All H-fields embed into Liouville closed H-fields having IVP, and the latter are exactly the existentially closed H-fields. Thus:

Theorem

The theory of Liouville closed H-fields having IVP is model complete.

Byproduct of the proof: *If K is a Liouville closed H-field having IVP, then K has no proper differential-algebraic H-field extension with the same constant field.*

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IVP refers to the ordering, but the valuation given by $\preccurlyeq$ is more robust and more useful.

IVP comes from two more fundamental properties: ω-freeness and newtonianity.
These properties make sense for more general differential fields with a valuation satisfying
$f \preccurlyeq g \iff f' \preccurlyeq g' \quad (0 \neq f, g \not\asymp 1)$

ω-freeness and newtonianity

To give an inkling of these somewhat technical notions, let $P(Y) \in K[Y, Y', Y'', \ldots]$ be a nonzero differential polynomial. We wish to understand how the function $y \mapsto P(y)$ behaves for $y \in K$ not too far from the constant field C.

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It turns out that this function only reveals its true colors after rewriting P in terms of a derivation $\phi\partial$ with $\phi \in K^\times$ sufficiently large (with respect to $\preccurlyeq$) subject to $\phi\partial$ being small, that is, $\phi\partial\mathcal{O} \subseteq \mathcal{O}$. Indeed, this rewritten P will eventually be of the form $a \cdot (N(Y) + R(Y))$ with $a \in K^\times$, nonzero $N(Y) \in C[Y, Y', Y'', \ldots]$ **independent of** ϕ, and where the coefficients of R are infinitesimal (in $\mathcal{O}$). We call $N(Y)$ the **Newton polynomial** of P.

K is **ω-free** $:\Longleftrightarrow$ for all P as above its Newton polynomial has the form $A(Y)(Y')^n$ with $A(Y) \in C[Y]$.

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K is **newtonian** $:\Longleftrightarrow$ for all P as above with Newton polynomial of degree 1 we have $P(y) = 0$ for some $y \preccurlyeq 1$.

For H-fields: IVP $\Longrightarrow$ ω-free + newtonian; the converse holds for Liouville closed H-fields.

Back to Hardy fields

From what I said in part II it is clear that maximal Hardy fields are Liouville closed. We have shown recently that they are ω-free, and we have a good outline for proving they are newtonian. (Idea: solve "quasilinear" differential equations in the realm of Hardy fields by constructing fixpoints.) If that works out it would settle Conjecture 1 from part II, with the consequence that all maximal Hardy fields are elementarily equivalent to $\mathbb{T}$.

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The analytically most important differential subfield of $\mathbb{T}$ is no doubt Écalle's field $\mathbb{T}_{\text{as}}$ of *accelero-summable transseries*. It comes with an isomorphism (called "accelero-summation") onto the Hardy field of so-called *analysable functions*. It would be highly desirable to show that $\mathbb{T}_{\text{as}}$ is ω-free and newtonian; this would confirm Écalle's conjecture that every solution in $\mathbb{T}$ of a non-trivial algebraic differential equation over $\mathbb{T}_{\text{as}}$ lies already in $\mathbb{T}_{\text{as}}$.

Thank you for your attention!