Exercises - Algebraic Topology 1. List 6

Classification of coverings, normal coverings and covering actions.

1. Find and describe all connected coverings of the space $(S^1 \times S^1) \cup (D^2 \times \{s_0\})$.
2. Let a and b be the generators of the group $\pi_1(S^1 \vee S^1)$ corresponding to the S^1-summands in the wedge.
   (1) Let Θ_4 be the graph with 2 vertices and four edges each of which connects those two vertices. Describe a covering map $p : \Theta_4 \to S^1 \vee S^1$ and show that the subgroup in the free group $\pi_1(S^1 \vee S^1) = F_{a,b}$ which corresponds to this covering coincides with the subgroup Q which consists of all elements represented by words (over the alphabet a, b, a^{-1}, b^{-1}) of even length.
   (2) Describe (and provide the justification for) the covering of the wedge $S^1 \vee S^1$ which corresponds to the normal subgroup generated by the elements a^2, b^2 and $(ab)^4$.
3. Describe all 2-fold and 3-fold connected coverings of the wedge $S^1 \vee S^1$ up to an isomorphism with and without a base point.
4. Recall that the projective plane RP^2 has the fundamental group of order 2, $\pi_1 RP^2 = Z_2$, and that its connected 2-fold covering is the 2-sphere S^2. Find all connected coverings of the wedge $RP^2 \vee RP^2$. Which of those coverings are normal?
5. Construct some not normal coverings of the Klein bottle: (1) by the torus, and (2) by the Klein bottle.
7. Let X be a connected, locally path connected and semi-locally simply connected space. We say that a covering $\widetilde{X} \to X$ is *abelian* if it is normal and its group of deck-transformations is abelian. Show that X has an abelian covering which covers all other abelian coverings of X (via projection compatible with projections to X), and that such covering is unique up to an isomorphism. We will call it *the universal abelian covering* of X. Describe the universal abelian coverings of the wedges $S^1 \vee S^1$ and $S^1 \vee S^1 \vee S^1$. Describe also all abelian coverings of the wedge $RP^2 \vee RP^2$ of two projective planes, and in particular its universal abelian covering.
8. Given covering actions of groups G_i on spaces X_i, for $i = 1, 2$, consider the action of $G_1 \times G_2$ on $X_1 \times X_2$ defined by

$$(g_1, g_2)(x_1, x_2) := (g_1(x_1), g_2(x_2)).$$

   Show that this new action is also a covering action, and that the quotient

$$(X_1 \times X_2)/(G_1 \times G_2)$$

   is homeomorphic to the product of quotients $(X_1/G_1) \times (X_2/G_2)$.
9. Given a covering action of a group G on a connected and locally path connected space X, any subgroup $H < G$ induces naturally two coverings:

$$X \to X/H \quad \text{and} \quad X/H \to X/G.$$

Justify that

(a) any connected *intermediate* covering $q : Y \to X/G$ (i.e. a covering admitting a lift $X \to Y$ of the map $X \to X/G$, which is also a covering) is isomorphic with the covering $X/H \to X/G$, for some subgroup $H < G$;

(b) two coverings $X/H_1 \to X/G$ and $X/H_2 \to X/G$ as above are isomorphic precisely when the subgroups H_i are conjugate in G;

(c) the covering $X/H \to X/G$ is normal precisely when H is a normal subgroup in G, and the group of deck-transformations of this covering coincides then with the quotient group G/H.