

## Exercises - Algebraic Topology 1. List 5

Coverings, lifts, and subgroups corresponding to coverings

1. Given a covering  $p : Y \rightarrow X$  and a subspace  $A \subset X$ , put  $B = p^{-1}(A)$ . Show that the restriction  $p : B \rightarrow A$  is then also a covering.
2. Let  $p_1 : Y_1 \rightarrow X_1$  and  $p_2 : Y_2 \rightarrow X_2$  be some coverings. Show that the product map  $p_1 \times p_2 : Y_1 \times Y_2 \rightarrow X_1 \times X_2$  is also a covering. What is the multiplicity of this covering (when  $X_1$  and  $X_2$  are connected)?
3. Let  $X$  be a *locally connected* space (which means that any open neighbourhood of any point contains some smaller connected open neighbourhood of the same point). Let  $p : Y \rightarrow X$  be a covering. Verify that the restriction of  $p$  to any connected component in  $Y$  is also a covering of  $X$ . Show that the local connectedness assumption is essential.
4. Let  $p : Y \rightarrow X$  be a covering whose all fibres  $p^{-1}(x)$ , for  $x \in X$ , are finite. Prove that if  $X$  is compact, then  $Y$  is also compact.
5. Consider the subspace  $\Sigma \subset \mathbb{R}^2$  (with the induced topology), called *the Warsaw circle*, given in the polar coordinates  $(r, \theta)$  as

$$\Sigma := \left\{ \left( 1 + \frac{1}{2} \sin \frac{4\pi^2}{\theta}, \theta \right) : \theta \in (0, 2\pi] \right\} \cup \left\{ (r, 0) : r \in \left[ \frac{1}{2}, \frac{3}{2} \right] \right\}.$$

The map  $(r, \theta) \rightarrow e^{i\theta}$  restricted to  $\Sigma$  yields a continuous map  $f : \Sigma \rightarrow S^1$ . Show that

- (a)  $\Sigma$  is not a locally pathwise connected space;
- (b) the map  $f$  has no lift in the covering  $\mathbb{R} \rightarrow S^1$ .

Using this example, show that the assumption of local path-connectedness is essential in the lifting criterion.

6. Use the Warsaw circle to show that a connected covering of a path-connected space needn't be path-connected.
7. Prove that any connected and locally path-connected space is path-connected. Deduce that any connected covering of a locally path-connected space is path-connected.
8. Consider the map  $p : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$  (where  $\mathbb{C}$  is the set of complex numbers) given by  $p(z) = z^2$ .
  - (1) Verify that  $p$  is a covering.
  - (2) The choice of a lift  $x \in p^{-1}(u)$  with respect to  $p$ , for a number  $u \in \mathbb{C} \setminus \{0\}$ , is a choice of one of the square roots of  $u$ . Let  $X$  be a connected and locally path-connected space, and let  $f : X \rightarrow \mathbb{C} \setminus \{0\}$  be a continuous complex function. Formulate a necessary and sufficient condition (in algebraico-topological terms) for the property that the function  $f$  admits a continuous square root (pointwise).
9. Let  $G$  be a connected and locally path-connected topological group, let  $p : \tilde{G} \rightarrow G$  be any connected covering of  $G$ , and let  $\tilde{e} \in p^{-1}(e)$ .
  - (1) Consider the map  $m : \tilde{G} \times \tilde{G} \rightarrow G$  given by  $m(x, y) = p(x) \cdot p(y)$ . Prove that the image  $m_*[\pi_1(\tilde{G} \times \tilde{G}, (\tilde{e}, \tilde{e}))]$  is contained in the image  $p_*[\pi_1(\tilde{G}, \tilde{e})]$ .
  - (2) Deduce that on the space  $\tilde{G}$  there is (a unique) structure of a topological group, for which  $\tilde{e}$  is the group unit, and for which  $p$  is a group homomorphism.
10. Let  $X$  be connected and locally path-connected space and let  $\tilde{X}$  be a simply connected covering of  $X$ .

- (1) Check that  $\tilde{X}$  is unique up to an isomorphism of based coverings.
- (2) Prove that any connected covering of  $X$  is covered by  $\tilde{X}$  (this justifies the term *universal covering*).
11. Describe connected and simply connected coverings of the following spaces (and the corresponding covering maps):
- union of the sphere  $S^2$  and a straight segment connecting its any two antipodal points;
  - union of the torus  $S^1 \times S^1$  and the disk  $D^2 \times \{s_0\}$ , for some  $s_0 \in S^2$ ;
  - union of a 2-sphere and a circle intersecting this sphere twice;
  - quotient of the sphere  $S^2$  obtained by gluing to each other its north pole and its south pole.
12. Let  $p : Y \rightarrow X$  be a path-connected covering, and let  $y_0, y_1 \in p^{-1}(x_0)$ . Prove that the subgroups  $p_*(\pi_1(Y, y_0))$  and  $p_*(\pi_1(Y, y_1))$  are conjugate in the group  $\pi_1(X, x_0)$ .
13. Let  $p : Y \rightarrow X$  be a covering by a simply connected space  $Y$ , let  $A \subset X$  be a connected and locally path-connected subspace, and let  $B$  be a path-connected component of the preimage  $p^{-1}(A)$ . Show that the restricted map  $p : B \rightarrow A$  is a covering, and that the corresponding subgroup  $p_*(\pi_1 B) < \pi_1 A$  of this covering coincides with the kernel of the homomorphism  $\pi_1 A \rightarrow \pi_1 X$  induced by the inclusion.
14. Let  $S_n$  be the circle on the plane  $R^2$  centered at the point  $(0, \frac{1}{n})$  and with radius  $\frac{1}{n}$ . Let  $X = \cup_{n=1}^{\infty} S_n$ , with the topology induced from the plane (this space  $X$  is called the *Hawaiian earring*). Prove that  $X$  does not admit a simply connected covering.
15. Consider a covering  $p : Y \rightarrow X \times [0, 1]$  of a product space  $X \times [0, 1]$ . Show that, for  $i = 0, 1$ , the restricted coverings  $p_i : p^{-1}(X \times \{i\}) \rightarrow X \times \{i\}$  are isomorphic as coverings of  $X$  (with respect to the natural identifications of the spaces  $X \times \{i\}$  with the space  $X$ ).
16. Let  $p : \tilde{X} \rightarrow X$  be a covering, and let  $f : Y \rightarrow X$  be any continuous map. Define the space

$$f^*(\tilde{X}) = \{(y, z) \in Y \times \tilde{X} \mid f(y) = p(z)\},$$

with topology induced from the product topology of  $Y \times \tilde{X}$ . Consider also the map  $f^*(p) : f^*(\tilde{X}) \rightarrow Y$ , as the restriction of the projection  $Y \times \tilde{X} \rightarrow Y$ .

- Show that  $f^*(p) : f^*(\tilde{X}) \rightarrow Y$  is a covering. It is called the *pullback covering* of the covering  $p : \tilde{X} \rightarrow X$  with respect to the map  $f$ .
- Show that if two maps  $f, f' : Y \rightarrow X$  are homotopic, then the pullback coverings of a covering  $p : \tilde{X} \rightarrow X$  with respect to  $f$  and  $f'$ , respectively, are isomorphic as coverings.
- Prove that if a map  $f : Y \rightarrow X$  is homotopic with a constant map then the pullback of any covering  $p : \tilde{X} \rightarrow X$  with respect to  $f$  is a trivial covering.
- Let  $y_0 \in Y$ ,  $x_0 = f(y_0)$ , and let  $\tilde{x}_0 \in p^{-1}(x_0)$ . Prove that the subgroup

$$[f^*(p)]_*[\pi_1(f^*(\tilde{X}), (y_0, \tilde{x}_0))] < \pi_1(Y, y_0)$$

corresponding to the pullback covering  $f^*(p) : (f^*(\tilde{X}), (y_0, \tilde{x}_0)) \rightarrow (Y, y_0)$  coincides with the subgroup  $f_*^{-1}[p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$ , where  $f_* : \pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$  is the induced homomorphism of the map  $f$ .